Invalidating Cantor's Continuum Hypothesis and Solving Hilbert's #1 Problem

Stephane H. Maes[1]

July 15, 2021

Abstract:

In this short paper, we provide a mathematical proof that in set theory, developed in a mathematical universe following the ZFC axioms, Cantor's continuum hypothesis does not hold and Gödel had the right hunch: the cardinality of the infinity of the set of all reals is $\aleph_2$, and not $\aleph_1$, i.e., two infinity orders away from the cardinality of the infinite set of naturals, $\aleph_0$.

The proof is derived from combinatorics, relying on ZFC solely for the model of Cantor and Gödel defining $\aleph_0$. It provides input to the still unresolved first of Hilbert famous 23 math problems of interest.

This paper, resolves the first of the 23 Hilbert problems with invalidation of the continuum hypothesis.

1. Introduction

The context of the of this discussion can be found in [1], that describes the continuum hypothesis (term typically used instead of conjecture) of Cantor's and Gödel analysis [2, 5,6]. It is also the still unresolved first of Hilbert famous 23 math problems of interest [4]. It is formulated as:

> *The continuum hypothesis is that there is no set whose cardinality is strictly between that of the integers and that of the real numbers.* *(1)*

Some argued that the work of Gödel [2] then Cohen [7] would have resolved it. But this merely hinted that the conjecture can not be proven or disproven within ZFC (as well as without the axiom of choice) [3] and assuming that ZFC is consistent. So the continuum hypothesis is independent of ZFC [8], that's all we knew so far. Our proof does not rely on ZFC other than for the definition of the cardinality of $\mathbb{N}$.

[1] is motivated by recent progresses in complementing ZFC with additional axioms to resolve the dilemma. Two were proposed: Martin's principle [9] and (*) [10-12]. It initially looked like different mathematical universes would exist depending on what additional axioms are added to ZFC to validate or invalidate the continuum hypothesis [1,8-12]. A recent result [13], shows that this may not be the case and seems to favor the invalidity of the continuum hypothesis [1]. Yet none of these work settle the continuum hypothesis.

We provide a proof that the cardinality of $\mathbb{R}$ is $\aleph_2$ as suspected by Gödel and not $\aleph_1$ proposed by Cantor with his continuum conjecture/hypothesis [1,2]. As such $\aleph_1$ is a candidate for set violating the continuum hypothesis.

---

[1] shmaes.physics@gmail.com

2. Proof – Computing the cardinality of the set of Reals

In this paper we only rely on ZFC to allow the definition of the cardinality of the infinite set of naturals, $\aleph_0$, as set cardinality and the cardinality of the infinite set of naturals, $\aleph_k$. as kth order of cardinality, that we interpret as in [2]. Beyond that, we do not use ZFC. So this brings most probably the additional assumptions that allow us to escape the work of Gödel.

It also is not affected by the methodology to count the cardinality of $\mathbb{R}$ (e.g. à la [7]) or by the current new axioms and their compatibility or incompatibilities [8-12].

The high-level sketch of the proof is:

- $\aleph_0$ is the cardinality of the naturals space, i.e. $\mathbb{N}$, which is a 1D discrete universe. $\aleph_0$ is the power of denumerably infinite sets [2].
- The cardinality of $\mathbb{R}$ is obtained as: $\aleph_0$ (for naturals before the decimal point) x $\aleph_0$ (for the position of first non-zero digit after the decimal point) x $\aleph_0$ (for value of the following digits, for the natural number that it consists of i.e. cardinality of $\mathbb{N}$) =>

  Cardinality of $\mathbb{R}$ is $\aleph_2 = \aleph_0 \times \aleph_0 \times \aleph_0 \neq \aleph_1 = \aleph_0 \times \aleph_0$ (2)

- x amounts to upping the order of ∞ as proposed in the definition of $\aleph_1$ [2].

On the other hand:

the cardinality of $\mathbb{N} \times \mathbb{N}$ is $\aleph_1$. (3)

(2) vs. (3) invalidate (1).

Note that it may seem that is doubling the same digits.. as 1._Any natural encompasses 1.1_any natural… But whatever natural is after the dot, this reasoning would miss also how many zeros before it hence it is in fact correct that it is $\aleph_2$ by simply saying that the position on where the natural number is put/added provides that extra dimension. Picking any other subset first would not change the cardinality just like $2\mathbb{N}$ and $2\mathbb{N}+1$ have all the same cardinality as $\mathbb{N}$.

It is important that these considerations are also key to the proof: the largest denumerable mechanism denotes the cardinality type for infinite denumerable systems: so just assuming say:

$\aleph_0$ (for naturals before the decimal point) x $\aleph_0$ (for natural after the decimal point) = $\aleph_1$ (4)

(4) is a smaller (just like $2\mathbb{N}$ vs. $\mathbb{N}$) and maps toa subset of the counting process leading to (2).On the other hand, there are no other processes, that is larger than the one associated to the derivation of (2): pick any other way to count somewhere after the decimal point and it results into count like (2) or like (4).

QED.

3. Conclusions

We think that, while rather obvious, this reasoning is a huge step forward as it was not apparently understood so far if [1] is to be believed. We do not make that latter claim, we just accept it.

Indeed, we note indeed the essential independence from ZFC as expected, the absence of the need to add axioms and the fact that the result appear true in mathematics, instead of possibly sometimes true and sometimes false depending on the axioms behind a model.

In fact, the work of Gödel and Cohen clearly identified that the continuum hypothesis is independent of the ZFC (or ZF per [8]). The mathematics community decided to therefore try to find additional axioms that can help decide. Our approach is different 1) somehow we dropped the need of axioms (other than as sustaining Mathematics and Logics with set theory) 2) showed in a framework that does not rely on them that in fact there is no degree of freedom: the continuum hypothesis is wrong when makes sense (i.e. defined); which of course maintains a link to set theory and ZFC/ZF axioms.

Of course, it would be of interest to see what additional axioms are actually equivalent to our proof. It is for future work or collaboration. On the basis of bypassing axioms, some may consider that it is an Physicist's or Engineer's proof. It is correct, that is what we have provided. We challenge others to help or produce the framework that they would desire, beyond this, to be satisfied.

On the basis of this paper, we argue that the first of the 23 Hilbert problems is now resolved with invalidation of the continuum hypothesis.

Interested readers will find in [14] the resulting cardinality multi-fold spacetime.

References

[1]: Natalie Wolchover, (2021), "How Many Numbers Exist? Infinity Proof Moves Math Closer to an Answer. For 50 years, mathematicians have believed that the total number of real numbers is unknowable. A new proof suggests otherwise.", https://www.quantamagazine.org/how-many-numbers-exist-infinity-proof-moves-math-closer-to-an-answer-20210715/. Retrieved on July 15, 2021.

[2]: Kurt Gödel, (1947), "What is Cantor's Continuum Problem?", The American Mathematical Monthly, Vol. 54, No. 9, pp. 515-525.

[3]: Wikipedia, "Zermelo–Fraenkel set theory", https://en.wikipedia.org/wiki/Zermelo%E2%80%93Fraenkel_set_theory. Retrieved on July 17, 2021.

[4]: Wikipedia, "Hilbert's problems", https://en.wikipedia.org/wiki/Hilbert%27s_problems. Retrieved on July 18, 2021.

[5] Gödel, K., (1938), "The consistency of the axiom of choice and of the generalized continuum-hypothesis, " Proceedings of the U.S. National Academy of Sciences, 24: 556–7.

[6] Gödel, K., (1938), "Consistency-proof for the generalized continuum-hypothesis, " Proceedings of the U.S. National Academy of Sciences, 25: 220–4.

[7]: Cohen, P., (1963), "The independence of the continuum hypothesis I" Proceedings of the U.S. National Academy of Sciences, 50: 1143–48.

[8]: Koellner, Peter, (2019), "The Continuum Hypothesis", The Stanford Encyclopedia of Philosophy, Edward N. Zalta (ed.), https://plato.stanford.edu/archives/spr2019/entries/continuum-hypothesis. Retrieved on July 17, 2021.

[9]: Matthew Dean Foreman, Menachem Magidor, Saharon Shelah, (1988), "Martin's Maximum, saturated ideals, and nonregular ultrafilters. Part I." Annals of Mathematics, Volume 127, Pages 1-47.

[10]: Woodin, W. H., (1999), "The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal", Vol. 1 of de Gruyter Series in "Logic and its Applications", de Gruyter, Berlin.

[11] Woodin, W. H., (2001), "The continuum hypothesis, part I", Notices of the American Mathematical Society 48(6): 567–576.

[12]: Woodin, W. H., (2001), "The continuum hypothesis, part II", Notices of the American Mathematical Society 48(7): 681–690.

[13]: David Asperó, Ralf Schindler, (2021), "Martin's Maximum++ implies Woodin's axiom (∗)" Annals of Mathematics, Volume 193, Pages 793-835.

[14]: Stephane H Maes, (2021), "Comments on Multi-fold spacetime cardinality", https://shmaesphysics.wordpress.com/2021/04/18/multi-fold-non-commutative-spacetime-higgs-and-the-standard-model-with-gravity/#comment-2505. July 17, 2021.