Magic Squares with Centrally Embedded Squares of Odd Order: A Construction Method

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Abstract

We present a method which modifies a magic square of odd order n and then adds two outer rows and two outer columns to produce a magic square of order $n + 2$. The modification of the original square will preserve the equality of sums of the rows, columns, and main diagonals as well as other significant properties. This modified square will be centrally embedded in the magic square of order $n + 2$.

Definitions

For the purposes of this paper, a **magic square** of order n shall mean an $n \times n$ arrangement of the integers 1 through n^2 such that the sums of each row, each column and both main diagonals all equal the magic sum $S = \frac{n^2(n^2+1)}{2n} = \frac{n}{2}\,(n^2+1)$. An **embedded square** of order n shall mean an $n \times n$ arrangement of distinct positive integers such that each row, each column and both main diagonals have the same sum.

106	91	134	93	95	130	129	98	99	126	102	103	122	121	146
81	107	166	63	69	70	72	73	152	151	150	149	78	169	145
82	59	108	187	42	45	48	176	51	173	54	171	188	167	144
83	62	52	109	27	28	195	32	191	36	196	203	174	164	143
87	64	47	33	110	15	209	18	20	205	214	193	179	162	139
94	67	43	29	19	111	8	9	216	221	207	197	183	159	132
101	71	41	25	13	7	112	225	2	219	213	201	185	155	125
147	170	189	204	215	220	3	113	223	6	11	22	37	56	79
142	168	180	202	212	222	224	1	114	4	14	24	46	58	84
141	165	182	200	210	5	218	217	10	115	16	26	44	61	85
140	161	186	192	12	211	17	208	206	21	116	34	40	65	86
138	158	177	23	199	198	31	194	35	190	30	117	49	68	88
137	160	38	39	184	181	178	50	175	53	172	55	118	66	89
136	57	60	163	157	156	154	153	74	75	76	77	148	119	90
80	135	92	133	131	96	97	128	127	100	124	123	104	105	120

Figure 1

Figure 1 shows an order-15 magic square with embedded squares of all odd dimensions from 3×3 to 13×13 . This square contains the integers 1 through $15^2 = 225$. The median value of these integers, the number 113, appears at the exact center of this magic square. The number 113 is also the median of the fifteen consecutive integers which appear on one main diagonal (from upper left to lower right) of this square. Each row and each column consists of one number from this main diagonal, seven numbers less than those on this main diagonal and seven numbers greater than those on this main diagonal. The other main diagonal also consists of one number from this main diagonal, seven numbers less than those on this main diagonal and seven numbers greater than those on this main diagonal. This example illustrates these crucial properties that will be preserved at each successive step in building our magic square. Each step will begin with a magic square having these properties, modify it to produce an embedded square of the same odd order and then add outer rows and columns to complete the next higher odd order magic square. A smallest nontrivial magic square with all these properties is shown in Figure 2. An order-5 magic square with these properties is show in Figure 3.

4	9	2
3	5	7
8	1	6

Figure 2

11	8	9	16	21
7	12	25	2	19
20	3	13	23	6
22	24	1	14	4
5	18	17	10	15

Figure 3

22						38
	23	8	9	40	45	
	7	24	49	2	43	
	44	3	25	47	6	
	46	48	1	26	4	
	5	42	41	10	27	
12						28

Figure 4

We will illustrate our process by starting with the square in Figure 3 and creating an order-7 magic square having all the specified properties. The median value of integers 1 through 49 is 25. This is how we know that the values on the main diagonal will each increase by 12. Numbers 1 through 10 will not change and numbers 16 through 25 will increase by 24. This gives us the new square shown in Figure 4. The numbers that will make up the surrounding rows and columns will be used in pairs as shown in Figure 5. The pair (22,28) must extend one main diagonal and we choose (12,38) to extend the other, as shown in Figure 4.

pair		difference	
11	39		
12	38		
13	37		
14	36		
15	35	20	x
16	34		
17	33	16	
18	32	14	x
19	31		
20	30	10	x
21	29	8	
22	28		

Figure 5

At this point the trial-and-error work begins. Since the magic sum for our desired square is 175, we see that the five numbers completing the new top row must have a sum of 115 and those completing the new rightmost column must sum to 109. The number 38 in the new upper right corner is from the second column in Figure 5 (i.e., larger than those on the main diagonal), so in each case our remaining five numbers must consist of two more from that column and three from the first column. With this in mind, we find that $109 = 31 + 37 + 11 + 14 + 16$. Now five pairs remain, the ones with a number in the difference column of Figure 5. The sum of the numbers in column 2 for these five pairs is 159. We need a sum of 115. The difference of these numbers is 44. Our attempt will succeed if we can find three of the indicated numbers in the difference column of our table whose total is 44. We do indeed find $20 + 14 + 10 = 44$ from the pairs marked by an x and thus we have $115 = 15 + 33 + 18 + 20 + 29$. These numbers may be inserted in any order to complete the top row and rightmost column and the corresponding numbers in their respective pairs will complete the bottom row and first column. This produces the order-7 magic square in Figure 6. We have preserved all the necessary properties so that the same process may now be used to enlarge to an order-9 square.

22	15	33	18	20	29	38
19	23	8	9	40	45	31
13	7	24	49	2	43	37
39	44	3	25	47	6	11
36	46	48	1	26	4	14
34	5	42	41	10	27	16
12	35	17	32	30	21	28

Figure 6

Observations and Suggestions for Further Investigation

The reader will now see that the process described here was used to move from the square in Figure 2 to that of Figure 3. We note that it can also be used to move from the trivial order-1 magic square (one-by-one square whose only entry is 1) to that of Figure 2. The trial-and-error aspects of our method become easier as the size of the square increases. Having more numbers to work with makes it easier to find those which make up the necessary sums.

At the final step in our example, we needed to find three numbers from the column of differences whose sum would be 44. Since all numbers in that column were even, we can only hope to succeed at this final step if the needed sum is even. So, there is an issue of parity here. Experience showed, and a careful analysis confirmed, that this method can succeed only if the pairs chosen for the corners consist of four numbers of the same parity as they did above.

Each iteration of our enlargement process begins with an order-n magic square. When this square is modified, the integers 1 through $\frac{n^2-n}{2}$ are not increased. This means that they will occupy the same positions relative to the center of the square from that point forward. In fact, the modification which converts the order-n square to become the embedded square is deterministic. This is sufficient knowledge to let us run this procedure backward from a known square produced by this process or to recognize that a given square could not have been produced by this process.

As noted above, considerations of parity play an important role in our construction method. Note that in Figure 2, the five odd numbers occupy the second row and second column. This pattern may be extended. By carefully exercising our choices as we build outward, we can extend this pattern as shown in Figure 7, where the shading corresponds to parity. A different arrangement of the values not on the main diagonals gives us the parity pattern seen in Figure 8.

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137	57	60	163	156	157	154	153	74	75	76	77	148	119	89
80	135	92	133	96	131	128	97	100	127	124	123	104	105	120

Figure 7

106	98	134	130	93	95	91	129	121	99	103	102	122	126	146
142	107	166	72	70	69	63	73	149	151	150	152	78	169	84
82	62	108	48	42	45	187	51	171	173	54	176	188	164	144
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Figure 8

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