Riemann Hypothesis Proof using an equivalent criterion of Balazard, Saias and Yor

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Abstract
In this manuscript we denote a unit disc by $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and a semi plane as $\mathbb{P} = \{s \in \mathbb{C} \mid \Re(s) > \frac{1}{2}\}$. We denote, $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$ and $\mathbb{R}_{\geq 1} = \{x \in \mathbb{R} \mid x \geq 1\}$. Considering non negative real axis as a branch cut, we define a map from slit unit disc to the slit plane as $s : \mathbb{D} \setminus \mathbb{R}_{\geq 0} \to \mathbb{P} \setminus \mathbb{R}_{\geq 1}$ defined as $s(z) = \frac{1}{1-\sqrt{z}}$ which is proved to be one-one and onto. Next, we define a function $f(z) = (s-1)\zeta(s)$ where $s = s(z)$ and both $s(z)$ and $f(z)$ are proved to be analytic in $\mathbb{D} \setminus \mathbb{R}_{\geq 0}$. Next we prove that $s = s(z)$ is a conformal map. We also show that f is continuous at 0. Using Cauchy's residue theorem to a keyhole contour and Lebesgue's dominated convergence theorem along with Schwarz reflection principle, we prove that,

$$\int_{-\infty}^{\infty} \frac{\log|\zeta(\frac{1}{2}+it)|}{\frac{1}{4}+t^2} dt = 0$$

This settles the Riemann Hypothesis because this relation is an equivalent version of Riemann Hypothesis as proved by Balazard, Saias and Yor [1].
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1 Introduction

The Riemann zeta function, $\zeta(s)$ is defined as the analytic continuation of the Dirichlet series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

which converges in the half plane $\Re(s) > 1$. The Riemann zeta function is a meromorphic function on the whole complex s-plane, which is holomorphic everywhere except for a simple pole at $s = 1$ with residue 1. All the non trivial zeros of the Riemann zeta function lie in the critical strip $0 < \Re(s) < 1$. The Riemann Hypothesis states that all the non trivial zeros of the Riemann zeta function lie on the critical line $\Re(s) = \frac{1}{2}$.
Levinson [6], in 1974 proved that more than one third of zeros of Riemann zeta function are on the critical line. Balazard et al.(see [1, p.1] or [12, p.136]) in 1999 proved an equivalent of the Riemann Hypothesis using the theory of Hardy spaces (see [3],[4],[5],[11]). Shaoji Feng [7], in 2012 proved that atleast 41.28 % of the zeros of Riemann zeta function are on the critical line. Pratt et al.[8] in 2020 proved that more than five-twelfths of the zeros are on the critical line.

2 Main Result

Let $\sum_{\Re(\rho)>\frac{1}{2}}$ be the sum over the hypothetical zeros with real part greater than $\frac{1}{2}$ of the Riemann zeta function, $\zeta(s)$. In the sum, zeros of multiplicity m are counted m times.

Balazard et al. (see [1, p.1] or [12, p.136]) proved that,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\log|\zeta(\frac{1}{2}+it)|}{\frac{1}{4}+t^2}dt=\sum_{\Re(\rho)>\frac{1}{2}}\log\left|\frac{\rho}{1-\rho}\right| \tag{1}$$

and the Riemann Hypothesis is true if and only if (see [1, p.1] or [12, p.136]),

$$\int_{-\infty}^{\infty}\frac{\log|\zeta(\frac{1}{2}+it)|}{\frac{1}{4}+t^2}dt=0 \tag{2}$$

The goal of this paper is to prove the following result.

Theorem 1: If $\zeta(s)$ denotes the Riemann zeta function then

$$\int_{-\infty}^{\infty}\frac{\log|\zeta(\frac{1}{2}+it)|}{\frac{1}{4}+t^2}dt=0$$

We start the proof of Theorem 1 as follows: Denote a unit disc as $\mathbb{D}=\{z\in\mathbb{C}\mid |z|<1\}$ and $\mathbb{D}\setminus\mathbb{R}_{\geq 0}$ (where $\mathbb{R}_{\geq 0}=\{x\in\mathbb{R}\mid x\geq 0\}$) as a slit disk which is simply connected [14, p.108] having all points that are in disc $\mathbb{D}$ except the non negative reals which means $\mathbb{D}\setminus[0,1)$. Let z denote an element of the disc $\mathbb{D}$. Considering the non negative real axis (i.e. $[0,\infty)$) as the branch cut and $0\leq\arg z<2\pi$ we define for $z=re^{i\theta}$,

$$\sqrt{z}:=\sqrt{r}e^{i\theta/2},\quad 0\leq\theta<2\pi$$

write,

$$s=s(z)=\frac{1}{2}+\frac{1+\sqrt{z}}{2(1-\sqrt{z})}=\frac{1}{1-\sqrt{z}} \tag{3}$$

Define a semi plane as $\mathbb{P}=\{s\in\mathbb{C}\mid\Re(s)>\frac{1}{2}\}$. For $R<1$, denote $\overline{\mathbb{D}_R}=\{z\in\mathbb{C}\mid|z|\leq R\}$ and $\mathbb{R}_{\geq 1}=\{x\in\mathbb{R}\mid x\geq 1\}$. We denote by f^* the function defined almost everywhere on the circle $\partial\mathbb{D}=\{z\in\mathbb{C}\mid|z|=1\}$ by $f^*(e^{i\theta})=\lim_{R\to 1^-}f(Re^{i\theta})$. We will now prove some Lemmas:

Lemma 1.1: Map $s:\mathbb{D}\setminus\mathbb{R}_{\geq 0}\to\mathbb{P}\setminus\mathbb{R}_{\geq 1}$ is one-one and onto.

Proof: For proving the map one-one, let $s(z)=s(z')$ where $z,z'\in\mathbb{D}\setminus\mathbb{R}_{\geq 0}$.
Write $z=re^{i\theta}$ and $z'=r'e^{i\theta'}$ so we get, $\sqrt{r}e^{i\theta/2}=\sqrt{r'}e^{i\theta'/2}$ and taking modulus we have $\sqrt{r}=\sqrt{r'}$ or $r=r'$ and hence $e^{i\theta/2}=e^{i\theta'/2}$. Hence we have, $\cos(\frac{\theta-\theta'}{2})=1$ and $\sin(\frac{\theta-\theta'}{2})=0$.
Since $\theta,\theta'\in(0,2\pi)$, so we get $\theta=\theta'$ and hence we have $z=z'$.
For onto, let $s_0\in\mathbb{P}\setminus\mathbb{R}_{\geq 1}$ then there exists $z_0\in\mathbb{D}\setminus\mathbb{R}_{\geq 0}$ such that $\sqrt{z_0}=\left(\frac{s_0-1}{s_0}\right)$ and $s(z_0)=s_0$.

Now we consider the function,

$$f(z)=(s-1)\zeta(s) \tag{4}$$

where

$$s=\frac{1}{1-\sqrt{z}} \tag{5}$$

then,

$$f(z)=\left(\frac{\sqrt{z}}{1-\sqrt{z}}\right)\zeta\left(\frac{1}{1-\sqrt{z}}\right) \tag{6}$$

Lemma 1.2: $s=s(z)=\frac{1}{1-\sqrt{z}}$ is analytic in $\mathbb{D}\setminus\mathbb{R}_{\geq 0}$ and $f(z)=(s-1)\zeta(s)$ is analytic in $\mathbb{D}\setminus\mathbb{R}_{\geq 0}$.

Proof: Any $z\in\mathbb{D}\setminus\mathbb{R}_{\geq 0}$ can be written uniquely as $z=re^{i\theta}$, where $r>0$ and $\theta\in(0,2\pi)$.
Next, we define a function $h:\mathbb{D}\setminus\mathbb{R}_{\geq 0}\to\mathbb{C}$ as $h(z):=\sqrt{z}$ and in polar form as:

$$\forall(r,\theta)\in\mathbb{R}_{>0}\times(0,2\pi):\quad h(re^{i\theta}):=\sqrt{r}e^{i\theta/2} \tag{7}$$

$$=\sqrt{r}\cos\left(\frac{\theta}{2}\right)+i\left[\sqrt{r}\sin\left(\frac{\theta}{2}\right)\right] \tag{8}$$

$$=u(r,\theta)+i\cdot v(r,\theta). \tag{9}$$

Now, functions u and v satisfy the polar version of the Cauchy-Riemann equations [10, p.232]:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

$u_r = \frac{1}{2\sqrt{r}}\cos\left(\frac{\theta}{2}\right)$, $u_\theta = -\frac{\sqrt{r}}{2}\sin\left(\frac{\theta}{2}\right)$, $v_r = \frac{1}{2\sqrt{r}}\sin\left(\frac{\theta}{2}\right)$ and $v_\theta = \frac{\sqrt{r}}{2}\cos\left(\frac{\theta}{2}\right)$. Since partial derivatives of u and v satisfy Cauchy-Riemann equations and these partial derivatives are continuous in $\mathbb{D}\setminus\mathbb{R}_{\geq 0}$, so h is analytic in $\mathbb{D}\setminus\mathbb{R}_{\geq 0}$.
Since, $s(z) = \frac{1}{1-\sqrt{z}}$ and $h(z) = \sqrt{z}$ is analytic in $\mathbb{C}\setminus\mathbb{R}_{\geq 0}$, also $q(z) = \frac{1}{1-z}$ is analytic in $\mathbb{C}\setminus\mathbb{R}_{\geq 0}$ and hence the composition, $s(z) = q(h(z))$ is analytic in $\mathbb{D}\setminus\mathbb{R}_{\geq 0}$. Now $k(z) = (z-1)\zeta(z)$ is analytic, so the composition $k(s(z)) = f(z)$ is analytic in $\mathbb{D}\setminus\mathbb{R}_{\geq 0}$.

Lemma 1.3: Map $s : \mathbb{D}\setminus\mathbb{R}_{\geq 0} \to \mathbb{P}\setminus\mathbb{R}_{\geq 1}$ is conformal which takes the slit disc $\mathbb{D}\setminus\mathbb{R}_{\geq 0}$ to the slit plane $\mathbb{P}\setminus\mathbb{R}_{\geq 1}$.

Proof: Since $s(z)$ is analytic in $\mathbb{D}\setminus\mathbb{R}_{\geq 0}$ so we have,

$$s'(z) = \frac{1}{2\sqrt{z}(1-\sqrt{z})^2}$$

Since $\mathbb{D}\setminus\mathbb{R}_{\geq 0}$ is an open set [14, p.108] and the derivative of $s(z)$ is non zero everywhere in $\mathbb{D}\setminus\mathbb{R}_{\geq 0}$ and also by Lemma 1.2 $s(z)$ is analytic in $\mathbb{D}\setminus\mathbb{R}_{\geq 0}$ and hence $s(z)$ is conformal.
Also $s = \frac{1}{1-\sqrt{z}}$, hence $z = \left(\frac{s-1}{s}\right)^2$ so that $|z| < 1$ if and only if $\Re(s) > \frac{1}{2}$. Since by Lemma 1.1, $s(z)$ is one-one and onto, so it takes the slit disc $\mathbb{D}\setminus\mathbb{R}_{\geq 0}$ to the slit plane $\mathbb{P}\setminus\mathbb{R}_{\geq 1}$.

Lemma 1.4: $f(z)$ is continuous at $z = 0$ and $\log|f(0)| = 0$.

Proof: Since $h(z) = \sqrt{z}$ is continuous at 0, so $s(z) = \frac{1}{1-\sqrt{z}}$ is continuous at 0.
Define $p(z) := (z-1)\zeta(z)$. Since $f(z) = (s-1)\zeta(s)$ where $s = \frac{1}{1-\sqrt{z}}$ so, $p(s(z)) = f(z)$. Since $s(z)$ is continuous at 0 and $p(z)$ is continuous at $s(0) = 1$, so we have the composition $p(s(z)) = f(z)$ is continuous at 0. Hence,

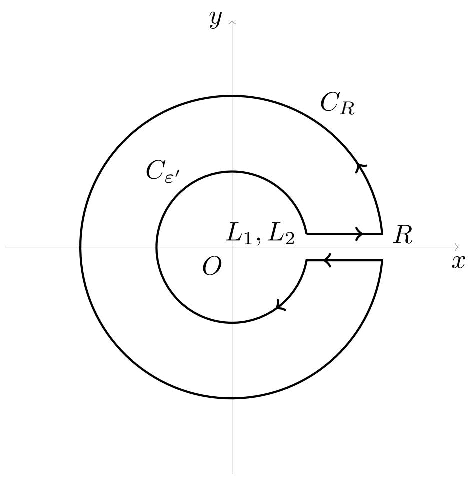
$$f(0) = \lim_{z\to 0} f(z) = \lim_{s\to 1}(s-1)\zeta(s)$$

So since $\lim_{s\to 1}(s-1)\zeta(s) = 1$ so we have,

$$f(0) = 1 \tag{10}$$

So,

$$\log|f(0)| = 0 \tag{11}$$



Consider a keyhole contour (simple closed contour) $C(\epsilon', R, \rho)$ consisting of two concentric circles, a bigger circle C_R of radius R unit, $0 < R < 1$ and a smaller circle $C_{\epsilon'}$ of radius ϵ' where $\epsilon' > 0$ arbitrarily small and having an infinitesimally small cross-cut to join C_R and $C_{\epsilon'}$. In this contour we exclude the non negative real axis (i.e. $[0,\infty)$). Let, this cross-cut be L_1 above the positive x-axis and L_2 below positive the x-axis. Let vertical distance between L_1 and x-axis be $\rho > 0$ and vertical distance between L_2 and x-axis be $\rho > 0$. Then we have ,

$$C(\epsilon', R, \rho) = C_R + L_1 - C_{\epsilon'} + L_2 \quad \text{where} \quad \epsilon' > 0 \quad \text{arbitrarily small} \quad \text{and} \quad 0 < R < 1$$

Let $\mathbb{I}(C)$ denote the interior of curve $C(\epsilon', R, \rho)$ and $\overline{\mathbb{I}(C)}$ denote the closure of interior of the curve $C(\epsilon', R, \rho)$.

<u>**Lemma 1.5**</u>:

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| d\theta = \log|f(0)| + \sum_{\alpha_n \in \overline{\mathbb{D}_R}, f(\alpha_n)=0} \log \frac{R}{|\alpha_n|}$$

Proof: By Lemma 1.2, since f is analytic in $\mathbb{D} \setminus \mathbb{R}_{\geq 0}$ so it is analytic in $\overline{\mathbb{D}_R} \setminus \mathbb{R}_{\geq 0}$ where $R < 1$ and hence f is analytic on and inside the simple closed contour C which is compact, so its zeros on and inside C are finite say, α_n. We define a finite product,

$$B(z) := \prod_{\alpha_n \in \overline{\mathbb{I}(C)}, f(\alpha_n)=0} \left(\frac{R^2 - \overline{\alpha_n} z}{R(z-\alpha_n)}\right) \frac{\alpha_n}{|\alpha_n|} \tag{12}$$

where in the above product, zeros of multiplicity m are counted m times. Define a function,

$$g(z) := f(z)B(z) = f(z) \prod_{\alpha_n \in \overline{\mathbb{I}(C)}, f(\alpha_n)=0} \left(\frac{R^2 - \overline{\alpha_n} z}{R(z-\alpha_n)}\right) \frac{\alpha_n}{|\alpha_n|} \tag{13}$$

By definition of $g(z)$, since $B(z)$ is a finite product whose denominators are the zeros of $f(z)$ and $f(z)$ is analytic in $\overline{\mathbb{I}(C)}$ (since f is analytic in $\overline{\mathbb{D}_R} \setminus \mathbb{R}_{\geq 0}$) so $g(z)$ is analytic and non zero in $\overline{\mathbb{I}(C)}$. By Cauchy's residue theorem [14, p.133] since $\frac{\log g(z)}{z}$ is analytic on and inside the simple closed contour C and $g(z)$ is non zero on and inside C so,

$$\oint_{C(\epsilon', R, \rho)} \frac{\log g(z)}{z} dz = 0$$

Since, $C(\epsilon', R, \rho) = C_R - C_{\epsilon'} + L_1 + L_2$ so we have

$$\Rightarrow \int_{C_R} \frac{\log g(z)}{z} dz - \int_{C_{\epsilon'}} \frac{\log g(z)}{z} dz + \int_{L_1} \frac{\log g(z)}{z} dz + \int_{L_2} \frac{\log g(z)}{z} dz = 0 \tag{14}$$

On C_R we have $z = Re^{i\theta}$, on $C'_\epsilon : z = Re^{i\theta}$, on $L_1 : z = x + i\rho$ and on $L_2 : z = x - i\rho$. Let ρ (which is the distance between L_1 and x-axis) tend to 0^+ so we have,

$$i.\int_0^{2\pi} \log g(Re^{i\theta}) d\theta - i.\int_0^{2\pi} \log g(\epsilon' e^{i\theta}) d\theta + \lim_{\rho \to 0^+} \left(\int_{\epsilon'}^{R} \frac{\log g(x+i\rho)}{x+i\rho} dx - \int_{\epsilon'}^{R} \frac{\log g(x-i\rho)}{x-i\rho} dx \right) = 0 \tag{15}$$

For $g(z)$ as defined in equation (13), we next prove using Schwarz reflection principle

$$\lim_{\rho \to 0^+} \left(\int_{\epsilon'}^{R} \frac{\log g(x+i\rho)}{x+i\rho} dx - \int_{\epsilon'}^{R} \frac{\log g(x-i\rho)}{x-i\rho} dx \right) = 0 \tag{16}$$

Define an open set $\Omega = \mathbb{D} \setminus \mathbb{R}_{\geq 0}$. Let Ω^+ denote the part of Ω which lies in the upper half-plane and Ω^- denote the part of Ω which lies in the lower half-plane. Also let $I = \Omega \cap \mathbb{R}$ so that I denotes the interior of that part of the boundary of Ω^+ and Ω^- that lies on the real axis. Then we have

$$\Omega = \Omega^+ \cup I \cup \Omega^-$$

Since by Lemma 1.2, f is holomorphic function in Ω^+ (since it is holomorphic in Ω) that extends continuously to I and such that f is real valued on I (since ζ is real valued on I) then since by the figure of contour C we have $x - i\rho \in \Omega^-$, so using Schwarz reflection principle [15, p.60] on Riemann zeta function we have for $f(z) = (s-1)\zeta(s)$ where $s = \frac{1}{1-\sqrt{z}}$, $\overline{f(x+i\rho)} = f(\overline{x+i\rho}) = f(x-i\rho)$. So

using this fact and since by equation (12) the finite product $B(z)$ satisfies, $\overline{B(x+i\rho)} = B(x-i\rho)$ so equation (13) gives $\overline{g(x+i\rho)} = g(x-i\rho)$.
Let us denote

$$T = \frac{1}{2i}\left(\frac{\log g(x+i\rho)}{x+i\rho} - \frac{\log g(x-i\rho)}{(x-i\rho)}\right)$$

then we have $T = \Im\left(\frac{\log g(x+i\rho)}{x+i\rho}\right)$. Since g is analytic on and inside the keyhole contour C, so it is continuous on and inside C and we have

$$\lim_{\rho\to 0^+} \Im\left(\frac{\log g(x+i\rho)}{x+i\rho}\right) = \frac{1}{2i}\lim_{\rho\to 0^+}\left(\frac{\log g(x+i\rho)}{x+i\rho} - \frac{\log g(x-i\rho)}{(x-i\rho)}\right)$$

$$\Rightarrow \lim_{\rho\to 0^+} \Im\left(\frac{\log g(x+i\rho)}{x+i\rho}\right) = \frac{1}{2i}\left(\lim_{\rho\to 0^+}\frac{\log g(x+i\rho)}{x+i\rho} - \lim_{\rho\to 0^+}\frac{\log \overline{g(x+i\rho)}}{\overline{(x+i\rho)}}\right)$$

$$\Rightarrow \lim_{\rho\to 0^+} \Im\left(\frac{\log g(x+i\rho)}{x+i\rho}\right) = \frac{1}{2i}\left(\lim_{\rho\to 0^+}\frac{\log g(x+i\rho)}{x+i\rho} - \lim_{\rho\to 0^+}\overline{\left(\frac{\log g(x+i\rho)}{(x+i\rho)}\right)}\right)$$

Since conjugation is a continuous function and g is analytic on C so we get

$$\lim_{\rho\to 0^+} \Im\left(\frac{\log g(x+i\rho)}{x+i\rho}\right) = \frac{1}{2i}\left(\frac{\log g(x)}{x} - \overline{\left(\frac{\log g(x)}{x}\right)}\right)$$

Since ζ is real on the real line so g is real on the real line and we have

$$\lim_{\rho\to 0^+} \Im\left(\frac{\log g(x+i\rho)}{x+i\rho}\right) = 0$$

So by epsilon-delta definition of limit, given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left|\Im\left(\frac{\log g(x+i\rho)}{x+i\rho}\right)\right| < \epsilon \quad \text{whenever} \quad \rho < \delta$$

$$\Rightarrow -\epsilon < \Im\left(\frac{\log g(x+i\rho)}{x+i\rho}\right) < \epsilon \quad \text{whenever} \quad \rho < \delta$$

On integrating both sides of above inequality,

$$-(R-\epsilon')\epsilon < \int_{\epsilon'}^{R} \Im\left(\frac{\log g(x+i\rho)}{x+i\rho}\right) dx < (R-\epsilon')\epsilon \quad \text{whenever} \quad \rho < \delta$$

$$\Rightarrow \frac{1}{(R-\epsilon')}\lim_{\rho\to 0^+}\int_{\epsilon'}^{R} \Im\left(\frac{\log g(x+i\rho)}{x+i\rho}\right) dx = 0$$

$$\Rightarrow \lim_{\rho\to 0^+}\left(\int_{\epsilon'}^{R}\frac{\log g(x+i\rho)}{x+i\rho}dx - \int_{\epsilon'}^{R}\frac{\log g(x-i\rho)}{x-i\rho}dx\right) = 0$$

which proves equation (16). So equation (15) gives

$$\int_0^{2\pi} \log g(Re^{i\theta})d\theta = \int_0^{2\pi} \log g(\epsilon' e^{i\theta})d\theta$$

Taking real parts on both sides,

$$\int_0^{2\pi} \log|g(R.e^{i\theta})|d\theta = \int_0^{2\pi} \log|g(\epsilon'.e^{i\theta})|d\theta$$

Taking limit as $\epsilon \to 0^+$ we get,

$$\int_0^{2\pi} \log|g(R.e^{i\theta})|d\theta = \lim_{\epsilon'\to 0^+}\int_0^{2\pi} \log|g(\epsilon'.e^{i\theta})|d\theta \tag{17}$$

By equation (13) putting $g(z) = f(z)B(z)$ in the left hand side of above equation we have,

$$\int_0^{2\pi} \log|f(R.e^{i\theta}).B(R.e^{i\theta})|d\theta = \lim_{\epsilon' \to 0^+} \int_0^{2\pi} \log|g(\epsilon'.e^{i\theta})|d\theta \tag{18}$$

On $|z| = R$, using equation (12) we have,

$$|B(z)| = \prod_{\alpha_n \in \overline{\mathbb{I}(C)}, f(\alpha_n)=0} \left| \frac{R^2 - \overline{\alpha_n} z}{R(z - \alpha_n)} \right| \tag{19}$$

On $|z| = R$,

$$R^2(z - \alpha_n)(\overline{z} - \overline{\alpha_n}) = R^2(z\overline{z} - (\overline{\alpha_n}z + \overline{z}\alpha_n) + \alpha_n\overline{\alpha_n})$$
$$\Rightarrow R^2(z - \alpha_n)(\overline{z} - \overline{\alpha_n}) = R^2(R^2 - (\overline{\alpha_n}z + \overline{z}\alpha_n) + \alpha_n\overline{\alpha_n})$$
$$\Rightarrow R^2(z - \alpha_n)(\overline{z} - \overline{\alpha_n}) = (R^2 - \overline{\alpha_n}z)(R^2 - \alpha_n\overline{z})$$

$$\left| \frac{R^2 - \overline{\alpha_n} z}{R(z - \alpha_n)} \right| = 1 \tag{20}$$

So, using equation (19) and (20),

$$|B(Re^{i\theta})| = 1 \tag{21}$$

Next we prove that in equation (18),

$$\lim_{\epsilon' \to 0^+} \int_0^{2\pi} \log|g(\epsilon'.e^{i\theta})|d\theta = 2\pi \log|g(0)|$$

Since by Lemma 1.4 and equation (13) g is continuous at 0, so we have $\lim_{z\to 0} g(z) = g(0)$.
Since modulus is a continuous function, so we have $\lim_{z\to 0} |g(z)| = |g(0)|$.
Since logarithm is a continuous function and $g(0) \neq 0$, so we have $\lim_{z\to 0} \log|g(z)| = \log|g(0)|$.
So given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|\log|g(z)| - \log|g(0)|| < \epsilon \quad \text{whenever } |z - 0| < \delta$$

Writing $z = \epsilon' e^{i\theta}$, we have

$$\log|g(0)| - \epsilon < \log|g(\epsilon'.e^{i\theta})| < \log|g(0)| + \epsilon \quad \text{whenever } \epsilon' < \delta$$

Integrating we get,

$$2\pi \log|g(0)| - 2\pi\epsilon < \int_0^{2\pi} \log|g(\epsilon'.e^{i\theta})|d\theta < 2\pi \log|g(0)| + 2\pi\epsilon \quad \text{whenever } \epsilon' < \delta$$

$$\Rightarrow \left| \left(\frac{1}{2\pi} \int_0^{2\pi} \log|g(\epsilon'.e^{i\theta})|d\theta \right) - \log|g(0)| \right| < \epsilon \quad \text{whenever } \epsilon' < \delta$$

So we have for $\epsilon' > 0$ arbitrarily small,

$$\lim_{\epsilon' \to 0^+} \int_0^{2\pi} \log|g(\epsilon'.e^{i\theta})|d\theta = 2\pi \log|g(0)| \tag{22}$$

Since g is continuous at 0 so $g(0) = \lim_{\epsilon' \to 0^+} g(\epsilon' e^{i\theta})$. By equation (13), as $\epsilon' \to 0^+$ the closure of interior of the curve C which is $\overline{\mathbb{I}(C)}$ becomes $\overline{\mathbb{D}_R}$, so we get

$$|g(0)| = |f(0)| \prod_{\alpha_n \in \overline{\mathbb{D}_R}, f(\alpha_n)=0} \frac{R}{|\alpha_n|} \tag{23}$$

putting the value of $|g(0)|$ from equation (23) in equation (22) we get,

$$\lim_{\epsilon' \to 0^+} \int_0^{2\pi} \log|g(\epsilon'.e^{i\theta})|d\theta = 2\pi \log|f(0)| + 2\pi \sum_{\alpha_n \in \overline{\mathbb{D}_R}, f(\alpha_n)=0} \log \frac{R}{|\alpha_n|} \tag{24}$$

Using equation (21) and (24) in equation (18) we have,

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|d\theta = \log|f(0)| + \sum_{\alpha_n\in\overline{\mathbb{D}_R}, f(\alpha_n)=0}\log\frac{R}{|\alpha_n|} \tag{25}$$

Lemma 1.6:

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f^*(e^{i\theta})|d\theta = \log|f(0)| + \sum_{\alpha_n\in\mathbb{D}\setminus\mathbb{R}_{\geq 0}, f(\alpha_n)=0}\log\frac{1}{|\alpha_n|}$$

Proof: Taking $R\to 1^-$ in equation (25) we get,

$$\lim_{R\to 1^-}\frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|d\theta = \log|f(0)| + \lim_{R\to 1^-}\sum_{\alpha_n\in\overline{\mathbb{D}_R}, f(\alpha_n)=0}\log\frac{R}{|\alpha_n|} \tag{26}$$

We first prove that

$$\lim_{R\to 1^-}\sum_{\alpha_n\in\overline{\mathbb{D}_R}, f(\alpha_n)=0}\log\frac{R}{|\alpha_n|} = \sum_{\alpha_n\in\overline{\mathbb{D}}, f(\alpha_n)=0}\log\frac{1}{|\alpha_n|}$$

On the one hand, when $\alpha_n\in\overline{\mathbb{D}_R}$ then $\frac{R}{|\alpha_n|}\geq 1$ and when $\alpha_n\in\overline{\mathbb{D}}$ then $\frac{1}{|\alpha_n|}\geq 1$. Also we have,

$$\sum_{\alpha_n\in\overline{\mathbb{D}_R}, f(\alpha_n)=0}\log\frac{R}{|\alpha_n|} \leq \sum_{\alpha_n\in\overline{\mathbb{D}}, f(\alpha_n)=0}\log\frac{1}{|\alpha_n|} \quad \forall\ R<1 \tag{27}$$

On the other hand, $\sum_{\alpha_n\in\overline{\mathbb{D}_R}, f(\alpha_n)=0}\log\frac{R}{|\alpha_n|}$ is monotonically increasing and is bounded above (for the latter see [1, p.2] and Lemma 1.8). Thus the limit $L := \lim_{R\to 1^-}\sum_{\alpha_n\in\overline{\mathbb{D}_R}, f(\alpha_n)=0}\log\frac{R}{|\alpha_n|}$ exists. Also,

$$L \geq \sum_{|\alpha_n|\leq R_1, f(\alpha_n)=0}\log\frac{R_2}{|\alpha_n|} \quad \forall R_1, R_2 < 1$$

Let $R_2\to 1^-$, we obtain

$$L \geq \sum_{|\alpha_n|\leq R_1, f(\alpha_n)=0}\log\frac{1}{|\alpha_n|} \quad \forall R_1 < 1$$

Let $R_1\to 1^-$, we obtain

$$L \geq \sum_{|\alpha_n|\leq 1, f(\alpha_n)=0}\log\frac{1}{|\alpha_n|}$$

So we get,

$$\lim_{R\to 1^-}\sum_{|\alpha_n|\leq R, f(\alpha_n)=0}\log\frac{R}{|\alpha_n|} = L = \sum_{|\alpha_n|\leq 1, f(\alpha_n)=0}\log\frac{1}{|\alpha_n|} \tag{28}$$

Since on $|\alpha_n| = 1$ we have $\log\frac{1}{|\alpha_n|} = 0$ so the above equation becomes,

$$\lim_{R\to 1^-}\sum_{|\alpha_n|\leq R, f(\alpha_n)=0}\log\frac{R}{|\alpha_n|} = \sum_{|\alpha_n|<1, f(\alpha_n)=0}\log\frac{1}{|\alpha_n|} \tag{29}$$

Also since by equation (4) $f(\alpha_n) = 0$ if and only if $\zeta(\rho_n) = 0$ and there exists no zero ρ_n such that $\rho_n\in\mathbb{R}$ and $\rho_n\in\mathbb{R}_{\geq 1}$ so there does not exists any zero α_n of f such that $\alpha_n\in\mathbb{R}$ and $\alpha_n\in\mathbb{R}_{\geq 0}$. Hence we get

$$\lim_{R\to 1^-}\sum_{|\alpha_n|\leq R, f(\alpha_n)=0}\log\frac{R}{|\alpha_n|} = \sum_{\alpha_n\in\mathbb{D}\setminus\mathbb{R}_{\geq 0}, f(\alpha_n)=0}\log\frac{1}{|\alpha_n|} \tag{30}$$

We next show that we can apply Lebesgue's dominated convergence theorem to move the limit inside the integral of the left hand side in equation (26).
Denote $\log^+|f| = \max(\log|f|, 0)$ and $\log^-|f| = \max(-\log|f|, 0)$.
Then we can write,

$$\log|f(Re^{i\theta})| = \log^+|f(Re^{i\theta})| - \log^-|f(Re^{i\theta})| \tag{31}$$

By equation (11), $\log|f(0)| = 0$ and so by equation (25) since $|\alpha_n| \leq R$ so,

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta \geq 0 \tag{32}$$

Using equation (31) and (32) we have,

$$\frac{1}{2\pi}\int_0^{2\pi} \log^-|f(Re^{i\theta})|d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(Re^{i\theta})|d\theta \tag{33}$$

Also note that we have,

$$\big|\log|f(Re^{i\theta})|\big| = \log^+|f(Re^{i\theta})| + \log^-|f(Re^{i\theta})| \tag{34}$$

$$\big|\log|f(Re^{i\theta})|\big| \leq 2\,(\log^+|f(Re^{i\theta})| + \log^-|f(Re^{i\theta})|) \tag{35}$$

So, we have $2\,(\log^+|f(Re^{i\theta})| + \log^-|f(Re^{i\theta})|)$ as the dominating function. Next we prove that this dominating function has a finite integral.
From equation (4),

$$f(z) = (s-1)\zeta(s)$$

where by equation (5), $s = \frac{1}{1-\sqrt{z}}$ and by Lemma 1.2, $f(z)$ is analytic in $\mathbb{D}\setminus\mathbb{R}_{\geq 0}$ where $\mathbb{D}$ is the unit disc. Hence, $\zeta(s)$ is analytic in $\mathbb{P}\setminus\mathbb{R}_{\geq 1}$ where $\mathbb{P}$ is the plane defined as $\mathbb{P} = \{s \in \mathbb{C} \mid \Re(s) > \frac{1}{2}\}$.

$$f(Re^{i\theta}) = \frac{\sqrt{R}e^{i\theta/2}}{1-\sqrt{R}e^{i\theta/2}} \cdot \zeta\left(\frac{1}{1-\sqrt{R}e^{i\theta/2}}\right) \tag{36}$$

and $s = \frac{1}{1-\sqrt{R}e^{i\theta/2}}$. Also, when $R < 1$, $\Re(s) > \frac{1}{2}$.
Since $\zeta(s)$ is analytic in $\mathbb{P}\setminus\mathbb{R}_{\geq 1}$ where $\mathbb{P} = \{s \in \mathbb{C} \mid \Re(s) > \frac{1}{2}\}$ so (see [9, p.29] or [13, p.547]),

$$\zeta(s) = \mathcal{O}(|s|) \quad \text{where} \quad s \in \mathbb{P}\setminus\mathbb{R}_{\geq 1} \quad \text{and} \quad |s| \to \infty \tag{37}$$

So using equation (36) and (37), there exists some constant $C > 0$ such that,

$$|f(Re^{i\theta})| \leq \frac{C\sqrt{R}}{|1-\sqrt{R}e^{i\theta/2}|^2} < \frac{C}{|e^{-i\theta/2}-\sqrt{R}|^2} \leq \frac{C}{\sin^2(\theta/2)}$$

Since we have $\int_0^{2\pi} \log(\sin^2(\theta/2))d\theta < \infty$ so we have,

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+|f(Re^{i\theta})|d\theta < \infty \tag{38}$$

By equation (33) we have,

$$\frac{1}{2\pi}\int_0^{2\pi} \log^-|f(Re^{i\theta})|d\theta < \infty \tag{39}$$

So we have

$$\frac{1}{2\pi}\int_0^{2\pi} 2(\log^+|f(Re^{i\theta})| + \log^-|f(Re^{i\theta})|)d\theta < \infty \tag{40}$$

Using Lebesgue's dominated convergence theorem in left hand side of equation (26) and substituting the value of summation from equation (30), we get,

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f^*(e^{i\theta})|d\theta = \log|f(0)| + \sum_{\alpha_n\in\mathbb{D}\setminus\mathbb{R}_{\geq 0}, f(\alpha_n)=0} \log\frac{1}{|\alpha_n|} \tag{41}$$

Lemma 1.7:

$$\int_0^{2\pi} \log|f^*(e^{i\theta})|d\theta = 2\int_0^{\infty} \frac{\log|\zeta(\frac{1}{2}+it)|}{\frac{1}{4}+t^2}dt$$

Proof: Let,

$$I = \int_0^{2\pi} \log|f^*(e^{i\theta})|d\theta \tag{42}$$

Since $f(z)$ is defined by equation (6) so,

$$I = \int_0^{2\pi} \log\left|\left(\frac{e^{\frac{i\theta}{2}}}{1-e^{\frac{i\theta}{2}}}\right)\zeta\left(\frac{1}{1-e^{\frac{i\theta}{2}}}\right)\right| d\theta$$

Observe that,

$$\frac{1}{1-e^{\frac{i\theta}{2}}} = \frac{1}{2} + \frac{i}{2}\cot\left(\frac{\theta}{4}\right) \quad \text{and} \quad |e^{\frac{i\theta}{2}}| = 1$$

$$\Rightarrow I = \int_0^{2\pi} \log\left|\left(\frac{1}{2} + \frac{i}{2}\cot\left(\frac{\theta}{4}\right)\right)\zeta\left(\frac{1}{2} + \frac{i}{2}\cot\left(\frac{\theta}{4}\right)\right)\right| d\theta$$

Substituting $t = \frac{1}{2}\cot(\frac{\theta}{4})$ we have $d\theta = \frac{-2}{\frac{1}{4}+t^2}dt$

$$I = 2\int_0^{\infty} \frac{\log|(\frac{1}{2}+it)\zeta(\frac{1}{2}+it)|}{\frac{1}{4}+t^2}dt$$

Since by contour integration or by substitution, $t = \frac{\tan\theta}{2}$ we have [13, p.550],

$$\int_{-\infty}^{\infty} \frac{\log|\frac{1}{2}+it|}{\frac{1}{4}+t^2}dt = 0$$

Since integrand is an even function so we have,

$$\int_0^{\infty} \frac{\log|\frac{1}{2}+it|}{\frac{1}{4}+t^2}dt = 0 \tag{43}$$

So we can write I as,

$$I = 2\int_0^{\infty} \frac{\log|\zeta(\frac{1}{2}+it)|}{\frac{1}{4}+t^2}dt$$

Putting the value of I from equation (42) we have,

$$\int_0^{2\pi} \log|f^*(e^{i\theta})|d\theta = 2\int_0^{\infty} \frac{\log|\zeta(\frac{1}{2}+it)|}{\frac{1}{4}+t^2}dt \tag{44}$$

Now since by equation (6), $f(z) = \left(\frac{\sqrt{z}}{1-\sqrt{z}}\right)\zeta\left(\frac{1}{1-\sqrt{z}}\right)$ and by equation (10), $f(0) \neq 0$, so $f(\alpha_n) = 0$ corresponds to $\zeta\left(\frac{1}{1-\sqrt{\alpha_n}}\right) = 0$. Let, ρ_n denote non trivial zeros of Riemann zeta function then,

$$\rho_n = \frac{1}{1-\sqrt{\alpha_n}} \tag{45}$$

<u>Lemma 1.8</u>:

$$\sum_{\alpha_n \in \mathbb{D}\setminus\mathbb{R}_{\geq 0}, f(\alpha_n)=0} \log\frac{1}{|\alpha_n|} = 2\sum_{\rho_n \in \mathbb{P}\setminus\mathbb{R}_{\geq 1},\ \zeta(\rho_n)=0} \log\left|\frac{\rho_n}{1-\rho_n}\right|$$

Proof: Since by equation (4), $f(z) = (s-1)\zeta(s)$ so we have $f(\alpha_n) = 0$ if and only if $\zeta(\rho_n) = 0$. By Lemma 1.3, the map $s : \mathbb{D}\setminus\mathbb{R}_{\geq 0} \to \mathbb{P}\setminus\mathbb{R}_{\geq 1}$ defined as $s(z) = \frac{1}{1-\sqrt{z}}$ is conformal so we have,

$$\sum_{\alpha_n \in \mathbb{D}\setminus\mathbb{R}_{\geq 0}, f(\alpha_n)=0} \log\frac{1}{|\alpha_n|} = 2\sum_{\alpha_n \in \mathbb{D}\setminus\mathbb{R}_{\geq 0}, f(\alpha_n)=0} \log\frac{1}{\sqrt{|\alpha_n|}}$$

$$\sum_{\alpha_n \in \mathbb{D}\setminus\mathbb{R}_{\geq 0}, f(\alpha_n)=0} \log\frac{1}{|\alpha_n|} = 2\sum_{\alpha_n \in \mathbb{D}\setminus\mathbb{R}_{\geq 0}, f(\alpha_n)=0} \log\left|\frac{\frac{1}{1-\sqrt{\alpha_n}}}{1-\frac{1}{1-\sqrt{\alpha_n}}}\right| \tag{46}$$

By Lemma 1.1, $s : \mathbb{D} \setminus \mathbb{R}_{\geq 0} \to \mathbb{P} \setminus \mathbb{R}_{\geq 1}$ defined as $s(z) = \frac{1}{1-\sqrt{z}}$ is injective and onto and since by equation (45), $\rho_n = \frac{1}{1-\sqrt{\alpha_n}}$ so equation (46) becomes,

$$\sum_{\alpha_n \in \mathbb{D}\setminus\mathbb{R}_{\geq 0}, f(\alpha_n)=0} \log \frac{1}{|\alpha_n|} = 2 \sum_{\rho_n \in \mathbb{P}\setminus\mathbb{R}_{\geq 1},\ \zeta(\rho_n)=0} \log \left| \frac{\rho_n}{1-\rho_n} \right| \tag{47}$$

Using equation (11), (44) and (47) in equation (41) we get,

$$\frac{1}{2\pi} \left(2 \int_0^\infty \frac{\log|\zeta(\frac{1}{2}+it)|}{\frac{1}{4}+t^2} dt \right) = 2 \sum_{\rho_n \in \mathbb{P}\setminus\mathbb{R}_{\geq 1},\ \zeta(\rho_n)=0} \log \left| \frac{\rho_n}{1-\rho_n} \right| \tag{48}$$

3 Proof of Theorem 1

Since the non trivial zeros of zeta function are countable so, equation (1) can be written as [13, p.549]

$$\frac{1}{2\pi} \int_{-\infty}^\infty \frac{\log|\zeta(\frac{1}{2}+it)|}{\frac{1}{4}+t^2} dt = \sum_{\Re(\rho_n)>\frac{1}{2},\ \zeta(\rho_n)=0} \log \left| \frac{\rho_n}{1-\rho_n} \right| \tag{49}$$

Since the non trivial zeros lie in the critical strip, $0 < \Re(\rho_n) < 1$ so we have,

$$\frac{1}{2\pi} \int_{-\infty}^\infty \frac{\log|\zeta(\frac{1}{2}+it)|}{\frac{1}{4}+t^2} dt = \sum_{\rho_n \in \mathbb{P}\setminus\mathbb{R}_{\geq 1},\ \zeta(\rho_n)=0} \log \left| \frac{\rho_n}{1-\rho_n} \right| \tag{50}$$

By Schwarz reflection principle the integrand is an even function and hence we have

$$\frac{1}{2\pi} \left(2 \int_0^\infty \frac{\log|\zeta(\frac{1}{2}+it)|}{\frac{1}{4}+t^2} dt \right) = \sum_{\rho_n \in \mathbb{P}\setminus\mathbb{R}_{\geq 1},\ \zeta(\rho_n)=0} \log \left| \frac{\rho_n}{1-\rho_n} \right| \tag{51}$$

Since the left hand sides of equation (48) and (51) are same so equating the right hand sides we get,

$$2 \sum_{\rho_n \in \mathbb{P}\setminus\mathbb{R}_{\geq 1},\ \zeta(\rho_n)=0} \log \left| \frac{\rho_n}{1-\rho_n} \right| = \sum_{\rho_n \in \mathbb{P}\setminus\mathbb{R}_{\geq 1},\ \zeta(\rho_n)=0} \log \left| \frac{\rho_n}{1-\rho_n} \right|$$

$$\Rightarrow \sum_{\rho_n \in \mathbb{P}\setminus\mathbb{R}_{\geq 1},\ \zeta(\rho_n)=0} \log \left| \frac{\rho_n}{1-\rho_n} \right| = 0$$

And equation (50) gives,

$$\int_{-\infty}^\infty \frac{\log|\zeta(\frac{1}{2}+it)|}{\frac{1}{4}+t^2} dt = 0 \tag{52}$$

Equation (52) completes the proof of Theorem 1. This resolves the Riemann Hypothesis because this relation is an equivalent version of Riemann Hypothesis as proved by Balazard, Saias and Yor [1].

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5 References

1. Balazard, M. Saias, E., and Yor, M. "Notes sur la fonction ζ de Riemann, 2", Advances in Mathematics , Volume 143 (1999) , 284-287.

2. Titchmarsh, E.C. The theory of functions, 2nd edition , Oxford science publication , 1939.

3. Hoffman, K. Banach Spaces of Analytic functions , Dover, New York (1988).

4. Peter Duren, Theory of H^p spaces, Vol. 38 , 1st edition, Academic Press (1970).

5. John B. Garnett, Bounded Analytic Functions, Vol.236 (2007).

6. Norman Levinson, More than one third of zeros of Riemann zeta function are on $\sigma = 1/2$, Advances in Mathematics (1974).

7. Shaoji Feng , Zeros of Riemann zeta function on the critical line, Journal of Number Theory (2012).

8. Kyle Pratt, Nicolas Robles, Alexandru Zaharescu , More than five-twelfths of zeros of ζ are on critical line , Research in the Mathematical Sciences, Springer (2020).

9. Titchmarsh, E.C. The theory of the Riemann zeta function , 2nd edition, revised by D.R. Heath Brown, Oxford university press (1986).

10. Rudin, W., Real and complex analysis, McGraw-Hill (1987),

11. Paul Koosis, Introduction to Hp Spaces, Cambridge University Press (1998).

12. Kevin Broughan, Equivalents of Riemann Hypothesis Volume 2, Analytic equivalents, Cambridge University press (2017).

13. K. Ilgar Eroglu and Iossif V. Ostrovskii, On an Application of the Hardy Classes to the Riemann Zeta-Function , Turk J Math (2001).

14. Joseph Bak, Donald J. Newman, Complex Analysis, Third edition, Springer (2010).

15. Elias M. Stein, Rami Shakarchi, Complex analysis, Princeton University Press (2003).