A new solvable quintic equation of the Bring-Jerrard form

$x^5 + ax + b = 0$

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Abstract

In the previous post, we gave one more irreducible equation of the shape $x^5+ax^2+b=0$ [3], which is solvable.In this paper, we give an irreducible equation of the shape $x^5+ax+b=0$, which is also solvable, contrary to some available arguments [1], [2].

We rewrite the Eq of the form $x^5 + ax^2 + b = 0$

It is known, up to scaling of the variable, there are exactly five solvable quintics of the shape $x^5 + ax^2 + b = 0$, which are (where s is a scaling factor) [1] :

$$x^5 - 2s^3x^2 - \frac{s^5}{5}$$
$$x^5 - 100s^3x^2 - 1000s^5$$
$$x^5 - 5s^3x^2 - 3s^5$$
$$x^5 - 5s^3x^2 + 15s^5$$
$$x^5 - 25s^3x^2 - 300s^5$$

However, the one below is also solvable [3].

$$x^5 - 5s^3x^2 + 2s^5 = 0$$

Some available arguments:

During the second half ofthe 19th century, john Stuart Glashan, George Paxton Young, and Carl Runge gave such a parameterization : *an irreducible quintic with rational coefficients in Bring-Jerrard form is solvable if and only if either a = 0 or it may be written [1]*

$$x^5 + \frac{5\mu^4(4\nu+3)}{\nu^2+1}x + \frac{4\mu^5(2\nu+1)(4\nu+3)}{\nu^2+1} = 0$$

And the theorem. [2] *Let a and b be rational numbers such that the quintic trinomial $x^5 + ax + b$ is irreducible. Then the equation $x^5 + ax + b = 0$ is solvable by radicals if and only if there exist rational numbers $\epsilon(=\pm 1), c(\geqslant 0)$and $e(\neq 0)$ such that*

$$a = \frac{5e^4(3-4\epsilon c)}{c^2+1}, b = \frac{-4e^5(11\epsilon+2c)}{c^2+1}$$

However, the irreducible equation below :

$$x^5 - 5x - 3 = 0$$

is not satisfy the arguments above, but solvable by radicals.

It is indeed based on existing theory (group theory...) to find solvable irreducible equations, specifically the above two forms. However, it omitted (invalid) the equations as presented above.

References

[1] Quintic function - Wikipedia

[2] B.K.Spearman, K.S.Williams: Characterization of solvable quintics $x^5 + ax + b$, The American Math Monthly - Volume 101,1994-Issue 10.

[3] Quang N V, A new solvable quintic equation of the shape $x^5 + ax^2 + b = 0$ Vixra:2011.0165 (AL), Semanticscholar.org:229488183

[4] Quang N V, Solvable form of the polynomial equation $x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0 = 0, (n = 2k + 1)$ Vixra:2104.0008

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