Mother Nature and the 32 Crystal Classes

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ABSTRACT

Starting from the 32 crystal classes, we find a complete classification scheme of the same with 5 bits. That's not simply reorganization, that's philosophy. The Nature follows a repetitive sequence. The repetitive sequence I consider is the whole 3-bit sequence 000, 00c, 0m0, 0mc, 200, 20c, 2m0, 2mc. The protagonist of this novel, Mother Nature, is so satisfied with this sequence, or with this physical behavior, that she repeats it 4 times. She does it with an additional 4 axis. Then she does it again with axis 3. And finally with a combination of axes 2 and 3, parallel or askew. I write everything as a novel, and try to replace the entire sequence of Hermann Mauguin and Schoenflies symbols too, with ad hoc symbols. Mathematical proofs are inside.

	000	00c	0m0	0mc	200	20c	2m0	2mc
00 No rotational symmetry	1	c	m	mc	2	2c	2m	2mc
04 (4)	4	4c	4m	4mc	42	42c	42m	42mc
30 (3)	3	3c	3m	3mc	32	32c	32m	32mc
34 (3 2)	23	23c	23m	23mc	232	232c	232m	232mc

PREMISE

This work arises from some subsequent intuitions, some of which are illustrated in previous articles, see for example. [1].
But the two main ideas that I believe have been decisive are the following.
The first: the demonstration that the gyroidal class 432 does not require between the generators the 3111 symmetry, but 4-fold rotational symmetries are enough [2] ;
The second: the intuition that the crystal classes can be described by a 8 classes sequence wich repeats, therefore 4x8 = 32. This intuition differs from previous versions, and in my opinion it definitely fixes the work.
I'll talk more about it later.
I intend to expose things in the form of a novel, in which there is a protagonist who is Nature who works in an almost personified way , gradually creating the 32 existing crystal classes.
In the following I tell the details of how our protagonist, Mother Nature , worked.

PRELIMINARY ANALYSIS

Here I arrive at a demonstration of how the 32 crystal classes can be classified in 5 bits, named 3, 4, 2, m, c starting from the least significant bit (c), up to the most significant bit (3).
Specifically, in the following:
A - we show how the 5-bit classification is born and why it is born;
B - we show case by case how each bit intervenes .
However, I can say that:
- in any case, bits 3 and 2 signify the presence of 3-fold rotation axis and 2-fold rotation axis;
- bit m always means the presence of a reflection plane;
- the bit c is always linked to a symmetry with respect to the center but it is so done that the presence in the crystal of an inversion center involves the presence of the bit c, but the opposite is not said;
- bit 4 means the presence of 4-fold rotation axis but in some cases it is more ambiguous: 4-fold rotation involves the presence of bit 4 but the reverse is not necessarily true.
That said, let's move on.
I recall from [1], [2] the classification of the first 8 triclinic + monoclinic + orthorhombic classes .

Hermann Mauguin	5 bit symbols	Symmetries
1	00000	none
1_	0000c	c
m	000m0	m
2/m	000mc	m,c
2	00200	2
222	0020c	2, c
mm2	002m0	2, m
mmm	002mc	2, m, c

Fig. 1 the 8 triclinic + monoclinic + orthorhombic classes classified at 3 bits

In previous versions I had concentrated on the sequence of bits 00, 01, 10 and 11, here named with the letters m and c, and therefore 00, 0c, m0, mc. To this sequence, repeated 8 times, I attributed a succession of actions, or properties, or something.
Now instead I do the same thing but the repetitive sequence that I consider is the entire 3-bit sequence 000, 00c, 0m0, 0mc, 200, 20c, 2m0, 2mc. This, or this physical behavior, will repeat itself 4 times.
Like in a mystery novel, to anticipate the murderess, before continuing I want to anticipate how and why I got there. Whoever wants, read, whoever does not want to skip to the next paragraph.

BEFORE CONTINUING

As I said, in the preceding versions see for example [3] had already arrived at a classification of 32 classes, made of 8 groups of 4. This

HM symbols	5 bit symbols
1	00000
1_	0000c
m	000m0
2/m	000mc
2	00200
222	0020c
mm2	002m0
mmm	002mc
4	04000
4_	0400c
4_2m	040m0
4/m	040mc
422	04200
432	0420c
4mm	042m0
4/mmm	042mc
3	30000
3_	3000c
3m	300m0
3_2/m	300mc
6	30200
6_	3020c
6_m2	302mc
6/m	302mc
32	34000
622	3400c
6mm	340m0
6/mmm	340mc
23	34200
m3	3420c
4_3m	342m0
m3m	342mc

The table here down shows the classification, together with the Hermann Mauguin and Schoenflies's symbols.
I say briefly, just to explain at least summarily, that this classification has a meaning: the matter aggregates with various rotational symmetries: no symmetry, binary symmetry , ternary, etc., simple or compound, and then adds to each rotational symmetry (here in yellow) the further symmetries c (center), m (planes) or c + m together.

	axes	c center	m plane	m + c
No rotational symmetry	monohedron no symmetry 1 C1 00000	parallellohedron 1_ Ci 0000c	dome m Cs=C1v 000m0	prism 2/m C2h 000mc
2	sphenoid 2 C2 00200	rhombic disphenoid 222 D2 0020c	rhombic pyramid mm2 C2v 002m0	rhombic dipyramid mmm D2h 002mc
4	tetragonal pyramid 4 C4 04000	tetragonal disphenoid 4_ S4 0400c	ditetragonal pyramid 4mm C4v 040m0	tetragonal dipyramid 4/m C4h 040mc
4 2	tetragonal trapezohedron 422 D4 04200	gyroidal 432 O 0420c	ditetragonal scalenohedron 4_2m D2d 042m0	ditetragonal dipyramid 4/mmm D4h 042mc
3	trigonal pyramid 3 C3 30000	rhombohedron 3_ C3i=S6 3000c	ditrigonal pyramid 3m C3v 300m0	hexagonal scalenohedron 3_2m D3d 300mc
3 2	hexagonal pyramid 6 C6 30200	trigonal dipyramid 6_ C3h 3020c	dihexagonal pyramid 6mm C6v 302m0	hexagonal dipyramid 6/m C6h 302mc
3 2	trigonal trapezohedron 32 D3 34000	hexagonal trapezohedron 622 D6 3400c	ditrigonal dipyramid 6_m2 D3h 340m0	dihexagonal dipyramid 6/mmm D6h 340mc
3 2	tetartoid 23 T 34200	diploid m3 Th 3420c	hextetrahedron 4_3m Td 342m0	hexoctahedron m3m Oh 342mc

Those in purple are also the 8 Laue classes with the highest symmetry.

Simplifying, with only the Hermann Mauguin symbols , the classification is this

	00 axes	0c center	m0 plane	mc m + c
No rotational symmetry	monohedron no symmetry 1 00000	parallellohedron 1_ 0000c	dome m 000m0	prism 2/m 000mc
2	sphenoid 2 00200	rhombic disphenoid 222 0020c	rhombic pyramid mm2 002m0	rhombic dipyramid mmm 002mc
4	tetragonal pyramid 4 04000	tetragonal disphenoid 4_ 0400c	ditetragonal pyramid 4mm 040m0	tetragonal dipyramid 4/m 040mc
4 2	tetragonal trapezohedron 422 04200	gyroidal 432 0420c	ditetragonal scalenohedron 4_2m 042m0	ditetragonal dipyramid 4/mmm 042mc
3	trigonal pyramid 3 30000	rhombohedron 3_ 3000c	ditrigonal pyramid 3m 300m0	hexagonal scalenohedron 3_2m 300mc
3 2	hexagonal pyramid 6 30200	trigonal dipyramid 6_ 3020c	dihexagonal pyramid 6mm 302m0	hexagonal dipyramid 6/m 302mc
3 2	trigonal trapezohedron 32 34000	hexagonal trapezohedron 622 3400c	ditrigonal dipyramid 6_m2 340m0	dihexagonal dipyramid 6/mmm 340mc
3 2	tetartoid 23 34200	diploid m3 3420c	hextetrahedron 4_3m 342m0	hexoctahedron m3m 342mc

Then one day I said to myself: let's try to change our point of view.

It is clear that in a binary numbering any kind of "bit repetition" can be taken into consideration, and then it can be given a meaning. Example:
the one-bit sequence (00, 0c) has the meaning "no symmetry, or a symmetry of behavior around the center".
And then I can repeat this 16 times, obtaining 2 rows x 16 columns = 32 .
If I consider a 3-bit sequence (000, 00c, 0m0, 0mc, 200, 20c, 2m0, 2mc) this sequence, or this physical behavior, will repeat itself 4 times.
4 rows x 8 columns = 32.
The same 32 classes, simply represented 4 rows x 8 columns, become like this

	000 axes	00c center	0m0 plane	0mc m + c	200 axes	20c center	2m0 plane	2mc m + c
00	1	1_	m	2/m	2	222	mm2	mmm
	00000	0000c	000m0	000mc	00200	0020c	002m0	002mc
04	4	4_	4mm	4/m	422	432	4_2m	4/mmm
	04000	0400c	040m0	040mc	04200	0420c	042m0	042mc
30	3	3_	3m	3_2m	6	6_	6mm	6/m
	30000	3000c	300m0	300mc	30200	3020c	302m0	302mc
34	32	622	6_m2	6/mmm	23	m3	4_3m	m3m
	34000	3400c	340m0	340mc	34200	3420c	342m0	342mc

This allows or requires a new meaning to bit 4 as well as 00, 04, 30, 34.
Furthermore: from this new 4 rows x 8 columns arrangement it seems to understand that the 4 classes with 6/m holohedria and the 4 with 6/mmm holohedria (in green) should be exchanged.

	000	00c	0m0	0mc	200	20c	2m0	2mc
00	1	1_	m	2/m	2	222	mm2	mmm
	00000	0000c	000m0	000mc	00200	0020c	002m0	002mc
04	4	4_	4mm	4/m	422	432	4_2m	4/mmm
	04000	0400c	040m0	040mc	04200	0420c	042m0	042mc
30	3	3_	3m	3_2m	6	6_	6mm	6/m
	30000	3000c	300m0	300mc	30200	3020c	302m0	302mc
34	32	622	6_m2	6/mmm	23	m3	4_3m	m3m
	34000	3400c	340m0	340mc	34200	3420c	342m0	342mc

I do it below

	000	00c	0m0	0mc	200	20c	2m0	2mc
00	1	1_	m	2/m	2	222	mm2	mmm
	00000	0000c	000m0	000mc	00200	0020c	002m0	002mc
04	4	4_	4mm	4/m	422	432	4_2m	4/mmm
	04000	0400c	040m0	040mc	04200	0420c	042m0	042mc
30	3	3_	3m	3_2m	32	622	6_m2	6/mmm
	30000	3000c	300m0	300mc	30200	3020c	302m0	302mc
34	6	6_	6mm	6/m	23	m3	4_3m	m3m
	34000	3400c	340m0	340mc	34200	3420c	342m0	342mc

	000	00c	0m0	0mc	200	20c	2m0	2mc
00	1	1_	m	2/m	2	222	mm2	mmm
	00000	0000c	000m0	000mc	00200	0020c	002m0	002mc
04	4	4_	4mm	4/m	422	432	4_2m	4/mmm
	04000	0400c	040m0	040mc	04200	0420c	042m0	042mc
30	3	3_	3m	3_2m	32	622	6_m2	6/mmm
	30000	3000c	300m0	300mc	30200	3020c	302m0	302mc
34	6	6_	6mm	6/m	23	m3	4_3m	m3m
	34000	3400c	340m0	340mc	34200	3420c	342m0	342mc

Once the exchange is completed, it is evident that the 32 classes follow each other in a much more reasonable sequence (observe how the classes placed in the columns are similar to each other).

So: I didn't realize that Nature doesn't repeat the sequence 00, 0c, m0, mc sequence.
Actually I think I can say that she tries to repeat the whole 000, 00c, 0m0, 0mc, 200, 20c, 2m0, 2mc sequence.
She does this with an additional 4 axis.
Then she does it again with axis 3.
And finally with a combination of axes 2 and 3, parallel or askew.
From here, I wanted to rewrite everything in a fictionalized (descriptive) form and even to replace the entire sequence of HM symbols with ad hoc symbols.
We can return to us, and at this point restart the novel.

HOW MOTHER NATURE WORKS

In a 5-bit classification 3,4,2,, m, c of the crystal classes what is the physical or 'operational' meaning of a particular sequence that is then repeated, i.e. for example the sequence

00 0c m0 mc?
I.e.:
is there a sense?
Which?
It should be noted that the sequence 00 0c m0 mc in the binary numbering is necessarly repeated, subsequently, with the 2
200 20c 2m0 2mc
and so on.
What does it mean? That is: is there a meaning?

There is no need to think of a tendentious question, since the sequence , which as I have pointed out is intrinsic in a binary numbering, actually implies that it repeats itself, and therefore the question necessarily arises.
Fortunately, the answer that comes with it is (or at least it seems) obvious:
due to this sequence 00 0c m0 mc there exists in some way a series of operations, or actions, which are repeated. These are the symmetry operations 00, 0c, m0, mc. With m I denote the symmetry with respect to a plane and with c the symmetry with respect to a center. The first class is the one where no m&c symmetry is introduced (bit 00). It is a class which evidently, as a consequence of the above, possesses only rotational symmetries.
The second class (0c) possesses only the symmetry c. And so on.
Let's imagine a step by step"birth"process of the various classes.
Let us consider a packing of points, or a composition of objects or points, or an arrangement of objects in space, whatever we want to say.
I take a packing of points: it is there where I put it.
I do not do anything.
This operation corresponds to the sequence 00000 or 00. No symmetry of rotation, or 360° rotational symmetry, indeed no symmetry at all.
Now I try a more complicated, or more 'rich', or more aesthetic arrangement, whatever you want to say:
to a point I associate another that is symmetrical (with respect to a center of symmetry).
This operation corresponds to the sequence 0000c or 0c.
Now I try this arrangement instead:
to a point I associate another that is symmetrical with respect to a plane of symmetry.
Or you can say it in a different but equivalent way, that is:
I create an object formed in such a way as to possess the property of symmetry with respect to a plane m.
This operation corresponds to the sequence 000m0 or m0.
Finally
I create an object formed in such a way as to possess both the property of symmetry with respect to a center c and the property of symmetry with respect to a plane m.
This operation corresponds to the sequence 000mc or mc.

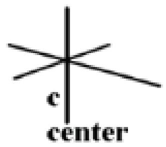
So, in order, the operations are:
00000
0000c
000m0
000mc.
Let's say that at this point I have exhausted the elementary symmetry operations that can be found in an object. Or: let's say that I have exhausted the elementary symmetry operations that can be found in an object excluding rotations.
I can therefore try more complicated arrangements, or more complicated objects, which also have rotational symmetries.
The first symmetry you can add is 2, two-fold rotation, 180 degree rotation.
This is exactly what happens in the crystals with the first 8 classes triclinic + monoclinic + orth rhombic, see the following table, which also reports the "generators" of the 8 classes, taken from [4] , and summarized in Appendix 2 . (In the 2/m class, note, the 2010 generator can very well be replaced by m010 given the presence of the center because, as crystallography teaches, c + m involves a 2-fold rotation axis perpendicular to m).

Hermann Mauguin	5 bit symbols	Symmetries	Generators ([4], Bilbao Crystallographic Server)
1	00000	none	$1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
1_	0000c	c	$1\_ = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
m	000m0	m	$m010 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
2/m	000mc	m,c	$1\_ = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $2010 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
2	00200	2	$2010 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
222	0020c	2, c	$2001 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $2010 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
mm2	002m0	2, m	$2001 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ldots\ldots m010 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
mmm	002mc	2, m, c	$2001 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ldots 2010 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} ..$ $1\_ = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

The 8 triclinic + monoclinic + orthorhombic classes and generators, [4].

As can be seen, therefore, up to 2/m only generators (3x3 matrices) relevant to the symmetries m and c intervene. After, intervenes 2.
With Axis 2 available, what does Mother Nature do? Initially she creates an object formed to possess the 2-fold rotational symmetry, bit 2.
Only this symmetry.
This operation corresponds to the sequence 00200.

Let's add the c, m and mc symmetries.
First the symmetry c :
I create an object formed to have both property 2 and property c.
Here Mother Nature has a difficulty, that is, an object that had both the property 2 that the symmetry properties c, also would own the m symmetry (because, as crystallography teaches, 2 + c implies a symmetry plane m perpendicular to 2). More, it would coincide with 2/m that is already there. This is not what Mother Nature intended , who wanted an arrangement of objects with only the 2 symmetry and the c symmetry , and no other.
How you do it?
There is no other possibility, to make the arrangement of objects maximally symmetrical with respect to a center, but without the inversion center c, than to arrange in a "centrosymmetrical" way the axes of symmetry 2, re-proposing them symmetrically . I memorize it like this: three orthogonal binary axes around the center.
(Note: this is quasi-centrosymmetry because the points on the three main planes xy, yz, zx are centrosymmetric).



This is the orthorhombic class 222.
All this corresponds to the sequence 0020c.
We continue.
Next I create an object formed to have both property 2 and symmetry property m.
This operation corresponds to the sequence 002m0.
Finally I create an object formed in such a way as to have both property 2 and both properties m and c.
This operation corresponds to the sequence 002mc.
So, in order, the operations are:
00200
0020c
002m0
002mc.
In all I created 8 operations
00000
0000c
000m0
000mc
00200
0020c
002m0
002mc.

Mother Nature is so fascinated by this sequence that she repeats it three more times.
She does it with an additional 4-fold rotation axis.
Then she does it again with axis 3.
And finally with a combination of axes 2 and 3, parallel or oblique.

It is easy to memorize the entire scheme, representing it with a table, which has two functions: one is to remember the symmetries, present in the class or that intervene in the formation of the class (*);
the other is to constitute a simple, mnemonic symbology, which in fact replaces, or can replace, those of Hermann Mauguin, Schoenflies, etc.

	000	00c	0m0	0mc	200	20c	2m0	2mc
00 No rotational symmetry	1	c	m	mc	2	2c	2m	2mc
04	4	4c	4m	4mc	42	42c	42m	42mc
30	3	3c	3m	3mc	32	32c	32m	32mc
34	6	6c	6m	6mc	232	232c	232m	232mc

For comparison, I superimpose the symbols of Hermann Mauguin of the 32 classes

	000	00c	0m0	0mc	200	20c	2m0	2mc
00 No rotational symmetry	1 1	1_ c	m m	2/m mc	2 2	222 2c	mm2 2m	mmm 2mc
04	4 4	4_ 4c	4mm 4m	4/m 4mc	422 42	432 42c	4_2m 42m	4/mmm 42mc
30	3 3	3_ 3c	3m 3m	3_2m 3mc	32 32	622 32c	6_m2 32m	6/mmm 32mc
34	6 6	6_ 6c	6mm 6m	6/m 6mc	23 232	m3 232c	4_3m 232m	m3m 232mc

The logic with which Mother Nature proceeds is reported in Appendix 1 in order not to weigh down the text. In Appendix 2 for ease of reference there are all the "generators" of the 32 classes, taken from [4] . Appendix 3 is with Schoenflies symbols.

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(*) I repeat: present in the class or who intervene in the formation of the class . Such is for example the case of 4c, class 4_ of the improper axis 4. The class doesn't have property 4, and it doesn't have property c either. However, there is no doubt that properties 4 and c must intervene in order to form property 4_.

CONCLUSIONS

In conclusion, I would say that the 5-bit classification that I have proposed has been contained in crystallography all over the time.
It was and is only a matter of expressing the basic symmetries differently.
The Nature repeats a sequence of 8 classes, so 4x8 = 32.
The repetitive sequence is the 3-bit sequence
000, 00c, 0m0, 0mc, 200, 20c, 2m0, 2mc.
Mother Nature is so pleased with this sequence, or with this physical behavior, that she repeats it 4 times.
She does by adding axis 4 .
Then she does it again with axis 3.
Then she does it with a combination of axes 2 and 3, parallel or askew.

We could say that it was the notations, both of Hermann Mauguin and of Schoenflies, that concealed this type of classification; whose main key is, I would say, in placing 432 among the classes 4, simply characterized by three orthogonal axes in the form 444.
Another key is to attribute a quasi-centrosymmetry property to the three classes 222, 432 and 622.
But the third key is philosophical, and it is more important: it is a Nature way of proceeding, in order, by successive symmetries.

REFERENCES

[1] Bettini, G, "32 Point Groups of Three Dimensional Crystal Cells Described by 5 Bits"
viXra:1012.0052

[2] Bettini, G, "A new classification proposed for Crystal Classes – UPDATE#2"
https://www.mindat.org/article.php/1887/A+new+classification+proposed+for+Crystal+Classes+-+UPDATE%231

[3] Bettini, G, "5 Bit, 32 Crystal Classes",
https://vixra.org/abs/1711.0365

[4] Bilbao Server http://www.cryst.ehu.es/cryst/get_point_genpos.html

[5] Schoenflies notation https://en.wikipedia.org/wiki/Crystallographic_point_group

APPENDIX 1
Mother Nature at work.
Mother Nature constructs the following table 4 x 8 = 32 formed by the generators of the 32 crystal classes.

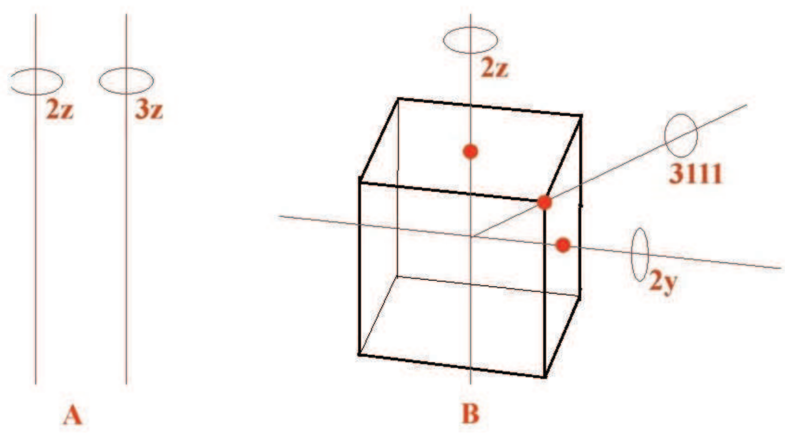
	c	m	m,c	2y	2y,c	2y,m	2y,m,c
4z	4z,c	4z,m	4z,m,c	4z,2y	4z,2y,c	4z,2y,m	4z,2y,m,c
3z	3z,c	3z,m	3z,m,c	3z,2y	3z,2y,c	3z,2y,m	3z,2y,m,c
3z,2z	3z,2z,c	3z,2z,m	3z,2z,m,c	3111,2z, 2y	3111,2z, 2y,c	3111,2z, 2y,m	3111,2z, 2y,m,c

For the construction of the table, the generators or the basic symmetries can be already written in the Bilbao Server [4] in the suitable form. If this is not the case, equivalent, ad hoc generators can be adopted. (Note: "y" is to remember "horizontal").
The equivalent generators are not the Bilbao Server generators but are precisely equivalent, in the sense that they produce the same class symmetries, and therefore their choice as alternative generators is entirely legitimate.
By "suitable form" we mean a form in which the generators of the first line are made to reappear, and are repeated in the following lines.

With the table made in this way, the repetition of the sequence at 8 (in red) is evident

	c	m	m,c	2y	2y,c	2y,m	2y,m,c
4z	4z,c	4z,m	4z,m,c	4z,2y	4z,2y,c	4z,2y,m	4z,2y,m,c
3z	3z,c	3z,m	3z,m,c	3z,2y	3z,2y,c	3z,2y,m	3z,2y,m,c
3z,2z	3z,2z,c	3z,2z,m	3z,2z,m,c	3111,2z, 2y	3111,2z, 2y,c	3111,2z, 2y,m	3111,2z, 2y,m,c

The repetition of the generators is done in the following way:
in the second row, a 4-fold rotation axis is added to the pre-existing generators (in red);
in the third row a 3-fold rotation axis is added to the existing generators;
in the last row the parallel rotations 2z and 3z are added (A) (hence a rotation 6). In the last four there are (B) the skewed rotations 2z and 3111, (then 2y etc.).



I have already seen in the text that Mother Nature finds the generators of the first line ready.

	000	00c	0m0	0mc	200	20c	2m0	2mc
00 No rotational symmetry	1 1	1_ c	m m	2/m mc	2 2	222 2c	mm2 2m	mmm 2mc

So I definitely write the first line

	c	m	m,c	2y	2y,c	2y,m	2y,m,c

Let the second line.

	000	00c	0m0	0mc	200	20c	2m0	2mc
04	4 4	4_ 4c	4mm 4m	4/m 4mc	422 42	432 42c	4_2m 42m	4/mmm 42mc

Mother Nature first introduces symmetry 4.
Subsequently Mother Nature passes to the 4c class (Hermann Mauguin symbol: 4_).
If class 4c had property 4 in addition to the center (therefore also axis 2 parallel to it), it would necessarily also have a plane m perpendicular to axis 4. This cannot be because the class must have only properties 4 and c. Also possessing 4, a perpendicular plane and the center of symmetry, the class would become the class 4mc, Hermann Mauguin symbol 4/m (see later).

So how does Mother Nature make sure that the class has 4 and c, even though it does not have 4 as a single property and not even c as a single property? Mother Nature is not a limited being like we humans are, the process she uses is a little more general. I can summarize it using the mnemonic concept: “I do this operation; if it's not possible, I'll do the one that most resembles". We have already seen an example in the case of the 222 class.

So she does like this:
4 and c exist but not as a single property , only together.
They intervene in the formation of an improper axis 4 , the rotoinversion axis 4_.

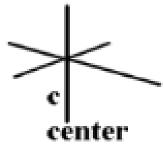
After that Mother Nature tries to add a symmetry m to 4. It does this with a vertical plane. This (see Appendix 2) produces the generators of the 4mm class.

Next class: 4mc. 4-fold rotation plus center c automatically produces m. The 4mc class is born, Hermann Mauguin symbol 4/m.

Next step: 42, adding a 2y horizontal axis perpendicular to 4z.
This (see Appendix 2) produces class 422 generators.

Next class: 42c.
Mother Nature is proposed here to create an object formed in order to have both property 42 (perpendicular axes 4 & 2) that the property c.
But an object that possessed both the property 42 (and thus also three 2-fold orthogonal axes) and also the further symmetry property c, would possess also various symmetry plans m, and ultimately it would coincide with 42mc.

The possibility of creating a center-like property with three orthogonal 2 axes makes no sense as well, because these are already there in the previous class 42.
How do you create an arrangement of objects with 42 symmetry and c symmetry but without m symmetry? The answer always lies in how Mother Nature works.
There is no other possibility, to make an arrangement of objects with 42 symmetry further centrosymmetric , than to arrange three orthogonal 4 axes.



Said and done, this is the gyroidal class 432.

Now Mother Nature tries to fix a series of objects with 4, 2 and m symmetries:
42m class.
But an added m plane , since there are already 4 and three orthogonal 2 axes , inevitably produces a center. It does not work this way. Try and try again, how does Mother Nature add an m plane to 42?
It goes like this:
symmetry 4 is used but as an improper axis, 4 _.
Thus the 42m class is born, Hermann Mauguin 4_2m .
Apparently there is a contradiction because looking at the generators of 4_2m (Appendix 2) there is no additional plan m. There is instead, because the vertical plane m110 appears between the symmetries of 4_2m.

Finally, the 42mc class remains to be tested.
This (see Appendix 2) produces class 4/mmm generators.

So I definitely write the second line

4z	4z,c	4z,m	4z,m,c	4z,2y	4z,2y,c	4z,2y,m	4z,2y,m,c

Now in the third row Mother Nature superimposes a 3-fold rotation on the triclinic monoclinic and orthorhombic sequence.

	000	00c	0m0	0mc	200	20c	2m0	2mc
30 3	3 3	3_ 3c	3m 3m	3_2m 3mc	32 32	622 32c	6_m2 32m	6/mmm 32mc

The first six classes that are generated in this way are obvious , and the symbology 3, 3c, 3m, 3mc, 32, 32c can be found perfectly in the generators of Appendix 2.
Thus the first six classes, Herman Mauguin symbols 3, 3_, 3m, 3_2m, 32, 622 are generated.

When Mother Nature tries to fix a series of objects with 3, 2 and m, 32m class symmetries, which should coincide with 6_m2, apparently there is a problem. The class should have a horizontal axis 2, which however does not appear among the generators of 6_m2 (see Appendix 2).
However, there is among the general positions of 6_m2 (see [4]) .
So it can be introduced between the generators.
Thus the 32m class was born, Hermann Mauguin symbol 6_m2 .

Finally, to produce 32mc there is no problem, the foreseen generators are already present (see Appendix 2) and thus the symmetries of class 6/mmm are born.

This is how the third row was born:

3z	3z,c	3z,m	3z,m,c	3z,2y	3z,2y,c	3z,2y,m	3z,2y,m,c

Now the fourth line, where Mother Nature goes to superimpose to the triclinic monoclinic and orthorhombic generators the axes 2 and 3, parallel or skewed respectively.

	000	00c	0m0	0mc	200	20c	2m0	2mc
34	6	6_	6mm	6/m	23	m3	4_3m	m3m
	6	6c	6m	6mc	232	232c	232m	232mc

The first four classes 6, 6c, 6m, 6mc, with axes 2 and 3 parallel, are obvious.
The generators are found perfectly already present in Appendix 2, and generate the Hermann Mauguin classes 6, 6_, 6mm, 6/m.
There remain the last four cubic classes 23, m3, 4_3m and m3m which could be thought to be the most difficult to interpret. Instead, they are among the easiest, as the generators are ready-made (see Appendix 2) in the desired shape 232, 23c, 23m, 23mc.

The last line also fits

3z,2z	3z,2z,c	3z,2z,m	3z,2z,m,c	3111,2z, 2y	3111,2z, 2y,c	3111,2z, 2y,m	3111,2z, 2y,m,c

APPENDIX 2

Generators of the 32 classes, taken from [4].

index

C_1	1	C_i	-1	C_2	2	C_s	m	C_{2h}	2/m	D_2	222	C_{2v}	mm2	D_{2h}	mmm
C_4	4	S_4	-4	C_{4h}	4/m	D_4	422	C_{4v}	4mm	D_{2d}	-42m	D_{4h}	4/mmm	C_3	3
C_{3i}	-3	D_3	32	C_{3v}	3m	D_{3d}	-3m	C_6	6	C_{3h}	-6	C_{6h}	6/m	D_6	622
C_{6v}	6mm	D_{3h}	-62m	D_{6h}	6/mmm	T	23	T_h	m-3	O	432	T_d	-43m	O_h	m-3m

Classe 1

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	x,y,z	1 0 0 0 1 0 0 0 1	1	1

Classe 1_

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-x,-y,-z	-1 0 0 0 -1 0 0 0 -1	-1 0,0,0	-1

Classe 2

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-x,y,-z	-1 0 0 0 1 0 0 0 -1	2 0,y,0	2_{010}

Classe m

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	x,-y,z	1 0 0 0 -1 0 0 0 1	m x,0,z	m_{010}

Classe 2/m

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-x,y,-z	-1 0 0 0 1 0 0 0 -1	2 0,y,0	2_{010}
2	-x,-y,-z	-1 0 0 0 -1 0 0 0 -1	-1 0,0,0	-1

Classe 222

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-x,-y,z	-1 0 0 0 -1 0 0 0 1	2 0,0,z	2_{001}
2	-x,y,-z	-1 0 0 0 1 0 0 0 -1	2 0,y,0	2_{010}

Classe mm2

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-x,-y,z	-1 0 0 0 -1 0 0 0 1	2 0,0,z	2_{001}
2	x,-y,z	1 0 0 0 -1 0 0 0 1	m 2x,x,z	m_{010}

Classe mmm

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-x,-y,z	-1 0 0 0 -1 0 0 0 1	2 0,0,z	2_{001}
2	-x,y,-z	-1 0 0 0 1 0 0 0 -1	2 0,y,0	2_{010}
3	-x,-y,-z	-1 0 0 0 -1 0 0 0 -1	-1 0,0,0	-1

Classe 4

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-x,-y,z	-1 0 0 0 -1 0 0 0 1	2 0,0,z	2_{001}
2	-y,x,z	0 -1 0 1 0 0 0 0 1	4^{+} 0,0,z	4^{+}_{001}

Classe 4_

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-x,-y,z	-1 0 0 0 -1 0 0 0 1	2 0,0,z	2_{001}
2	y,-x,-z	0 1 0 -1 0 0 0 0 -1	-4^{+} 0,0,z; 0,0,0	-4^{+}_{001}

Classe 4/m

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-x,-y,z	-1 0 0 0 -1 0 0 0 1	2 0,0,z	2_{001}
2	-y,x,z	0 -1 0 1 0 0 0 0 1	4^{+} 0,0,z	4^{+}_{001}
3	-x,-y,-z	-1 0 0 0 -1 0 0 0 -1	-1 0,0,0	-1

Classe 422

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-x,-y,z	-1 0 0 0 -1 0 0 0 1	2 0,0,z	2_{001}
2	-y,x,z	0 -1 0 1 0 0 0 0 1	4^+ 0,0,z	4^+_{001}
3	-x,y,-z	-1 0 0 0 1 0 0 0 -1	2 0,y,0	2_{010}

Classe 4mm

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-x,-y,z	-1 0 0 0 -1 0 0 0 1	2 0,0,z	2_{001}
2	-y,x,z	0 -1 0 1 0 0 0 0 1	4^+ 0,0,z	4^+_{001}
3	x,-y,z	1 0 0 0 -1 0 0 0 1	m x,0,z	m_{010}

Classe 4_2m

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-x,-y,z	-1 0 0 0 -1 0 0 0 1	2 0,0,z	2_{001}
2	y,-x,-z	0 1 0 -1 0 0 0 0 -1	-4^{+} 0,0,z; 0,0,0	-4^{+}_{001}
3	-x,y,-z	-1 0 0 0 1 0 0 0 -1	2 0,y,0	2_{010}

Classe 4/mmm

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-x,-y,z	-1 0 0 0 -1 0 0 0 1	2 0,0,z	2_{001}
2	-y,x,z	0 -1 0 1 0 0 0 0 1	4^{+} 0,0,z	4^{+}_{001}
3	-x,y,-z	-1 0 0 0 1 0 0 0 -1	2 0,y,0	2_{010}
4	-x,-y,-z	-1 0 0 0 -1 0 0 0 -1	-1 0,0,0	-1

Classe 3

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-y,x-y,z	0 -1 0 1 -1 0 0 0 1	3^+ 0,0,z	3^+_{001}

Classe 3_

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-y,x-y,z	0 -1 0 1 -1 0 0 0 1	3^+ 0,0,z	3^+_{001}
2	-x,-y,-z	-1 0 0 0 -1 0 0 0 -1	-1 0,0,0	-1

Classe 32

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-y,x-y,z	0 -1 0 1 -1 0 0 0 1	3^+ 0,0,z	3^+_{001}
2	-y,-x,-z	0 -1 0 -1 0 0 0 0 -1	2 x,-x,0	2_{1-10}

Classe 3m

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-y,x-y,z	0 -1 0 1 -1 0 0 0 1	3^+ 0,0,z	3^+_{001}
2	-y,-x,z	0 -1 0 -1 0 0 0 0 1	m x,-x,z	m_{110}

Classe 3_m

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-y,x-y,z	0 -1 0 1 -1 0 0 0 1	3^+ 0,0,z	3^+_{001}
2	-y,-x,-z	0 -1 0 -1 0 0 0 0 -1	2 x,-x,0	2_{1-10}
3	-x,-y,-z	-1 0 0 0 -1 0 0 0 -1	-1 0,0,0	-1

classe 6

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-y,x-y,z	0 -1 0 1 -1 0 0 0 1	3^+ 0,0,z	3^+_{001}
2	-x,-y,z	-1 0 0 0 -1 0 0 0 1	2 0,0,z	2_{001}

Classe 6_

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-y,x-y,z	0 -1 0 1 -1 0 0 0 1	3^+ 0,0,z	3^+_{001}
2	x,y,-z	1 0 0 0 1 0 0 0 -1	m x,y,0	m_{001}

Classe 6/m

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-y,x-y,z	0 -1 0 1 -1 0 0 0 1	3^+ 0,0,z	3^+_{001}
2	-x,-y,z	-1 0 0 0 -1 0 0 0 1	2 0,0,z	2_{001}
3	-x,-y,-z	-1 0 0 0 -1 0 0 0 -1	-1 0,0,0	-1

Classe 622

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-y,x-y,z	0 -1 0 1 -1 0 0 0 1	3^{+} 0,0,z	3^{+}_{001}
2	-x,-y,z	-1 0 0 0 -1 0 0 0 1	2 0,0,z	2_{001}
3	y,x,-z	0 1 0 1 0 0 0 0 -1	2 x,x,0	2_{110}

Classe 6mm

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-y,x-y,z	0 -1 0 1 -1 0 0 0 1	3^{+} 0,0,z	3^{+}_{001}
2	-x,-y,z	-1 0 0 0 -1 0 0 0 1	2 0,0,z	2_{001}
3	-y,-x,z	0 -1 0 -1 0 0 0 0 1	m x,-x,z	m_{110}

Classe 6_2m

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-y,x-y,z	0 -1 0 1 -1 0 0 0 1	3^{+} 0,0,z	3^{+}_{001}
2	x,y,-z	1 0 0 0 1 0 0 0 -1	m x,y,0	m_{001}
3	-y,-x,z	0 -1 0 -1 0 0 0 0 1	m x,-x,z	m_{110}

Classe 6/mmm

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-y,x-y,z	0 -1 0 1 -1 0 0 0 1	3^{+} 0,0,z	3^{+}_{001}
2	-x,-y,z	-1 0 0 0 -1 0 0 0 1	2 0,0,z	2_{001}
3	y,x,-z	0 1 0 1 0 0 0 0 -1	2 x,x,0	2_{110}
4	-x,-y,-z	-1 0 0 0 -1 0 0 0 -1	-1 0,0,0	-1

Classe 23

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-x,-y,z	-1 0 0 0 -1 0 0 0 1	2 0,0,z	2_{001}
2	-x,y,-z	-1 0 0 0 1 0 0 0 -1	2 0,y,0	2_{010}
3	z,x,y	0 0 1 1 0 0 0 1 0	3^{+} x,x,x	$3^{+}{}_{111}$

Classe m3_

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-x,-y,z	-1 0 0 0 -1 0 0 0 1	2 0,0,z	2_{001}
2	-x,y,-z	-1 0 0 0 1 0 0 0 -1	2 0,y,0	2_{010}
3	z,x,y	0 0 1 1 0 0 0 1 0	3^{+} x,x,x	3^{+}_{111}
4	-x,-y,-z	-1 0 0 0 -1 0 0 0 -1	-1 0,0,0	-1

classe 432

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-x,-y,z	-1 0 0 0 -1 0 0 0 1	2 0,0,z	2_{001}
2	-x,y,-z	-1 0 0 0 1 0 0 0 -1	2 0,y,0	2_{010}
3	z,x,y	0 0 1 1 0 0 0 1 0	3^+ x,x,x	3^+_{111}
4	y,x,-z	0 1 0 1 0 0 0 0 -1	2 x,x,0	2_{110}

classe 4_3m

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-x,-y,z	-1 0 0 0 -1 0 0 0 1	2 0,0,z	2_{001}
2	-x,y,-z	-1 0 0 0 1 0 0 0 -1	2 0,y,0	2_{010}
3	z,x,y	0 0 1 1 0 0 0 1 0	3^+ x,x,x	3^+_{111}
4	y,x,z	0 1 0 1 0 0 0 0 1	m x,x,z	m_{1-10}

classe m3_m

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-x,-y,z	-1 0 0 0 -1 0 0 0 1	2 0,0,z	2_{001}
2	-x,y,-z	-1 0 0 0 1 0 0 0 -1	2 0,y,0	2_{010}
3	z,x,y	0 0 1 1 0 0 0 1 0	3^+ x,x,x	3^+_{111}
4	y,x,-z	0 1 0 1 0 0 0 0 -1	2 x,x,0	2_{110}
5	-x,-y,-z	-1 0 0 0 -1 0 0 0 -1	-1 0,0,0	-1

APPENDIX 3
A3.1
It is very interesting to proceed with the Schoenflies symbols

As a first step, I superimpose the Schoenflies symbols on the mnemonic table .

	000	00c	0m0	0mc	200	20c	2m0	2mc
00 No rotational symmetry	C1 1	Ci c	Cs m	C2h mc	C2 2	D2 2c	C2v 2m	D2h 2mc
04	C4 4	S4 4c	C4v 4m	C4h 4mc	D4 42	O 42c	D2d 42m	D4h 42mc
30	C3 3	C3i 3c	C3v 3m	D3d 3mc	D3 32	D6 32c	D3h 32m	D6h 32mc
34	C6 6	C3h 6c	C6v 6m	C6h 6mc	T 232	Th 232c	Td 232m	Oh 232mc

Once the operation is completed , also in this case, as well as with the HM symbols, it is evident that the 32 classes follow each other in a very reasonable sequence (observe how the classes placed in a column are similar to each other). Even more evident if I change certain symbols. This is not in contradiction with Schoenflies symbology, because it is equivalent to memorizing the symmetries differently. Which is legitimate, because certain basic symmetries can be replaced by "equivalent symmetries". Here I report the meaning of the Schoenflies symbols (from [5]).
Quote.
"Schoenflies notation.
In Schoenflies notation, point groups are denoted by a letter symbol with a subscript. The symbols used in crystallography mean the following:

Cn (for cyclic) indicates that the group has an n-fold rotation axis. Cnh is Cn with the addition of a mirror (reflection) plane perpendicular to the axis of rotation. Cnv is Cn with the addition of n mirror planes parallel to the axis of rotation.
S2n (for Spiegel, German for mirror) denotes a group with only a 2n-fold rotation-reflection axis.
Dn (for dihedral, or two-sided) indicates that the group has an n-fold rotation axis plus n twofold axes perpendicular to that axis. Dnh has, in addition, a mirror plane perpendicular to the n-fold axis. Dnd has, in addition to the elements of Dn, mirror planes parallel to the n-fold axis.
The letter T (for tetrahedron) indicates that the group has the symmetry of a tetrahedron. Td includes improper rotation operations, T excludes improper rotation operations, and Th is T with the addition of an inversion.
The letter O (for octahedron) indicates that the group has the symmetry of an octahedron (or cube), with (Oh) or without (O) improper operations (those that change handedness)."

The first change I make is in the second column (in red). What is memorized is that in the class the rotation and the symmetry center are involved, either separately or at the same time.

	000	00c	0m0	0mc	200	20c	2m0	2mc
00 No rotational symmetry	C1 1	Ci c	Cs m	C2h mc	C2 2	D2 2c	C2v 2m	D2h 2mc
04	C4 4	C4i 4c	C4v 4m	C4h 4mc	D4 42	O 42c	D2d 42m	D4h 42mc
30	C3 3	C3i 3c	C3v 3m	D3d 3mc	D3 32	D6 32c	D3h 32m	D6h 32mc
34	C6 6	C6i 6c	C6v 6m	C6h 6mc	T 232	Th 232c	Td 232m	Oh 232mc

Going forward, I realize that the third column includes the indication "v" of the vertical plane. That's redundant, because if the class must be only rotation + plane m, it is automatic that the plane is vertical. So we can assume that the class symbol is Cnm. Meaning: "Cnm is Cn with the addition of specular planes only" .
Therefore it is permissible that "Cnm is Cn with the addition of mirror planes only" replaces the Schoenflies definition" Cnv is Cn with the addition of n mirror planes parallel to the axis of rotation. "

I do it in symbols.

	000	00c	0m0	0mc	200	20c	2m0	2mc
00 No rotational symmetry	C1 1	Ci c	Cs m	C2h mc	C2 2	D2 2c	C2v 2m	D2h 2mc
04	C4 4	C4i 4c	C4m 4m	C4h 4mc	D4 42	O 42c	D2d 42m	D4h 42mc
30	C3 3	C3i 3c	C3m 3m	D3d 3mc	D3 32	D6 32c	D3h 32m	D6h 32mc
34	C6 6	C6i 6c	C6m 6m	C6h 6mc	T 232	Th 232c	Td 232m	Oh 232mc

Then it happens that when you want to add to the starting class (first column C1, C4, C3, C6) specular planes + a true symmetry center, the class becomes Cnmi.
I do it in symbols (mathematical details in A3.2).

	000	00c	0m0	0mc	200	20c	2m0	2mc
00 No rotational symmetry	C1 1	Ci c	Cs m	C2h mc	C2 2	D2 2c	C2v 2m	D2h 2mc
04 4	C4 4	C4i 4c	C4m 4m	C4mi 4mc	D4 42	O 42c	D2d 42m	D4h 42mc
30 3	C3 3	C3i 3c	C3m 3m	C3mi 3mc	D3 32	D6 32c	D3h 32m	D6h 32mc
34 3 2	C6 6	C6i 6c	C6m 6m	C6mi 6mc	T 232	Th 232c	Td 232m	Oh 232mc

At this point I realize in the first line that in fact Cs and C2h are nothing more than the symmetry m and the symmetry m + i added to the rotation C1 (C1 which means 360° rotational symmetry, that is: nothing).
So I change the symbols accordingly.

	000	00c	0m0	0mc	200	20c	2m0	2mc
00 No rotational symmetry	C1 1	Ci c	Cm m	Cmi mc	C2 2	D2 2c	C2v 2m	D2h 2mc
04 4	C4 4	C4i 4c	C4m 4m	C4mi 4mc	D4 42	O 42c	D2d 42m	D4h 42mc
30 3	C3 3	C3i 3c	C3m 3m	C3mi 3mc	D3 32	D6 32c	D3h 32m	D6h 32mc
34 3 2	C6 6	C6i 6c	C6m 6m	C6mi 6mc	T 232	Th 232c	Td 232m	Oh 232mc

I can well say at this point that I have well represented the basic symmetries, coherently with the Schoenflies symbols, having simply decided to represent them in a different way.

At this point a "lateral" axis 2 is born, that is the symbology D.
" Dn (for dihedral, or two-sided) indicates that the group has an n-fold rotation axis plus n twofold axes perpendicular to that axis. "
Let's say that a lateral 2-fold rotation axis is added to the main rotation .
If by itself the symbol "D" means the added lateral axis, I can use it combined with the previous rotation, which is given by 1, 4, 3, 23.
I memorize it in symbols

	000	00c	0m0	0mc	200	20c	2m0	2mc
00 No rotational symmetry	C1 1	Ci c	Cm m	Cmi mc	D1 2	D2 2c	C2v 2m	D2h 2mc
04 4	C4 4	C4i 4c	C4m 4m	C4mi 4mc	D4 42	O 42c	D2d 42m	D4h 42mc
30 3	C3 3	C3i 3c	C3m 3m	C3mi 3mc	D3 32	D6 32c	D3h 32m	D6h 32mc
3 2 34 3 2	C6 6	C6i 6c	C6m 6m	C6mi 6mc	D23 232	Th 232c	Td 232m	Oh 232mc

I can say that as well (in red) is well represented and explained the meaning of the fifth column, including the point group T .
The group T, rightly, according to the definition of Schoenflies, has an axis 2 perpendicular to the original vertical rotation 23 (here the axis 2z forming part of the axes 2z and 3111 of the cube).
Schoenflies D4 and D3 symbols (in black) do not need to be changed .

For the following columns it is advisable to start from the last one.
A horizontal plane is added to the dihedral group Dn so in the symbols it is sufficient to update Oh in the form D23h.
D2h, D4h, D6h (in black) do not need to be changed for the moment .
I memorize in red the symmetry Oh which is written D23h.

	000	00c	0m0	0mc	200	20c	2m0	2mc
00 No rotational symmetry	C1 1	Ci c	Cm m	Cmi mc	D1 2	D2 2c	C2v 2m	D2h 2mc
04 4	C4 4	C4i 4c	C4m 4m	C4mi 4mc	D4 42	O 42c	D2d 42m	D4h 42mc
30 3	C3 3	C3i 3c	C3m 3m	C3mi 3mc	D3 32	D6 32c	D3h 32m	D6h 32mc
3 2 34 3 2	C6 6	C6i 6c	C6m 6m	C6mi 6mc	D23 232	Th 232c	Td 232m	D23h 232mc

If in the last column we examine the numerous symmetries that necessarily arise with the addition of a horizontal plane (and therefore with a symmetry center, with various axes 2 and consequent planes m), it is easy to realize that the last column contains the holohedral classes mmm, 4 / mmm, 6 / mmm, together with the maximum symmetry class Oh.

At this point, to proceed, it is advisable to carefully examine the third last column.
I say that this contains the simmetries of the class Dn (as previously defined) to which the center or a triad of orthogonal axes is added which is a triplication of the main rotation axis 4 or 2.
(A kind of "centrosymmetry" that is what I have indicated as "bit c").
The classes are therefore indicated as Dnc , defined as follows:
"The Dnc class is the Dn class plus the center or three orthogonal axes that triple the rotation axes 2 or 4 already present"
I draw them in red.

	000	00c	0m0	0mc	200	20c	2m0	2mc
00 No rotational symmetry	C1 1	Ci c	Cm m	Cmi mc	D1 2	D1c 2c	C2v 2m	D2h 2mc
04	C4 4	C4i 4c	C4m 4m	C4mi 4mc	D4 42	D4c 42c	D2d 42m	D4h 42mc
30	C3 3	C3i 3c	C3m 3m	C3mi 3mc	D3 32	D3c 32c	D3h 32m	D6h 32mc
34	C6 6	C6i 6c	C6m 6m	C6mi 6mc	D23 232	D23c 232c	Td 232m	D23h 232mc

It is easy to see that all these four classes contain such a triplet.
The first is 222.
The second is the gyroidal class 432 which I can indicate as 444.
The third is 622 which contains the axis 3 + the triad 222.
The fourth also contains the triad 222, along with also a true inversion center i

$$1\_ = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(in fact it is T plus an inversion center).

Finally we come to the penultimate column.
This column contains the symmetries of the Dn classes (as previously defined) to which symmetries m, planes, are added.
The column therefore contains classes that I certainly indicate with Dnm, characterized by the fact of not to have the inversion center i, just planes.
I write them in red.

	000	00c	0m0	0mc	200	20c	2m0	2mc
00 No rotational symmetry	C1 1	Ci c	Cm m	Cmi mc	D1 2	D1c 2c	D1m 2m	D2h 2mc
04 4	C4 4	C4i 4c	C4m 4m	C4mi 4mc	D4 42	D4c 42c	D4m 42m	D4h 42mc
30 3	C3 3	C3i 3c	C3m 3m	C3mi 3mc	D3 32	D3c 32c	D3m 32m	D6h 32mc
3 2 34 3 2	C6 6	C6i 6c	C6m 6m	C6mi 6mc	D23 232	D23c 232c	D23m 232m	D23h 232mc

At this point, adding an inversion center, we certainly arrive at the maximum symmetry classes of the last column. Their symbols were provisional, so I change them to Dnmc. I do it in red.

	000	00c	0m0	0mc	200	20c	2m0	2mc
00 No rotational symmetry	C1 1	Ci c	Cm m	Cmi mc	D1 2	D1c 2c	D1m 2m	D1mc 2mc
04 4	C4 4	C4i 4c	C4m 4m	C4mi 4mc	D4 42	D4c 42c	D4m 42m	D4mc 42mc
30 3	C3 3	C3i 3c	C3m 3m	C3mi 3mc	D3 32	D3c 32c	D3m 32m	D3mc 32mc
3 2 34 3 2	C6 6	C6i 6c	C6m 6m	C6mi 6mc	D23 232	D23c 232c	D23m 232m	D23mc 232mc

In conclusion, I would say that the 5-bit classification that I have proposed has always been contained in the Schoenflies symbology. It was and is only a matter of expressing the basic symmetries differently, with a symbology that is "quasi-Schoenflies".

We could say that it was the notations, either by Hermann Mauguin or by Schoenflies, which hid this type of classification; whose main key is , I would say, in placing 432 among classes 4, simply characterized by three orthogonal axes in the form 444.
Another key is to attribute a quasi-centrosymmetry property to the three classes 222, 432 and 622.

A3.2
Mathematical details as to the dihedral class D3d (example: calcite).

I'll show here how you may though of the dihedral class D3d as C3mi (C3 + plane, m + inversion center, i).
To do this, I substitute "equivalent generators" to the Bilbao generators.
The D3d class (Hermann Mauguin 3_2m) has the following generators

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-y,x-y,z	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	3^+ 0,0,z	3^+_{001}
2	-y,-x,-z	$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	2 x,-x,0	2_{1-10}
3	-x,-y,-z	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	-1 0,0,0	-1

Following the Schoenflie symbols this class is dixedral, Dnd.
Rimember: "Dn (for dihedral, or two-sided) indicates that the group has an n-fold rotation axis plus n twofold axes perpendicular to that axis. Dnh has, in addition, a mirror plane perpendicular to the n-fold axis. Dnd has, in addition to the elements of Dn, mirror planes parallel to the n-fold axis."

Howeverit can be though of as C3mi (C3 + plane, m + inversion center, i).
Remember: "Cn (for cyclic) indicates that the group has an n-fold rotation axis. Cnh is Cn with the addition of a mirror (reflection) plane perpendicular to the axis of rotation. Cnv is Cn with the addition of n mirror planes parallel to the axis of rotation."

To show this, we note first that among the class symmetries there is the vertical plane m1-10.

As a cosequence, the 2-fold rotational generator, namely, 21-10

-y,-x,-z	$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	2 x,-x,0	2_{1-10}

can be substituted by m1-10

y,x,z	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	m x,x,z	$m_{1\text{-}10}$

due to the inversion center

-x,-y,-z	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	-1 0,0,0	-1

This because

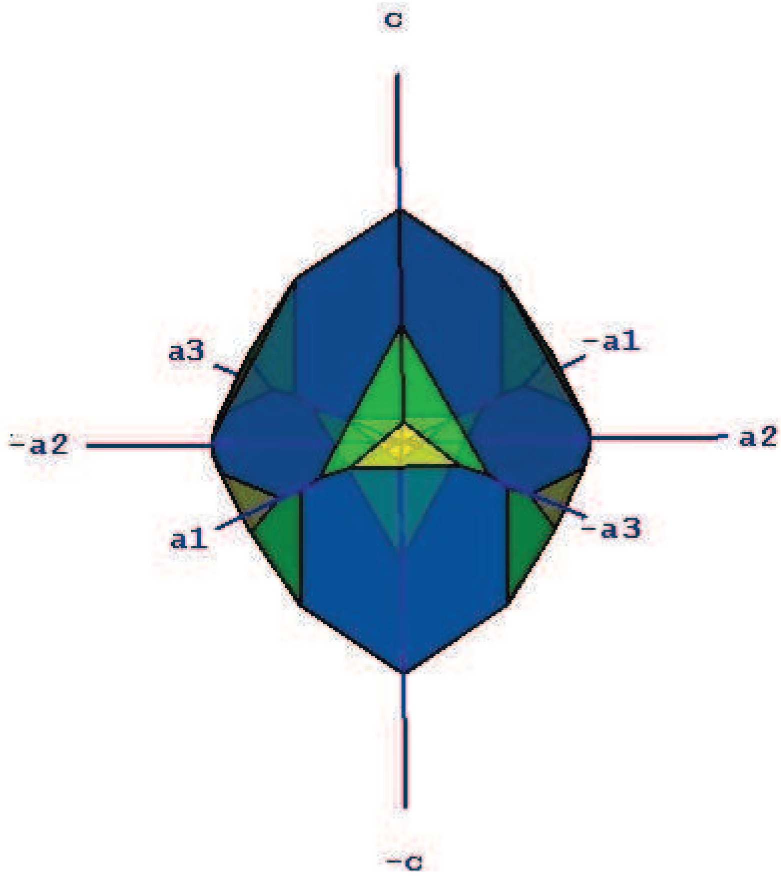
$$\text{(plane m) x (center i)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{x} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \text{axis 2}$$

So insted of Bilbao generators it's legitimate to assume the equivalent generators

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	-y,x-y,z	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	3^{+} 0,0,z	3^{+}_{001}
	y,x,z	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	m x,x,z	$m_{1\text{-}10}$
3	-x,-y,-z	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	-1 0,0,0	-1

Conclusion:
the dihedral group D3d appears to be (or may be though of as) C3 wich, after the 3, 3c, 3m symmetries, adds the symmetry 3mc too.

A calcite crystal (3_2m class) clearly shows the vertical symmetry planes and the inversion center.



c
a3
-a1
-a2
a2
a1
-a3
-c