On 6k ± 1 Primes in Goldbach Strong Conjecture

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Abstract

Goldbach strong conjecture, still unsolved, states that all even integers $n>2$ can be expressed as the sum of two prime numbers (Goldbach partitions of n). Each prime $p>3$ can be expressed as $6k \pm 1$. This work is devoted to studies of $6k \pm 1$ primes in Goldbach partitions and enhanced Goldbach strong conjecture with the lesser of twin primes of form $6k-1$ used as a baseline.

1 Introduction

Goldbach strong conjecture (GSC, also called binary) asserts that all positive even integer $n \geq 4$ can be expressed as the sum of two prime numbers. This hypothesis, formulated by Goldbach in 1742 in letter to Euler [1] and then updated by Euler to the form above is one of the oldest and still unsolved problems in number theory. Empirical verification showed that it is true for all $n \leq 4 \text{ x } 10^{18}$ [2] [3].

The expression of a given positive even number n as a sum of two primes p_1 and p_2 is called a Goldbach Partition (GP) of n. Let's denote this relation as $GSC(n, p_1, p_2)$. Then Goldbach strong conjecture can be written as (1):

$$\forall_{x>1, x\in\mathbb{N}} \exists_{p_1, p_2 \in \mathbb{P}} GSC(2x, p_1, p_2) \quad (1)$$

2 6k ± 1 primes in GSC

Every prime $p>3$ can be written as $6k \pm 1$, where $k \in \mathbb{N}$ (lemma proved in [4]). There are exactly two primes that are not of form $6k \pm 1$: 2 and 3. Prime 2 is present in one partition only: $GSC(4, 2, 2)$, while prime $p = 3$ plays much important role in GSC - it is the most frequent prime in the partitions for even $n<10^6$ [4].

Let's exclude both primes 2 and 3 from a set of primes used to fulfill GSC. All remaining primes are of form $6k \pm 1$. It would not be possible to build neither 4 nor 6 nor 8 from a sum of two such primes (because these numbers always have GP with either 2 or 3: $GSC(4, 2, 2)$, $GSC(6, 3, 3)$, $GSC(8, 3, 5)$, $GSC(8, 5, 3)$), but situation is changing for bigger even numbers. Let $R(n)$ be a set of GPs of n, while $R_{6k\pm1}(n)$ a set of GPs of n but without using primes 2 and 3 in any GP. As shown above: $R_{6k\pm1}(4) = \emptyset$, $R_{6k\pm1}(6) = \emptyset$, $R_{6k\pm1}(8) = \emptyset$.

Lemma 1. $0 \leq |R(n)| - |R_{6k\pm1}(n)| \leq 1$

Proof. There are exactly two primes 2 and 3 that are not of form $6k \pm 1$, where $k \in \mathbb{N}$. Let's analyze two cases: $n = 4$ and $n>4$. For first case we have: $R(4) = (2, 2)$, $R_{6k\pm1}(4) = \emptyset$, thus $|R(4)| - |R_{6k\pm1}(4)| = 1$ which fulfills the lemma. 2 is not a part of any other GP. Let's take a look at even $n>4$. There are n for which 3 is present in GP (i.e. $R(22) = (3, 19), (5, 17), (11, 11)$, $R_{6k\pm1}(22) = (5, 17), (11, 11)$) or missing (i.e. $R(24) = R_{6k\pm1}(24) = (5, 19), (7, 17), (11, 13)$. 3 can exist in at least one GP for $n>4$ because in $GSC(n, 3, p_1)$ we have just one way to express p_1: $p_1 = n - 3$. Thus for $n>4$ we have that $|R(n)| - |R_{6k\pm1}(n)|$ is either 0 or 1, and this fulfills the remaining part of the lemma. □

Let $R_{6k+1}(n)$ be a set of GPs of n that both factors are primes of form $6k + 1$, and $R_{6k-1}(n)$ be a set of GPs of n that both factors are primes of form $6k - 1$. By definition $R_{6k+1}(n) \subseteq R_{6k\pm1}(n)$ and $R_{6k-1}(n) \subseteq R_{6k\pm1}(n)$.

Lemma 2.

$$\forall_{n\in\mathbb{N}} |R_{6k-1}(6n)| = |R_{6k+1}(6n)| = 0 \quad (2)$$

Proof. Every number of form $6n$, $n \in \mathbb{N}$, is divisible by both 2 and 3. Let's assume that p_1 is of form $6k_1 - 1$ and p_2 is of form $6k_2 - 1$ ($k_1, k_2 \in \mathbb{N}$). Then $s = p_1 + p_2 = 6k_1 - 1 + 6k_2 - 1 = 6(k_1 + k_2) - 2 = 2(3k_1 + 3k_2 - 1)$. s is divisible by 2 but is not divisible by 3 because 3 does not divide $3k_1 + 3k_2 - 1$. Similar reasoning can be done for a case when both p_1 and p_2 are of form $6k + 1$. This means that $6n$ cannot be built from a sum of neither two primes of form $6k - 1$ nor $6k + 1$. □

3 GSC broken down into three

Original GSC does not say anything particular about primes. Let's take a look at even numbers $n>8$. Each such number can be expressed as either $3x$ or $3x + 1$ or $3x + 2$, where $x \in \mathbb{N}$. Calculations run for small n show that original GSC can be extended to a form (3):

$$\forall_{m>4, m\in\mathbb{N}} \begin{cases} GSC(2m, p_{6k-1}, p_{6k+1}) & \text{if } m \bmod 3 = 0 \\ GSC(2m, p_{6k+1}, p_{6k+1}) & \text{if } m \bmod 3 = 1 \\ GSC(2m, p_{6k-1}, p_{6k-1}) & \text{if } m \bmod 3 = 2 \end{cases} \quad (3)$$

where p_{6k-1} is a prime of form $6a - 1$ [5] and p_{6k+1} is a prime of form $6b + 1$ [6] ($a, b \in \mathbb{N}$). Conjecture (3) uses limited set of prime numbers in GSC - primes 2 and 3 are excluded.

Every twin prime pair different than $(3, 5)$ is of form $(6k - 1, 6k + 1)$, where $k \in \mathbb{N}$ [4]. This gives a hint that yet stronger version of conjecture (3) is potentially possible. If we assume that k is the same in all three conditions for the same n, and both p_{6k-1} are the lesser of twin primes ($\mathbb{P}_{\mathbb{LT}}$), then we can articulate the following hypothesis (4):

$$\underset{A\in\mathbb{N}}{\exists}\ \underset{n>A,n\in\mathbb{N}}{\forall}\ \underset{p_1,p_2\in\mathbb{P}_{\mathrm{LT}}}{\exists} GSC(6n-2,p_1,p_2) \qquad (4)$$

where A is a constant to be provided.

Lemma 3. *If conjecture (4) is true, then we have a method to proof or invalidate GSC.*

Proof. If both p_1 and p_2 in $GSC(6n-2,p_1,p_2)$ are the lesser of twin primes, then we have $GSC(6n,p_1+2,p_2)$ and $GSC(6n+2,p_1+2,p_2+2)$. This is true because both p_1+2 and p_2+2 are the greater of twin primes. GSC is formulated for even numbers >2. If $n \in \mathbb{N}$, then numbers $6n-2$, $6n$ and $6n+2$ can build every even number >2. Conjecture (4) starts from point $A+1$. If A is finite, then we have a finite number of additional cases ($\leq A$) to verify against GSC. □

4 Results of experiments

Experiments were focused, firstly, on confirmation of conjecture (3) for bigger even numbers, secondly, on search for value of A in conjecture (4), and thirdly, on looking for possible patterns between $R_{6k\pm1}(n)$ and $R(n)$, and inside $R_{6k\pm1}(n)$.

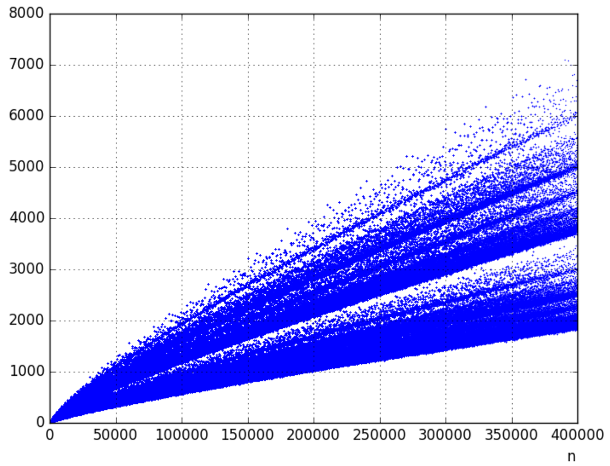

Figure 1: $R_{6k\pm1}(n)$ ($4 < n < 2\times10^6$, $n=2k, k\in\mathbb{N}$)

Conjecture (3) was confimed for $4 \leq m \leq 2\times10^6$ - this means that all even numbers n that $8 < n < 4\times10^6$ have (3) fulfilled. Figure 1 depicts number of GPs of even $n>4$ built from primes $p>3$. There is only one non-$6k\pm1$-like prime which can be a member of such partition, 3, but for s given n it can be present in one GP only (Lemma 1). This means that Figure 1 is very close to shape of original Goldbach's comet.

Calculations run for $1 \leq n \leq 4\times10^6$ confirmed that there are only 12 known cases when even number of form $6n-2$>2 is not a sum of two the lesser of twin primes: 4, 94, 400, 514, 784, 904, 1114, 1144, 1264, 1354, 3244, 4204. This sequence was submitted to OEIS database as OEIS A321221 [7]. A in conjecture (4) is taken from last term: if $6n-2=4204$, then $n=701$, thus $A=701$. A321221 is a subset of sequence A007534 [9] described in [10]. 701 is also the last term of related sequence [8].

Figure 2 illustrates number of GPs of n (4 mod 6) with two primes that are the lesser of twin primes. Let $R_{LTP}(n)$

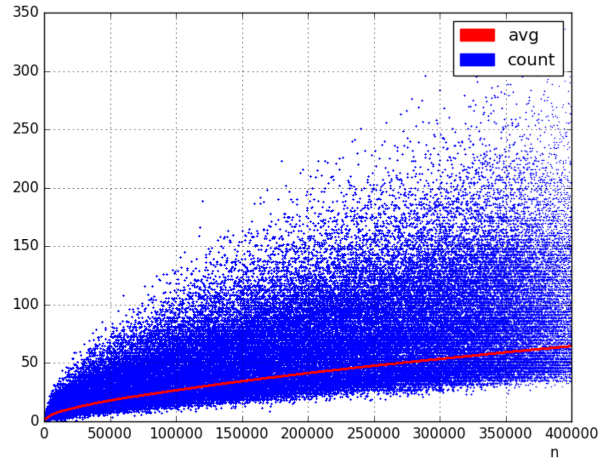

Figure 2: Number of GPs for n with both primes that are the lesser of twin primes, with average values ($n=4$ *mod* 6, $2 < n < 4\times10^5$, $n=2k, k\in\mathbb{N}$)

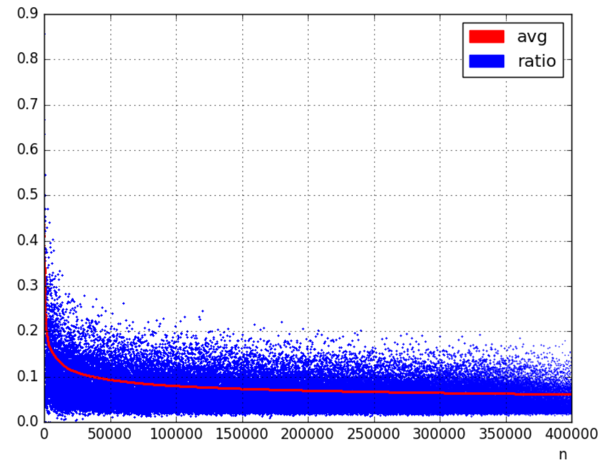

Figure 3: Ratio of $|R_{LTP}(n)|$ to $|R(n)|$, with average values ($n=4$ *mod* 6, $2 < n < 4\times10^5$, $n=2k, k\in\mathbb{N}$)

be a set of all partitions of n where both primes are the lesser of twin primes. Figure 3 depicts ratio of number of elements of $R_{LTP}(n)$ to number of elements of $R(n)$. Obviously this ratio is 0 only for n from OEIS A321221.

It has been computationally verified that the following even numbers of form $6n-2$ have just one partition with two primes that are the lesser of twin primes: 10, 16, 28, 40, 52, 64, 106, 124, 136, 172, 184, 226, 262, 304, 394, 412, 442, 484, 544, 556, 604, 634, 664, 682, 694, 724, 736, 754, 772, 802, 874, 934, 976, 994, 1012, 1984, 1174, 1204, 1324, 1384, 1414, 1534, 1564, 1594, 1606, 1744, 1786, 1852, 1864, 1996, 2074, 2164, 2584, 2674, 3052, 3424, 3502, 3844, 9844, 12742, 15124, 15814, 24094, 24532 - no further terms were found so far. Figure 2 demonstrates ascending trend of average number of partitions of $6n-2$.

Figures 4 and 5 are devoted to differences between primes of form $6k-1$ and $6k+1$ in GPs. In general we can observe two cases: the first one with difference close to 0, and the second one - with difference either positive or negative (and with generally ascending trend for bigger n).

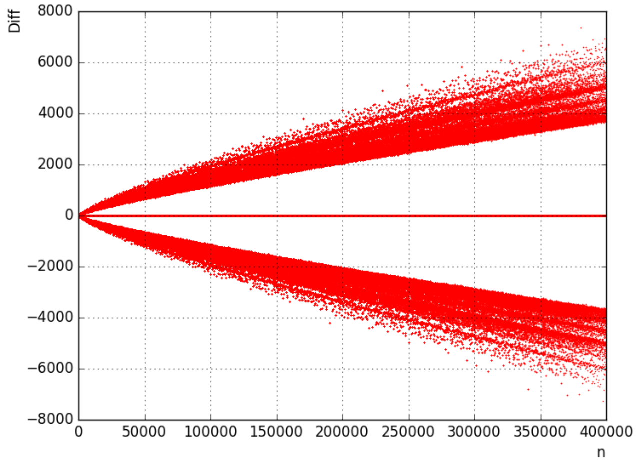

Figure 4: Number of primes of form $6k-1$ in $R(n)$ - number of primes of form $6k+1$ in $R(n)$ ($2 < n < 4 \times 10^5$, $n = 2k, k \in \mathbb{N}$)

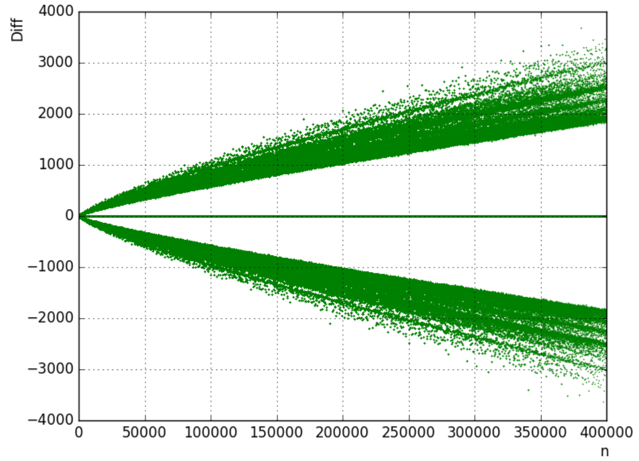

Figure 5: $|R_{6k-1}(n)| - |R_{6k+1}(n)|$ ($2 < n < 4 \times 10^5$, $n = 2k, k \in \mathbb{N}$)

5 Summary and next steps

Executed experiments confirmed that (3) is true for $4 \leq n \leq 4 \times 10^6$ and (4) with $A = 701$ is true at least for $1 \leq n \leq 4 \times 10^6$. As a result this work led to more precise conjecture (5):

$$\forall_{n>701, n\in\mathbb{N}} \exists_{p_1,p_2\in\mathbb{P}_{LT}} GSC(6n-2, p_1, p_2) \tag{5}$$

If (5) is true, then GSC is true.

Furthermore, even if 3 looks to be the most common prime in GPs, executed experiments revealed that all even $n > 8$ have at least one partition without prime 3, with both primes of form $6k\pm1$. This observation raised another open question which can be foundation of further research work: which primes can be skipped in GSC? Maybe prime set is much bigger than required to fullfil GSC?

References

[1] Christian Goldbach, *On the margin of a letter to Leonard Euler*, 1742.

[2] Tomás Oliveira e Silva, *Goldbach conjecture verification.* http://sweet.ua.pt/tos/goldbach.html, 2012.

[3] Tomás Oliveira e Silva, Siegfried Herzog, and Silvio Pardi, *Empirical verification of the even Goldbach conjecture and computation of prime gaps up to 4* $\times 10^{18}$., Mathematics of Computation, vol. 83, no. 288, pp. 2033-2060, July 2014 (published electronically on November 18, 2013.

[4] Marcin Barylski, *Studies on Twin Primes in Goldbach Partitions of Even Numbers*, http://tasmoto.org/research/TwinPrimesInGoldbachPartitions.pdf, 2018.

[5] OEIS Foundation Inc. (2018), The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A007528. Primes of form 6n-1.

[6] OEIS Foundation Inc. (2018), The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A002476. Primes of the form 6m+1.

[7] OEIS Foundation Inc. (2018), The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A321221. Numbers of the form 6n-2 which are not a sum of two numbers that are the lesser of twin primes.

[8] OEIS Foundation Inc. (2018), The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A243956. Positive numbers n without a decomposition into a sum n = i+j such that 6i-1, 6i+1, 6j-1, 6j+1 are twin primes.

[9] OEIS Foundation Inc. (2019), The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A007534. Even numbers that are not the sum of a pair of twin primes.

[10] Dan Zwillinger, *A Goldbach Conjecture Using Twin Primes*, Math. Comp. 33, No.147, p.1071, 1979.