TWO-DIMENSIONAL FOURIER TRANSFORMATIONS AND DOUBLE MORDELL INTEGRALS II

MARTIN NICHOLSON

Abstract. Several Fourier transforms of functions of two variables are calculated. They enable one to calculate integrals that contain trigonometric and hyperbolic functions and also evaluate certain double Mordell integrals in closed form.

1. Introduction

This is continuation of the analysis that has been started in [7]. Among one of the formulas proved there was the following integral

$$\int_0^\infty \tanh(\pi x)\tanh(\alpha x)\cos(2\alpha x^2)\,dx = 0, \qquad \alpha > 0. \tag{1.1}$$

The fact that this integral converges can be seen by comparing it to the Fresnel integrals

$$\int_0^\infty \cos(\alpha x^2)\,dx = \int_0^\infty \sin(\alpha x^2)\,dx = \sqrt{\frac{\pi}{8\alpha}}, \qquad \alpha > 0,$$

using the asymptotics

$$\tanh(\pi x)\tanh(\alpha x) = 1 + O(e^{-cx}), \quad c > 0, \text{ when } x \to +\infty.$$

Integrals that contain trigonometric functions of the argument αx^2 and hyperbolic functions of both of the arguments πx and αx have been known, for example [1]

$$\int_0^\infty \frac{\cosh(\alpha x)}{\cosh(\pi x)}\cos(\alpha x^2)\,dx = \frac{1}{2}\cos\frac{\alpha}{4}, \qquad -\pi < \alpha < \pi. \tag{1.2}$$

Generalization of this integral with interesting applications was given in [3],[4]

$$\int_0^\infty \frac{\cosh(\pi x)\cosh(\alpha x)}{\cosh(2\pi x)+\cosh(2b)}\cos(\alpha x^2)\,dx = \frac{\cos\left(\frac{\alpha}{4}+\frac{\alpha b^2}{4\pi^2}\right)}{4\cosh(b)}, \qquad \alpha > 0. \tag{1.3}$$

What makes the formulas interesting is the fact that similar looking integrals do not always have closed form for all $\alpha > 0$. This can be demonstrated by the integral [5],[9]

$$\int_0^\infty \frac{e^{i\alpha x^2}}{\cosh(\pi x)}\cos(bx)\,dx = \sum_{k=0}^\infty (-1)^k e^{-b\left(k+\frac{1}{2}\right)-i\alpha\left(k+\frac{1}{2}\right)^2}$$
$$+\sqrt{\frac{\pi}{\alpha}}\sum_{k=0}^\infty (-1)^k e^{-\frac{\pi b}{\alpha}\left(k+\frac{1}{2}\right)+\frac{i\pi}{4}-\frac{ib^2}{4\alpha}+\frac{i\pi^2}{\alpha}\left(k+\frac{1}{2}\right)^2}, \quad \alpha > 0,\, b > 0. \tag{1.4}$$

One can notice that when $\alpha/\pi \in \mathbb{Q}$, the two series can be expressed in terms of finite sums, for example

$$\int_0^\infty \frac{e^{i\pi x^2}}{\cosh(\pi x)}\cos(bx)\,dx = \frac{e^{-\frac{\pi i}{4}} + ie^{-\frac{ib^2}{4\pi}}}{2\cosh\frac{b}{2}}.$$

However, no apparent closed form exists for general α.

One can notice that the poles of the integrand in formulas 1.2, 1.3 form an arithmetic progression with the common difference i. However, unless α is a rational multiple of π, the integrand in 1.1 has two sets of poles that form arithmetic progressions with incommensurate common differences i and $i\pi/\alpha$. For simplicity, in this paper, the integrals of the first type (equations 1.2, 1.3, 1.4) will be called type I, and

Date: August 27, 2021.

of the second type with two incommensurate sets of poles (equation 1.1 and others to be considered in section 3) will be called type II.

The fact that not all integrals of type II have closed form is demonstrated by the transformation formula

$$\sqrt{2}\int_0^\infty \frac{\cos(\alpha x^2)}{\cosh(\pi x)\cosh(\alpha x)}\,dx = \int_0^\infty \frac{\cosh(\frac{\pi x}{2})}{\cosh(\pi x)}\cdot\frac{\cosh(\frac{\alpha x}{2})}{\cosh(\alpha x)}\,dx, \qquad \alpha>0. \tag{1.5}$$

It is evident that the right hand side of 1.5 can not have a closed form unless $\alpha/\pi\in\mathbb{Q}$. It is worth mentioning here that the function $\frac{\cosh(\frac{\pi x}{2})}{\cosh(\pi x)}$ is an eigenfunction of the cosine Fourier transform (up to rescaling). By applying Plancherel type argument to two different eigenfunctions of the cosine Fourier transform, Ramanujan derived transformation formulas for integrals of products of self-reciprocal hyperbolic functions (e.g., equation 10 in [8]).

Sometimes a type II integral can be expressed in terms of integrals of type I

$$\int_0^\infty \frac{\sin(\alpha x^2)}{\sinh(\pi x)\sinh(\alpha x)}\cos(bx)\,dx = \left|\int_0^\infty \frac{e^{i\alpha x^2}}{\cosh(\pi x)}\cos(bx)\,dx\right|^2, \qquad \alpha>0, b>0. \tag{1.6}$$

Although in [7] this identity was proved for $b=0$, the proof easily can be extended to the case $b>0$.

The organization of the paper is as follows. Following the same logic that have been used in [7], we evaluate double Fourier transforms of certain functions of two variables in section 2. We find that the result of these Fourier transforms is the same function taken with minus sign, plus two terms with Dirac delta functions. In section 3 we use these formulas to calculate two type II integrals in closed form. Two more type II integrals will follow by taking linear combinations of the first two. In section 4, two double Mordell integrals will be evaluated. In the Appendix we give a proof of the formula 1.4 using Poisson summation formula.

2. Two auxiliary integrals

We will need the Sokhotski–Plemelj theorem:

$$\lim_{\varepsilon\to+0}\frac{1}{x\pm i\varepsilon} = \mathcal{P}\left(\frac{1}{x}\right)\mp i\pi\delta(x),$$

where $\mathcal{P}$ denotes the Cauchy principal value, and δ is the Dirac delta function.

Lemma 1. *For $a,b\in\mathbb{R}$*

$$\frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{\sin(xy)}{\tanh(x)\tanh(\pi y)}\cos(ax)\cos(by)\,dxdy = -\frac{\sin(ab)}{\tanh(\pi a)\tanh(b)}+\delta(a)+\pi\delta(b). \tag{2.1}$$

Proof. Let

$$I_{\varepsilon,\omega}(a,b) = \frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{\cosh((1-\varepsilon)x)\cosh(\pi(1-\omega)y)}{\sinh(x)\sinh(\pi y)}\sin(xy)\cos(ax)\cos(by)\,dxdy,$$

where $0<\varepsilon<1$, $0<\omega<1$. The integral in the lemma is clearly divergent. It will be regularized as

$$\lim_{\substack{\varepsilon\to+0\\ \omega\to+0}} I_{\varepsilon,\omega}(a,b).$$

Writing $\sin(xy)\cos(by)=\frac12\sin(y(x+b))+\frac12\sin(y(x-b))$ and calculating the integral over y using the formula 3.981.8 from [5]

$$\int_0^\infty \frac{\cosh(\theta y)}{\sinh(\pi y)}\sin(ay)\,dy = \frac12\cdot\frac{\sinh(a)}{\cosh(a)+\cos(\theta)}, \qquad 0<\theta<\pi,\ a>0, \tag{2.2}$$

yields

$$I_{\varepsilon,\omega}(a,b) = \frac{1}{2\pi}\int_0^\infty \frac{\cosh((1-\varepsilon)x)}{\sinh(x)}\left(\frac{\sinh(x+b)}{\cosh(x+b)-\cos(\pi\omega)}+\frac{\sinh(x-b)}{\cosh(x-b)-\cos(\pi\omega)}\right)\cos(ax)\,dx$$

$$= \frac{1}{\pi}\int_0^\infty \frac{(\cosh(x)-\cosh(b)\cos(\pi\omega))\cosh((1-\varepsilon)x)}{(\cosh(x+b)-\cos(\pi\omega))(\cosh(x-b)-\cos(\pi\omega))}\cos(ax)\,dx$$

$$= \operatorname{Re}\left\{\frac{1}{2\pi}\int_{-\infty}^\infty \frac{\cosh(x)-\cosh(b)\cos(\pi\omega)}{(\cosh(x+b)-\cos(\pi\omega))(\cosh(x-b)-\cos(\pi\omega))}e^{(1-\varepsilon)x+iax}dx\right\}.$$

Next, apply contour integration along a rectangular contour with vertices $(-R,0)$, $(R,0)$, $(R,2\pi i)$, $(-R,2\pi i)$, where $R\to\infty$. The integral over the line $\operatorname{Im} z = 2\pi$ will be equal to the integral over the real axis times $-e^{2(1-\varepsilon)\pi i-2\pi a}$. After tedious but quite straightforward calculation one obtains

$$I_{\varepsilon,\omega}(a,b) = \operatorname{Im}\left\{\frac{\sinh(\pi a+i\pi\epsilon-(ia-\epsilon+1)(b-i\pi\omega))}{2\sinh(\pi a+i\pi\epsilon)\sinh(b-\pi i\omega)} - \frac{\sinh(\pi a+i\pi\epsilon+(ia-\epsilon+1)(b+i\pi\omega))}{2\sinh(\pi a+i\pi\epsilon)\sinh(b+i\pi\omega)}\right\}.$$

Hence

$$\lim_{\substack{\varepsilon\to+0\\ \omega\to+0}} I_{\varepsilon,\omega}(a,b) = -\frac{\sin(ab)}{\tanh(\pi a)\tanh(b)} + \delta(a) + \pi\delta(b),$$

from which the claim follows. □

Lemma 2. *For $a,b\in\mathbb{R}$*

$$\frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{\sin(2xy)}{\tanh(x)\tanh(\pi y)}\cos(ax)\cos(by)\,dxdy = -\frac{\sin\frac{ab}{2}}{2\tanh\frac{\pi a}{2}\tanh\frac{b}{2}} + \delta(a) + \pi\delta(b).$$

Proof. Let

$$I_{\varepsilon,\omega}(a,b) = \frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{\cosh((1-\varepsilon)x)\cosh(\pi(1-\omega)y)}{\sinh(x)\sinh(\pi y)}\sin(2xy)\cos(ax)\cos(by)\,dxdy,$$

where $0<\varepsilon<1$, $0<\omega<1$. The integral in the lemma is

$$\lim_{\substack{\varepsilon\to+0\\ \omega\to+0}} I_{\varepsilon,\omega}(a,b).$$

$$I_{\varepsilon,\omega}(a,b) = \operatorname{Re}\left\{\frac{1}{\pi}\int_{-\infty}^\infty \frac{(\cosh(2x)-\cosh(b)\cos(\pi\omega))\cosh(x)}{(\cosh(2x+b)-\cos(\pi\omega))(\cosh(2x-b)-\cos(\pi\omega))}e^{(1-\varepsilon)x+iax}dx\right\}.$$

After simple calculation

$$I_{\varepsilon,\omega}(a,b) = \operatorname{Im}\left\{\frac{\sinh\frac{\pi a+i\pi\epsilon-(ia-\epsilon+1)(b-i\pi\omega)}{2}}{4\sinh\frac{\pi a+i\pi\epsilon}{2}\sinh\frac{b-\pi i\omega}{2}} - \frac{\sinh\frac{\pi a+i\pi\epsilon+(ia-\epsilon+1)(b+i\pi\omega)}{2}}{4\sinh\frac{\pi a+i\pi\epsilon}{2}\sinh\frac{b+i\pi\omega}{2}}\right\}.$$

Using contour integration, or the formula from the proof of the previous lemma one finds. It follows that

$$\lim_{\substack{\varepsilon\to+0\\ \omega\to+0}} I_{\varepsilon,\omega}(a,b) = -\frac{\sin\frac{ab}{2}}{2\tanh\frac{\pi a}{2}\tanh\frac{b}{2}} + \delta(a) + \pi\delta(b)$$

as required. □

3. Type II integrals

Theorem 3. *For $\alpha>0$*

$$\int_0^\infty \frac{\sin(\alpha x^2)\,dx}{\tanh(\pi x)\tanh(\alpha x)} = \frac{1}{4} + \frac{\pi}{4\alpha}.$$

Proof. In lemma 1, put $b=\alpha a$ to obtain

$$\frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{\sin(xy)}{\tanh(x)\tanh(\pi y)}\cos(ax)\cos(\alpha ay)\,dxdy = -\frac{\sin(\alpha a^2)}{\tanh(\pi a)\tanh(\alpha a)} + \left(1+\frac{\pi}{\alpha}\right)\delta(a).$$

Integrating with respect to a from 0 to ∞ using the formulas

$$\int_0^\infty \delta(a)\, da = \frac{1}{2},$$

$$\frac{2}{\pi}\int_0^\infty \cos(ax)\cos(\alpha a y)\, da = \delta(x-\alpha y) + \delta(x+\alpha y),$$

one finds

$$\int_0^\infty\int_0^\infty \frac{\sin(xy)}{\tanh(x)\tanh(\pi y)} \left(\delta(x-\alpha y)+\delta(x+\alpha y)\right) dxdy$$

$$= -\int_0^\infty \frac{\sin(\alpha a^2)}{\tanh(\pi a)\tanh(\alpha a)}\, da + \frac{1}{2}\left(1+\frac{\pi}{\alpha}\right).$$

After calculating the integral over x one finds the same integral both on the right and left hand side. Thus

$$2\cdot\int_0^\infty \frac{\sin(\alpha a^2)}{\tanh(\pi a)\tanh(\alpha a)}\, da = \frac{1}{2}\left(1+\frac{\pi}{\alpha}\right),$$

as required. □

Theorem 4. *For $\alpha > 0$*

$$\int_0^\infty \frac{\sin(2\alpha x^2)\, dx}{\tanh(\pi x)\tanh(\alpha x)} = \frac{1}{4} + \frac{\pi}{4\alpha}.$$

Proof. Proof follows from lemma 2 along the same lines as in the proof of the previous theorem. □

Theorem 5. *For $\alpha > 0$*

$$\int_0^\infty \frac{\tanh(\alpha x)}{\tanh(\pi x)} \sin(2\alpha x^2)\, dx = \frac{1}{4}.$$

Proof. Use the elementary identity

$$2\coth(2x) - \coth(x) = \tanh(x) \tag{3.1}$$

and the previous two theorems. □

Note that when $\alpha = \pi$, the formula in theorem 5 reduces to one of the Fresnel integrals.

Theorem 6. *For $\alpha > 0$*

$$\int_0^\infty \frac{\sin(\alpha x^2)\, dx}{\sinh(\pi x)\tanh(\alpha x)} = \frac{1}{4}.$$

Proof. Use the elementary identity

$$\coth(x) - \coth(2x) = \frac{1}{\sinh(2x)} \tag{3.2}$$

and theorems 3 and 4. □

4. Double Mordell integrals

Lemma 7. *Let $f(x)$ be an eigenfunction of the cosine Fourier transform, and $\alpha\beta = \pi$ or $\alpha\beta = \frac{\pi}{2}$. Then*

$$\int_0^\infty\int_0^\infty \frac{\sin(xy)}{\tanh(\alpha x)\tanh(\beta y)} f(x)f(y)\, dxdy = \frac{\pi^{3/2}}{4\sqrt{2}}\left(\alpha^{-1}+\beta^{-1}\right)(f(0))^2.$$

Proof. The fact that $f(x)$ is an eigenfunction of the cosine Fourier transform means that

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(bx)\,dx = f(b).$$

Multiplying 2.1 by $f(sa)f(tb)$ and integrating with respect to a and b from 0 to ∞ one obtains

$$\begin{aligned}
&\frac{1}{st}\int_0^\infty\int_0^\infty \frac{\sin(xy)}{\tanh(x)\tanh(\pi y)} f(x/s)f(y/t)\,dxdy \\
&= -\int_0^\infty\int_0^\infty \frac{\sin(ab)}{\tanh(\pi a)\tanh(b)} f(sa)f(tb)\,dadb + \frac{1}{2}\sqrt{\frac{\pi}{2}}\left(t^{-1}+\pi s^{-1}\right)(f(0))^2 .
\end{aligned}$$

It is not hard to notice that when $st=1$ both double integrals are equal to each other. Redefining the parameters according to $s=\alpha$, $\pi t=\beta$, we complete the proof of the case with $\alpha\beta=\pi$. The case $\alpha\beta=\frac{\pi}{2}$ is derived from lemma 2 in a similar manner. □

Theorem 8. *If $\alpha\beta=2\pi$ or $\alpha\beta=\pi$, then*

$$\int_0^\infty\int_0^\infty \frac{\sin(2xy)}{\tanh(\alpha x)\tanh(\beta y)}\, e^{-x^2-y^2}\,dxdy = \frac{\pi^{3/2}}{8}\left(\alpha^{-1}+\beta^{-1}\right).$$

Proof. This is direct consequence of the previous theorem and the fact that $f(x)=e^{-x^2/2}$ is an eigenfunction of the cosine Fourier transform. □

Theorem 9. *If $\alpha\beta=\pi$, then*

$$\int_0^\infty\int_0^\infty \frac{\tanh(\alpha x)}{\tanh(\beta y)}\,\sin(2xy)\,e^{-x^2-y^2}\,dxdy = \frac{\sqrt{\pi}}{8}\alpha.$$

Proof. Use identity 3.1 and the previous theorem. □

Theorem 10. *If $\alpha\beta=2\pi$, then*

$$\int_0^\infty\int_0^\infty \frac{\sin(2xy)}{\tanh(\alpha x)\sinh(\beta y)}\, e^{-x^2-y^2}\,dxdy = \frac{\sqrt{\pi}}{16}\alpha.$$

Proof. Use identity 3.2 and theorem 8. □

Appendix: Proof of 1.4

Ramanujan gave a proof of 1.4 using Laplace transform. Proof using contour integration can be found in [6]. The proof below is based on Poisson summation formula.

Proof. Using the partial fractions expansion

$$\frac{1}{\cosh(\pi x)} = \frac{1}{\pi}\sum_{k=0}^\infty \frac{(-1)^k(2k+1)}{x^2+\left(k+\frac{1}{2}\right)^2},$$

and integrating termwise one obtains

$$I(\alpha) = \frac{1}{2\pi}\sum_{k=0}^\infty (-1)^k(2k+1)\,I_k(\alpha),$$

where

$$I_k(\alpha) = \int_{-\infty}^\infty \frac{e^{i\alpha x^2+ibx}}{x^2+\left(k+\frac{1}{2}\right)^2}\,dx.$$

Integrals of this form can be reduced to error function. $I_k(\alpha)$ satisfies the following differential equation

$$I_k'(\alpha) + i\left(k+\frac{1}{2}\right)^2 I_k(\alpha) = i\int_{-\infty}^\infty e^{i\alpha x^2+ibx}\,dx = \sqrt{\frac{\pi}{\alpha}}\,e^{\frac{3i\pi}{4}-\frac{ib^2}{4\alpha}},$$

with the initial condition $I_k(0)=\frac{2\pi}{2k+1}\,e^{-b\left(k+\frac{1}{2}\right)}$. One can check by direct calculation (using integration by parts) that the solution is given by

$$I_k(\alpha)=\frac{2\pi}{2k+1}\,e^{-b\left(k+\frac{1}{2}\right)-i\alpha\left(k+\frac{1}{2}\right)^2}+\frac{4\sqrt{\pi}}{2k+1}\,e^{\frac{i\pi}{4}-\frac{ib^2}{4\alpha}}\int\limits_0^\infty e^{iy^2-\frac{yb}{\sqrt{\alpha}}}\sin\left(y\sqrt{\alpha}(2k+1)\right)\,dy.$$

Thus

$$I(\alpha)=\sum_{k=0}^{\infty}(-1)^k e^{-b\left(k+\frac{1}{2}\right)-i\alpha\left(k+\frac{1}{2}\right)^2}$$
$$+\,2e^{\frac{i\pi}{4}-\frac{ib^2}{4\alpha}}\sqrt{\frac{\pi}{\alpha}}\sum_{k=0}^{\infty}(-1)^k\int\limits_0^\infty e^{\frac{i\pi^2}{\alpha}y^2-\frac{\pi b}{\alpha}y}\sin\left(\pi(2k+1)y\right)\,dy.$$

Next, we apply Poisson summation formula in the form

$$2\sum_{k=0}^{\infty}(-1)^k\sin\left(\pi(2k+1)y\right)=\sum_{k=-\infty}^{\infty}(-1)^k\delta\left(y-k-\tfrac{1}{2}\right).$$

This completes the proof. □

References

[1] B. Berndt and G.E. Andrews, *Ramanujan's lost notebook, part IV*, Springer New York (2005).
[2] B. Cais, *On the transformation of infinite series* (unpublished), (1999).
[3] M.L. Glasser, *Generalization of a Definite Integral of Ramanujan*, J. Ind. Math. Soc. 37, 351 (1974).
[4] M.L. Glasser, https://arxiv.org/abs/1308.6361
[5] I.S. Gradshteyn, and I.M. Ryzhik, *Table of Integrals, Series, and Products*, 6th ed., Academic Press, Boston (2000).
[6] L. J. Mordell, *The definite integral* $\int_0^\infty \frac{e^{ax^2+bx}}{e^{cx}+d}\,dx$ *and the analytic theory of numbers*, Manchester. Acta Math. 61: 323-360 (1933).
[7] M. Nicholson, *Two-dimensional Fourier transformations and double Mordell integrals I*, https://arxiv.org/abs/1712.10324 (unpublished)
[8] S. Ramanujan, *Some definite integrals*, J. Indian Math. Soc. 11 (1915), 81-87.
[9] S. Ramanujan, *Some definite integrals connected with Gauss's sums*, Mess. Math. 44 (1915), 75-85.
[10] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, Cambridge university press (1996).