Einstein-Rosen proposition (1935) revisited.

©Thierry PERIAT - no affiliation

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This document proves the existence of a link between (i) each Bowen-York solution for the Einstein's equations and (ii) the main part of the decomposition of a deformed angular momentum. The intrinsic method is the tool supporting the demonstration. The strange and unlikely concept of deformed angular momentum is explained in a formal way referring to quantum physics. This explanation suggests that any tetra-polar polarization is related to a quantified spin field. The thematic of the asymptotic flatness is examined at the end of this exploration; ©Thierry PERIAT, Einstein-Rosen proposition in 1935 revisited. This version is deepening and hopefully reformulating the first version [a] with more precision.

Contents

1 Introduction

1.1 Context

Einstein's master work [00; (a) for the original version; (b) for a translation into the English language] is published in 1916. In 1935, Einstein and Rosen propose in [01] a very original concept for the description of particles within a specific context which can be obtained in starting from the prescriptions exposed in [00]. The proposition should presumably allow a correct understanding of the atomic structures; at least the ones which was known at this time.

In 1944, A. Lichnerowicz writes his famous equation [02], [03(b); §8.2.4, pp. 130-131]. One may appreciate the very easily understandable presentation of the initial data problem and of its treatment in [07; pp. 109-112].

Between 1979 and 1982, Lichnerowicz approach is reworked and significantly extended by Bowen in [03(a-1), (a-3)]. The progression is also explained in [03(b); chapter 6, pp. 83-102] and [03(b); chapter 8; §8.2.6, pp. 136-139]. This extension allows focusing attention on black holes [03(a-2)].

Recent explorations are applying former considerations to the neutrons stars [08]. At a more general level, the initial data problem gives rise to numerous articles, see e.g.: [09]. For some authors, there are conformally flat initial data with explicitly given analytic extrinsic curvature solving the vacuum momentum constraints. The cylindrical symmetric sub-case of the Bowen-York solutions is a sub-class of this general configuration [10]. Other authors suggest to manage the concept of conformally flat slices cautiously [11].

1.2 Claim

As a matter of historical facts, one can discover a set of solutions for the Einstein's equations with what will be called for simplicity the Lichnerowicz-York-Bowen (L-Y-B) initial data. The formalism of these solutions can be read, e.g., in [03(b); chapter 8, §.8.2.6, p.136, (8.69)]:

$$X^i_{LYB} = -\frac{1}{4\,.\,r}\,.\,(7\,.\,f^{ij}\,.\,P_j + \frac{P_j\,.\,x^j\,.\,x^i}{r^2} - \frac{1}{r^3}\,.\,\epsilon^{ij}{}_k\,.\,S_j\,.\,x^k$$

Let recall that $\mathbf{P}^*{}_{LYB}$ (P_1, P_2, P_3) coincides with the co-variant version of the ADM linear momentum associated with the initial hyper-surface Σ_0 [03(b); chapter 8, §.8.2.6, p.138, (8.82)] whilst $\mathbf{S}^*{}_{LYB}$ (S_1, S_2, S_3) coincides with the co-variant version of the ADM angular momentum associated with the same initial hyper-surface Σ_0 [03(b); chapter 8, §.8.2.6, p.139, (8.84)].

At a first glance, it is not easy to find a concrete and physical interpretation for these solutions. This is the reason why the first claim of this document will be to prove that these solutions can be indirectly related to matrices resulting from the decomposition of deformed angular momentum. The tools to reach that purpose will be (i) the intrinsic method which has been explained in [b] and (ii) the so-called extrinsic method [c]. These methods are the basic stones of what has been called the theory of the (E) question.

1.3 Recalling the basic results of the theory of the (E) question

Let $V = \mathbb{C} \otimes E(3, \mathbb{R})$ be the set in which the discussion is developed. This set is equipped with a deformed cross product characterized by its deforming matrix [A] in M(3, $\mathbb{C}$). Let this deformed cross product act on some pair of elements ($\mathbf{a}$, $\mathbf{b}$) in V × V. Let consider the image of that deformed cross product in the dual space of V.

Any given deformed cross product has at least one simple decomposition without residual part [b; proposition 2.2 and its demonstration, pp.7-8]. Sometimes, this product has non-trivial divisions. In that case, the image of the deformed cross product is a pair ([P], $\mathbf{z}$) in M(3, $\mathbb{C}$) × V such that:

$$|[\mathbf{a}, \mathbf{b}]_{[A]} >= [P] \,.\, |\mathbf{b} > + |\mathbf{z} >$$

Because of the so-called initial theorem [b] any non-trivial decomposition is associated with a polynomial form of degree at most equal to two which is nothing but the determinant measuring the difference between the simplest decomposition without residual part and the main part of this non-trivial decomposition. This polynomial depends on the components of $\mathbf{a}$:

$$\Lambda(\mathbf{a}) = \Lambda(a^1, a^2, a^3) = |_{[A]}\Phi(\mathbf{a}) - [P]| = \sum_{ij} d_{ij} \,.\, a^i \,.\, a^j + \sum_i d_i \,.\, a^i - |P|$$

It allows the definition of two pertinent mathematical objects, precisely:

1. **A classical Hessian**; when this polynomial has constant coefficients and is a smooth derivable function, then there exists a (3-3) matrix [D] containing all coefficients of degree two such that:

$$[S_0] = [Hess_{(\mathbf{a},0)}\Lambda(\mathbf{a})] = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{12} & h_{22} & h_{23} \\ h_{13} & h_{23} & h_{33} \end{bmatrix} = [d_{ij}] + [d_{ij}]^t = [D] + [D]^t$$

The subscript ($\mathbf{a}$, 0) means that that Hessian has been calculated in operating the partial derivations by respect for the diverse components of the vector **a** in a geometrically flat environment (symbolized through a vanishing Ricci scalar: R = 0).

2. and a vector $\mathbf{d}^*$ of which the components are the coefficients of degree one:

$$|\mathbf{d}^* >= |d_1\,,\,d_2\,,\,d_3>$$

The kernel of any main part belongs either to the class I when the Hessian of the polynomial is not degenerated or to the class II when the Hessian of the polynomial is degenerated.

- The case of a degenerated Hessian has been treated in the original version[1] of [b] and will no longer be developed here.
- In opposition, if the polynomial is not degenerated, its Hessian is reversible; this situation is characterized with the condition:

$$|S_0|$$
$$=$$
$$h_{11}\,.\,h_{22}\,.\,h_{33} + 2\,.\,h_{12}\,.\,h_{23}\,.\,h_{13} - h_{11}.(h_{23})^2 - h_{22}.(h_{13})^2 - h_{33}.(h_{12})^2 \neq 0$$

In that case, the main part of any non-trivial decomposition writes [b; p.31]:

$$[P]_{|A|}$$
$$=$$
$$\frac{1}{|A|}\,.\,\underbrace{\{[A]^t\,.\,[J]\}}_{effective\,deforming\,matrix}\,.\,\underbrace{\{\frac{1}{2}\,.\,[Hess_{(\mathbf{a},0)}\Lambda(\mathbf{a})] - \frac{1}{|A|}\,.\,{}_{[J]}\Phi({}_\Lambda\mathbf{s})\}}_{=[N]_{|A|},\,the\,kernel}$$

Where:

- Per convention:

$$[A] = \begin{bmatrix} A^1_{12} & A^2_{12} & A^3_{12} \\ A^1_{23} & A^2_{23} & A^3_{23} \\ A^1_{13} & A^2_{13} & A^3_{13} \end{bmatrix},\ |A| \neq 0$$

  and:

$$[J] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

- ${}_\Lambda\mathbf{s}$ denotes the singular vector of the polynomial Λ:

$$|{}_\Lambda\mathbf{s} >= -[Hess_{(\mathbf{a},0)}\Lambda(\mathbf{a})]^{-1}\,.\,|\mathbf{d}^* >$$

[1] In the French language

- The determinant of the kernel can be precisely calculated:

$$|N| \,=\, \frac{|S_0|}{8} \,+\, \frac{1}{|A|^2}\,.\, <\boldsymbol{\Lambda}\mathbf{S}, \boldsymbol{\Lambda}\mathbf{S}>_{[D]}$$

  In this expression the coefficients of degree two of the polynomial Λ generate a (local) scalar product:

$$< \ldots, \ldots >_{[d_{ij}]}$$

  Since (i) the determinant of the transposed of a given matrix is equal to the determinant of that matrix and (ii) |J| = -1, it is obvious that:

$$|P| \,=\, -|N|$$

  Remark: The determinant of the kernel may eventually be null although the determinant of the Hessian does not vanish.

As an evident matter of facts, these basic results can be applied to a whole set of real situations and of mathematical problems. Indeed, within that theory, any proper polynomial is now a sufficient tool allowing the discovery of a family of non-trivially decomposed deformed cross products. The immediate purpose of this document is to apply them to a very important physical object: the angular momentum. Hence, this exploration considers deformed cross products of the following type[2]:

$$[d\mathbf{x}, \ldots]_{[A]}$$

The main part [P] of a non-trivial decomposition differs from the most simple one without residual part:

$$_{[A]}\Phi(d\mathbf{x})$$

The polynomial form of interest measuring the difference is the determinant:

$$\Lambda(d\mathbf{x}) \,=\, |_{[A]}\Phi(d\mathbf{x}) \,-\, [P]|$$

This determinant is a polynomial of degree at most equal to two [b; the initial theorem, p.14] depending on d**x**.

Because of two unexpected coincidences, these results can be applied to the solutions proposed by Bowen-York for a specific set of initial data.

1.4 The main proposition

Proposition 1.1. *Provided:*

1. *The polynomial $\Lambda(d\boldsymbol{x})$ can be identified with at least one Taylor - MacLaurin development up to the second order of some numerical function f depending on the three spatial components (x, y, z) of a given position **x**:*

$$\exists\, f \,:\, \Lambda(d\mathbf{x}) \,=\, df(\mathbf{x})$$

[2] Here "d" denotes an ordinary derivation (see Descartes or Leibniz), **x** is the spatial position for some event, **...** is any spatial vector and [A] represents an element in M(3, $\mathbb{R}$) or in M(3, $\mathbb{C}$). In the semantic of the theory of the (E) question, [A] is a *deforming matrix*.

2. *The spatial gradient of $f(\boldsymbol{x})$ is a:*

$$\frac{1}{r^2}\, field$$

With:

$$r \,=\, (< \mathbf{x}, \mathbf{x} >_{Id_3})^{1/2} \,=\, \sqrt{x^2 \,+\, y^2 \,+\, z^2}$$

Then:

1. *The singular vector of any non-degenerated polynomial form $\Lambda(d\boldsymbol{x})$ coincides with the spatial position $\boldsymbol{x}$;*

2. *Any deformed cross product $[d\boldsymbol{x}, \ldots]_{[A]}$ accepts a non-trivial decomposition of which the kernel [N] of the main part [P] is such that the spatial vector:*

$$k \,.\, \underbrace{{}^{(3)}[T]^{-1} \,.\, {}^{(3)}[N]}_{=[P]} \,.\, |^{(3)}\mathbf{P}^{*}_{LYB} >$$

... can be identified in a coherent manner with a solution [03(b); §8.2.6, p. 136, (8.69)] for some initial data of the "Bowen-York type":

$$= \mathbf{X}_{LYB}$$

Here, for coherence:

- k is some scalar of which the precise formulation will be given below during the demonstration;
- the matrix $[T]^{-1}$ represents the *effective deforming matrix* which has been introduced in [b];
- the spatial vector $\mathbf{P}^{*}{}_{LYB}$ coincides with the co-variant version of the ADM linear momentum of the initial hyper-surface Σ_0

2 Demonstration- part 1

2.1 Prerequisites

For the deformed cross products at hand, the results which have been obtained in [b] write:

$$[P]_{|A|}$$

$$=$$

$$\frac{1}{|A|} \cdot \{[A]^t \,.\, [J]\} \,.\, \{\frac{1}{2} \cdot [Hess_{(d\mathbf{x},0)}\Lambda(d\mathbf{x})] + \frac{1}{|A|} \cdot {}_{[J]}\Phi([Hess_{(d\mathbf{x},0)}\Lambda(d\mathbf{x})]^{-1} \,.\, |\mathbf{d}^{*} >)\}$$

And:

$$\Lambda(d\mathbf{x})$$

$$=$$

$$\Lambda(dx^1,\, dx^2,\, dx^3) \,=\, |_{[A]}\Phi(d\mathbf{x}) \,-\, [P]| \,=\, \sum_{ij} d_{ij}\,.\,dx^i\,.\,dx^j \,+\, \sum_i d_i\,.\,dx^i \,+\, |N|$$

$$|_{\mathbf{\Lambda}}\mathbf{s}> = \,-[Hess_{(d\mathbf{x},0)}\Lambda(d\mathbf{x})]^{-1}\,.\,|\mathbf{d}^*>$$

Let now consider the two prerequisites of proposition 1.1:

1. Let for example focus attention on Newtonian fields:

$$d_i \,=\, -\frac{G.m}{r^3}.x^i \iff \mathbf{d}^* \,=\, -\frac{G.m}{r^3}\,.\,\mathbf{x}$$

with:

$$r \,=\, (<\mathbf{x},\mathbf{x}>_{Id_3})^{1/2}$$

2. ... and let suppose that the polynomial form Λ can be rewritten in the following way:

$$\Lambda(d\mathbf{x}) \,=\, df(\mathbf{x}) \,=\, \frac{\partial f(\mathbf{x})}{\partial x^i}\,.\,dx^i \,+\, \frac{1}{2}\,.\,\frac{\partial^2 f(\mathbf{x})}{\partial x^i \partial x^j}\,.\,dx^i\,.\,dx^j \,+\, 0(3)$$

This hypothesis is naturally yielding the following identifications :

$$[D] \,=\, \frac{1}{2}\,.\,[Hess_{(\mathbf{x},0)}f(\mathbf{x})]$$

$$\mathbf{d}^* \,=\, \mathbf{Grad_x}f(\mathbf{x})$$

$$|N| \,=\, 0(3)$$

2.2 Plausibility of the prerequisites

The polynomial Λ(d$\mathbf{x}$) can only be identified with the development of df($\mathbf{x}$) if and when the determinant |N| has a relatively small value. Let verify if this condition is realized in the case at hand. Concretely, the determinant |N| must be calculated. Due to the fact that (recall):

$$[S_0] \,=\, [Hess_{(d\mathbf{x},0)}\Lambda(d\mathbf{x})] \,=\, [d_{ij}] \,+\, [d_{ij}]^t$$

When f is a continuous function, its Hessian is a symmetric matrix and an eventual identification between the coefficients of degree two implies:

$$[S_0] \,=\, [Hess_{(d\mathbf{x},0)}\Lambda(d\mathbf{x})] \,=\, Hess_{(\mathbf{x},0)}f(\mathbf{x})$$

In the case at hand, the classical Hessian of f($\mathbf{x}$) can be written with more precision; since:

- $$\frac{\partial^2 f(\mathbf{x})}{\partial x^i \partial x^j} \,=\, \frac{\partial d_i}{\partial x^j} \,=\, -\,G.m.\frac{\partial(\frac{x^i}{r^3})}{\partial x^j} \,=\, -\,\frac{G.m}{r^6}\,.\,(\delta^i_j\,.\,r^3 \,-\, x^i\,.\,3\,.\,r^2\,.\,\frac{\partial r}{\partial x^j})$$
- $$\frac{\partial r}{\partial x^j} \,=\, \frac{x^j}{r}$$

It is easy to state that:

$$[S_0] = -\frac{G.m}{r^3}.\{Id_3 - \frac{3}{r^2}.T_2(\otimes)(\mathbf{x},\mathbf{x})\} = -\frac{G.m}{r^3}.\{Id_3 - \frac{3}{r^2}.\phi\}$$

With, per convention:

$$T_2(\otimes)(\mathbf{x},\mathbf{x}) = \begin{bmatrix} x^1.x^1 & x^2.x^1 & x^3.x^1 \\ x^1.x^2 & x^2.x^2 & x^3.x^2 \\ x^1.x^3 & x^2.x^3 & x^3.x^3 \end{bmatrix} = [x^i.x^j] = \phi$$

It can be proved that:

$$|Id_3 - \frac{3}{r^2}.\phi| = -2$$

The determinant of this Hessian is:

$$|S_0| = (-2).(-\frac{G.m}{r^3})^3 = 2.(\frac{G.m}{r^3})^3$$

The purpose is to prove the plausibility of the condition:

$$|N| = \frac{|S_0|}{8} + \frac{1}{|A|^2}.<{}_{\mathbf{\Lambda}}\mathbf{s},{}_{\mathbf{\Lambda}}\mathbf{s}>_{[d_{ij}]} = 0(3)$$

The proof can only be obtained in calculating the singular vector. Fortunately, except for vanishing sources (m = 0) or at infinity (r $\to \infty$), the determinant of the Hessian never vanishes. Therefore, the Hessian is most of the time a reversible matrix and the singular vector of the polynomial Lambda can be calculated; concretely:

$$[S_0]^{-1} = -\frac{2.r^3}{G.m}.\{Id_3 - \frac{3}{2.r^2}.\phi\}$$

Working with this result, the singular vector is:

$$|{}_{\mathbf{\Lambda}}\mathbf{s}>$$
$$=$$
$$-Hess^{-1}_{(d\mathbf{x},0)}\Lambda(d\mathbf{x}).|\mathbf{d}^*>$$
$$=$$
$$\frac{G.m}{r^3}.[S_0]^{-1}.|\mathbf{x}>$$
$$=$$
$$-2.\{Id_3 - \frac{3}{2.r^2}.\phi\}.|\mathbf{x}>$$

But since:

$$\phi.|\mathbf{x}> = r^2.|\mathbf{x}>$$

One gets the important coincidence:

$$|{}_{\mathbf{\Lambda}}\mathbf{s}> = -2.|\mathbf{x}> + 3.|\mathbf{x}> = |\mathbf{x}>$$

This result allows ending the calculation of |N|:

$$-|P| = d = |N| = \frac{2.(\frac{G.m}{r^3})^3}{8} + \frac{1}{2.|A|^2} . < \mathbf{x}, \mathbf{x} >_{[S_0]}$$

Here the Hessian $[S_0]$ is not coinciding with the identity matrix $[Id_3]$. Therefore, the second term must be calculated separately:

$$< \mathbf{x}, \mathbf{x} >_{[S_0]}$$

$$=$$

$$< \mathbf{x}, \mathbf{x} >_{-\frac{G.m}{r^3} . \{Id_3 - \frac{3}{r^2} . \phi\}}$$

$$=$$

$$-\frac{G.m}{r^3} . < \mathbf{x}| . \{\{Id_3 - \frac{3}{r^2} . \phi\} . |\mathbf{x} >\}$$

$$=$$

$$-\frac{G.m}{r^3} . -2 . r^2$$

$$=$$

$$\frac{2.G.m}{r}$$

All these calculations are giving:

$$|N| = \frac{1}{|A|^2} . \frac{G.m}{r} + \frac{1}{4} . (\frac{G . m}{r^3})^3$$

Within the context given at the beginning of this exploration, any deformed cross product $[d\mathbf{x}, \mathbf{...}]_{[A]}$ can be decomposed and the determinant of the main part of a decomposition seems to be a modified expression of the Newtonian potential:

$$|P| = -\frac{1}{|A|^2} . \frac{G.m}{r} - \frac{1}{4} . \frac{G^3 . m^3}{r^9}$$

The prerequisites for the validation of the condition:

$$\exists f : \Lambda(d\mathbf{x}) = df(\mathbf{x})$$

... are realized when:

$$d = |N| = \frac{1}{|A|^2} . \frac{G.m}{r} + \frac{1}{4} . \frac{G^3 . m^3}{r^9} = 0(3)$$

...

3 Demonstration - part 2

3.1 Conventions

Up to now, the prerequisites which have been exposed at the beginning of this document are supposed to be realized. Concretely: the discussion concerns a $1/\mathrm{r}^2$ field, the condition $\Lambda(\mathrm{d}\mathbf{x}) = \mathrm{df}(\mathbf{x})$ is a reality, the Hessian of Λ is not degenerated and its singular vector coincides with a spatial position. Therefore, the main part of the decomposition of any deformed cross product $[\mathrm{d}\mathbf{x}, \mathbf{...}]_{[A]}$ can be written:

$$[P]_{|A|}$$
$$=$$
$$\frac{1}{|A|} . \{[A]^t . [J]\} . \{\frac{1}{2} . [Hess_{(d\mathbf{x},0)}\Lambda(d\mathbf{x})] + \frac{1}{|A|} . {}_{[J]}\Phi([Hess_{(d\mathbf{x},0)}\Lambda(d\mathbf{x})]^{-1} . |\mathbf{d}^* >)\}$$
$$=$$
$$\frac{1}{|A|} . \{[A]^t . [J]\} . \{\frac{1}{2} . [S_0] \; + \; \frac{1}{|A|} . {}_{[J]}\Phi(\underbrace{[S_0]^{-1} . |\mathbf{d}^* >}_{=-_\Lambda \mathbf{s}}\}$$
$$=$$
$$\frac{1}{|A|} . \{[A]^t . [J]\} . \{-\frac{G . m}{2 . r^3} . \{Id_3 \; - \; \frac{3}{r^2} . \phi\} \; - \; \frac{1}{|A|} . {}_{[J]}\Phi(\mathbf{x})\}$$

At this stage, an important result which has been obtained in [b; theorem 3.2, p.20] must be recalled. More precisely one must reintroduce the matrix [T] and write:

$$[A]^t . [J] \; = \; [T]^{-1}, \; [A] \; \in \; GL_3(\mathbb{C})$$

The matrix [P] can now be written as:

$${}_{|A|}p_{ij} \; = \; \frac{1}{|A|} . T_{ik} . \{\frac{G . m}{2 . r^5} . \underbrace{\{3 . x^k . x^j \; - \; r^2 . \delta^k_j\}}_{tetra-polar\; momentum\; tensor} \; - \frac{1}{|A|} . \underbrace{\epsilon_{kjm} . x^m}_{axial\; rotation} \}$$

3.2 The link with the Bowen-York solutions

Starting from the matrix [P], let now build the following vector components:

$$\frac{r^2}{6 . G . m} . {}_{|A|}p_{ij} . P_{j-LYB}$$
$$=$$
$$\frac{1}{|A|} . \frac{r^2}{6 . G . m} . T_{ik} . \{-\frac{G . m}{2.r^3} . \{\delta^k_j \; - \; \frac{3}{r^2} . x^k . x^j\} \; - \; \frac{1}{|A|} . \epsilon_{kjm} . x^m\} . P_{j-LYB}$$
$$=$$
$$-\frac{1}{12 . |A| . r} . T_{ik} . \{\delta^k_j - \frac{3}{r^2} . x^k . x^j\} . P_{j-LYB} - \frac{1}{|A|} . \frac{r^2}{6 . G . m} . T_{ik} . \epsilon_{kjm} . x^m . P_{j-LYB}$$
$$=$$

$$-\frac{1}{12\,.\,|A|\,.\,r}\,.\,T_{ij}\,.\,P_{j-LYB}$$
$$+\frac{1}{4\,.\,|A|\,.\,r^3}\,.\,T_{ik}\,.\,x^k\,.\,x^j\,.\,P_{j-LYB}$$
$$-\frac{1}{|A|}\cdot\frac{r^2}{6\,.\,G\,.\,m}\,.\,T_{ik}\,.\,\epsilon_{kjm}\,.\,x^m\,.\,P_{j-LYB}$$

Let now consider the solutions resulting from the Bowen-York initial data as for example exposed in [03; (b) §8.2.6, p.136, (8.69)], [03; (c) p. 23, (69)] and in [08; p.3, (28) and (29)]:

$$X^i_{LYB} = -\frac{1}{4\,.\,r}\,.\,(7\,.\,f^{ij}\,.\,P_{j-LYB} + \frac{P_{j-LYB}\,.\,x^j\,.\,x^i}{r^2}) - \frac{1}{r^3}\,.\,\epsilon^{ij}{}_k\,.\,S_{j-LYB}\,.\,x^k$$

The second point of proposition 1.1 is true:

$$X^i_{LYB} = \frac{r^2}{6\,.\,G\,.\,m}\cdot{}_{|A|}p_{ij}\,.\,P_{j-LYB}$$

... if and only if one can write simultaneously:

$$-\frac{1}{12\,.\,|A|\,.\,r}\,.\,T_{ij}\,.\,P_{j-LYB} = -\frac{7}{4\,.\,r}\,.\,f^{ij}\,.\,P_{j-LYB}$$
$$\frac{1}{4\,.\,|A|\,.\,r^3}\,.\,T_{ik}\,.\,x^k\,.\,x^j\,.\,P_{j-LYB} = -\frac{1}{4\,.\,r^3}\,.\,P_{j-LYB}\,.\,x^j\,.\,x^i$$
$$-\frac{1}{|A|}\cdot\frac{r^2}{6\,.\,G\,.\,m}\,.\,T_{ik}\,.\,\epsilon_{kjm}\,.\,x^m\,.\,P_{j-LYB} = -\frac{1}{r^3}\,.\,\epsilon^{ij}{}_k\,.\,S_{j-LYB}\,.\,x^k$$

Let reorganize these identifications and get for each i = 1, 2, 3:

$$\frac{1}{|A|}\,.\,T_{ij}\,.\,P_{j-LYB} = 21\,.\,f^{ij}\,.\,P_{j-LYB}$$
$$\frac{1}{|A|}\,.\,T_{ik}\,.\,x^k\,.\,x^j\,.\,P_{j-LYB} = -P_{j-LYB}\,.\,x^j\,.\,x^i$$
$$\frac{1}{|A|}\,.\,T_{ik}\,.\,\epsilon_{kjm}\,.\,x^m\,.\,P_{j-LYB} = \frac{6\,.\,G\,.\,m}{r^5}\,.\,\epsilon^{ij}{}_l\,.\,S_{j-LYB}\,.\,x^l$$

Are the proposed identifications coherent?

3.3 The admissible metrics

Let examine the relation:

$$\frac{1}{|A|}\,.\,T_{ij}\,.\,P_{j-LYB} = 21\,.\,f^{ij}\,.\,P_{j-LYB}$$

The f^{ij} represent the entries of the inverse of some conformally flat metric which has been introduced by Bowen and York. Metrics are tensors usually represented through matrices. The entries T_{ij} are those of a non-degenerated matrix $[T]^{-1}$ which has been introduced in [b] (recall):

$$[T_{ij}] = [T]^{-1} = [A]^t . [J]$$

Hence, for any non-vanishing kinetic momentum, the first relation resulting from the proposed identification can be rewritten as:

$$\frac{1}{|A|} . [T]^{-1} = 21 . [f]^{-1}$$

The effective deforming matrix coincides with the inverse of the Bowen-York flat and conformal metric.

One remarks that if [A] = [J], then |A| = -1:

$$[f]^{-1} = -\frac{1}{21} . Id_3$$

3.4 The admissible deforming matrices

The second relation can first be rewritten as:

$$(\sum_k T_{ik} . x^k) . (\sum_j x^j . P_{j-LYB}) = -|A| . (\sum_j P_{j-LYB} . x^j) . x^i$$

And then as:

$$(\sum_k T_{ik} . x^k + |A| . x^i) . (\sum_j P_{j-LYB} . x^j) = 0$$

At this stage, one must:

- recall the obligatory condition $\mathbf{x} \neq \mathbf{0}$ within the Bowen-York approach;
- state that if the kinetic momentum does not vanish, this relation can be true when:

$$< \mathbf{P}^*_{LYB}, \mathbf{x} >_{Id_3} = 0$$

- state that if:

$$< \mathbf{P}^*_{LYB}, \mathbf{x} >_{Id_3} \neq 0$$

  Then the second relation is reduced to:

$$\sum_k T_{ik} . x^k + |A| . x^i = 0$$

  It is equivalent to:

$$\{[T]^{-1} + |A| . Id_3\} . |\mathbf{x} >= \mathbf{0} >$$

  It can only be true when:

$$|[T]^{-1} + |A| . Id_3| = 0$$

  On remarks that this relation is trivially verified when [A] = [J] and |A| = -1 because in that case $[T]^{-1}$ = Id_3:

$$|[T]^{-1} - Id_3| = |Id_3 - Id_3| = 0$$

3.5 The link between the kinetic momentum and the angular momentum

Let now observe the third relation resulting from the identification. Note that it involves the Levi-Civita alternating tensor associated with the flat metric [f] and recall the link existing between the effective deforming matrix and the flat metric. After some reorganizations one gets:

$$f^{ik} \,.\, \epsilon_{kqm} \,.\, x^m \,.\, P_{q-LYB} \;=\; \frac{2\,.\,|A|\,.\,G\,.\,m}{7\,.\,r^5} \,.\, \epsilon_{kqm} \,.\, f^{ik} \,.\, f^{jq} \,.\, S_{j-LYB} \,.\, x^m$$

This relation is equivalent to:

$$f^{ik} \,.\, \epsilon_{kqm} \,.\, x^m \,.\, (P_{q-LYB} \;-\; \frac{2\,.\,|A|\,.\,G\,.\,m}{7\,.\,r^5} \,.\, f^{jq} \,.\, S_{j-LYB}) \;=\; 0$$

... which may also be written as:

$$[f]^{-1} \,.\, |\mathbf{x} \wedge (\mathbf{P}^*_{LYB} \;-\; \frac{2\,.\,|A|\,.\,G\,.\,m}{7\,.\,r^5} \,.\, [f]^{-1} \,.\, |\mathbf{S}^*_{LYB} >) >= \, |\mathbf{0} >$$

Since the effective deforming matrix which has been introduced in [b] is not degenerated, this relation is equivalent to:

$$\mathbf{x} \wedge (\mathbf{P}^*_{LYB} \;-\; \frac{2\,.\,|A|\,.\,G\,.\,m}{7\,.\,r^5} \,.\, [f]^{-1} \,.\, |\mathbf{S}^*_{LYB} >) \;=\; \mathbf{0}$$

Once again, since $\mathbf{x} \neq \mathbf{0}$ within the Bowen-York approach, this condition is equivalent to:

$$\exists\, \lambda \neq 0 : \mathbf{P}^*_{LYB} \;-\; \frac{2\,.\,|A|\,.\,G\,.\,m}{7\,.\,r^5} \,.\, [f]^{-1} \,.\, |\mathbf{S}^*_{LYB} >= \lambda \,.\, |\mathbf{x} > \neq \, |\mathbf{0} >$$

In our classical three-dimensional Euclidean world, the cross product is characterized by the matrix [A] = [J], |A| = -1 and $[f]^{-1}$ = -(1/21).Id_3; in that case, the third relation writes:

$$\mathbf{P}^*_{LYB} \;-\; \frac{2\,.\,G\,.\,m}{147\,.\,r^5} \,.\, \mathbf{S}^*_{LYB} \;=\; \lambda \,.\, \mathbf{x}$$

The formalism of this relation obviously corresponds to a degenerated situation because, in a classical three-dimensional Euclidean geometry, the vector $\mathbf{S}^*$ is orthogonal to the position and to the kinetic momentum... it does not in the plane which is generated by these vectors!

3.6 Conclusions concerning the demonstrations

At this stage, one may conclude that the second part of proposition 1.1 is true in general but that the Euclidean case is a problematic limit case. At the end of the day, it has been proved that a whole family of deformed cross products, precisely the $[d\mathbf{x}, \ldots]_{[A]}$, allow a formal recovery of solutions for the Einstein's equations when one works with the L-Y-B initial data and the prerequisites of proposition 1.1.

4 Discussion

4.1 Comments

Let analyze the way of thinking which has been followed until now:

$$|[d\mathbf{x}, ...]_{[A]} >= [P] \,.\, |\mathbf{...}> + |\mathbf{z}>, \; conditions\, of\, proposition\, 1.1\, plus\, [b]$$

$$\Downarrow$$

$$|^{(3)}\mathbf{X}_{LYB}> = \frac{r^2}{6 \,.\, G \,.\, m} \,.\, ^{(3)}[P]_{|A|} \,.\, |^{(3)}\mathbf{P}^*_{LYB}>$$

With:

$$[P]_{|A|}$$

$$=$$

$$-\frac{1}{|A|} \,.\, \{[A]^t \,.\, [J]\} \,.\, \{\frac{1}{|A|} \,.\, _{[J]}\Phi(\mathbf{x}) + \frac{G \,.\, m}{2 \,.\, r^3} \,.\, \{Id_3 - \frac{3}{r^2} \,.\, \phi\}\}$$

This result is strongly intrinsic method dependent.

4.2 Analyzing the polynomial Λ further

Let write the polynomial Λ when the prerequisites of proposition 1.1 are realized:

$$\Lambda(d\mathbf{x})$$

$$=$$

$$\Lambda(dx^1, \, dx^2, \, dx^3)$$

$$=$$

$$|_{[A]}\Phi(d\mathbf{x}) - [P]|$$

$$=$$

$$\sum_{ij} d_{ij} \,.\, dx^i \,.\, dx^j + \sum_i d_i \,.\, dx^i - |P|$$

$$=$$

$$-\frac{G \,.\, m}{2 \,.\, r^3} \,.\, \sum_{ij} (\delta^i_j - \frac{3}{r^2} \,.\, x^i \,.\, x^j) \,.\, dx^i \,.\, dx^j$$

$$-\frac{G \,.\, m}{r^3} \,.\, \sum_i x^i \,.\, dx^i$$

$$+ \underbrace{\frac{G.m}{r} + \frac{1}{4} \,.\, \frac{G^3 \,.\, m^3}{r^9}}_{=0(3),\, due\, to\, the\, prerequisites}$$

This polynomial can be reformulated as:

$$\Lambda(d\mathbf{x})$$

$$=$$

$$\frac{3\,.\,G\,.\,m}{2\,.\,r^5}\,.\,\sum_{ij} dx^i\,.\,dx^j\,.\,x^i\,.\,x^j \;-\; \frac{G\,.\,m}{r^3}\,.\,\sum_{i} dx^i\,.\,x^i - \frac{G\,.\,m}{2\,.\,r^3}\,.\,d(r^2)$$

Therefore, it is also a polynomial form of degree two depending on the position $\mathbf{x}$. Due to the initial theorem [b], this formulation allows to interpret this polynomial as a proof for the existence of deformed cross product in some $[\mathbf{x},\ldots]_{[-]}$ family. Since this polynomial Λ is related to the decomposition of deformed cross products in the $[\mathrm{d}\mathbf{x},\ldots]_{[A]}$ family, this exploration is presumably studying the decomposition of any deformed angular momentum $[\mathbf{x},\mathrm{d}\mathbf{x}]_{[A]}$.

Remark: This affirmation is more revolutionary than the simplicity of its formulation suggests because, in quantum physics, an angular momentum is not deformed or modified, it is only quantified! This crucial topic will be deepen below.

4.3 Confronting the intrinsic method and the extrinsic method

Let consider the deformed cross product $[\mathrm{d}\mathbf{x},\,\mathbf{x}]_{[A]}$ and let decompose it successively with the intrinsic method and then with the extrinsic method. At a first glance, the main part of a generic decomposition depends on the method because:

$$[P_{intrinsic}]_{|A|}$$

$$=$$

$$-\frac{1}{|A|}\,.\,[T]^{-1}\,.\,\{\frac{1}{|A|}\,.\,{}_{[J]}\Phi(\mathbf{x}) \;+\; \frac{G\,.\,m}{2\,.\,r^3}\,.\,\{Id_3 \;-\; \frac{3}{r^2}\,.\,\phi\}\}$$

... whilst, if [B] represents the non-degenerated bi-linear form involved in the extrinsic method:

$$[P_{extrinsic}] \;=\; {}_{[A]}\Phi(d\mathbf{x}) \;-\; \frac{1}{2}\,.\,[B]^{-1}\,.\,[Hess_{(\mathbf{x},0)}P_2(\mathbf{x})]$$

Let ask if it is possible to write:

$$[P_{extrinsic}] \;=\; [P_{intrinsic}]\,?$$

Proposition 4.1. *The intrinsic method [b] and the extrinsic method [c] furnish the same results when a small set of three relations is valid.*

Proof. Let observe both expressions attentively:

1. In a first step let write:

$$\frac{1}{|A|} \cdot [T]^{-1} = [B]^{-1}$$

The non-degenerated bi-linear form which has been involved in the extrinsic method is now proportional to the effective deforming matrix.

2. Let then write:

$$[Hess_{(\mathbf{x},0)} P_2(\mathbf{x})] = \frac{G \,.\, m}{r^3} \cdot \{Id_3 - \frac{3}{r^2} \cdot \phi\}$$

It is easy to recognize that this identification is equivalent to:

$$[Hess_{(\mathbf{x},0)} P_2(\mathbf{x})] = -[S_0]$$

Since here:

$$[S_0] = [Hess_{(d\mathbf{x},0)} \Lambda(d\mathbf{x})] = Hess_{(\mathbf{x},0)} f(\mathbf{x})$$

One states that:

$$[Hess_{(\mathbf{x},0)} P_2(\mathbf{x})] = -[Hess_{(\mathbf{x},0)} f(\mathbf{x})]$$

The polynomial $P_2(\mathbf{x})$ coincides with $f(\mathbf{x})$ up to a minus sign.

3. The third necessary relation is:

$$_{[A]}\Phi(d\mathbf{x}) = -\frac{1}{|A|^2} \cdot [T]^{-1} \cdot {}_{[J]}\Phi(\mathbf{x})$$

Since:

- (recall):

$$[T]^{-1} = [A]^t \cdot [J]$$

- the lecture of [b] brings the information:

$$_{[A]}\Phi(\mathbf{x}) = [T]^{-1} \cdot {}_{[J]}\Phi(\mathbf{x})$$
$$_{[A]}\Phi(d\mathbf{x}) = [T]^{-1} \cdot {}_{[J]}\Phi(d\mathbf{x})$$

... one must write:

$$[T]^{-1} \cdot {}_{[J]}\Phi(d\mathbf{x} + \frac{1}{|A|^2} \cdot \mathbf{x}) =^{(3)} [0]$$

The general formalism of this third relation is relating a spatial position to a difference of spatial positions in a way depending on the effective deforming matrix at hand.

The Euclidean case ([A] = [J], |A| = -1 and $[T]^{-1} = Id_3$) represents a limit case characterized by the decreasing exponential relation:

$$d\mathbf{x} = -\mathbf{x}$$

□

4.4 Intermezzo

A systematic exploration of the properties of deformed cross products and of their decomposition is a rich branch of mathematics. For example, it can be proved that any $[\mathbf{a}, ...]_{[A]}$ is a derivation acting on the elements of $L_{[A]} = \{V = \mathbb{C} \otimes E(3, \mathbb{R}), [..., ...]_{[A]}\}$ when $L_{[A]}$ is a Lie algebra. The conceptual difficulty accompanying this affirmation is due to the fact that this specific type of derivations is done by respect for the pair ([A], $\mathbf{a}$) and no more by respect for a scalar or for a vector.

For now, the theory studying the deformation and the decomposition of classical cross products is able to incorporate an elastic three-dimensional geometry in defining the interplay between that changing geometry and the diverse deformations of the cross products. It also has revealed a mathematical coincidence with the Bowen York solutions for the theory of relativity when the necessary prerequisites are realized.

For the completeness of this approach, it must be remarked that the residual part of each non-trivial decomposition has not been studied. Its expression can be calculated with the help of the extrinsic method.

4.5 Angular momentum: deformation or quantification?

As mentioned earlier in this document: *...in quantum physics, an angular momentum is not deformed or modified, it is only quantified!* This unavoidable fact imposes the search for a plausible explanation justifying this unlikely concept of deformed angular momentum.

For that purpose, let consider the subset of deforming matrices [A] with a determinant equal to minus one (|A| = -1) and let then consider the main part of a deformed angular momentum which has been obtained during the above demonstration:

$$[P_{intrinsic}]_{(|A|=-1)}$$

$$=$$

$$[T]^{-1} . \{-_{[J]}\Phi(\mathbf{x}) + \frac{G . m}{2 . r^3} . \{Id_3 - \frac{3}{r^2} . \phi\}\}$$

In that case:

$$|[d\mathbf{x}, \mathbf{x}]_{[A]} >$$

$$=$$

$$[T]^{-1} . \{-_{[J]}\Phi(\mathbf{x}) + \frac{G . m}{2 . r^3} . \{Id_3 - \frac{3}{r^2} . \phi\}\} . |\mathbf{x} > + |\mathbf{z} >$$

For the pedagogy only, let suppose that the third relation insuring the coincidence between both methods (intrinsic and extrinsic) holds true. Then, one can write:

$$|[d\mathbf{x}, \mathbf{x}]_{[A]} >$$

$$=$$

$$-[T]^{-1}\,.\,_{[J]}\Phi(\mathbf{x})\,.\,|\mathbf{x}> + \frac{G\,.\,m}{2\,.\,r^3}\,.\,[T]^{-1}\,.\,\{Id_3 - \frac{3}{r^2}\,.\,\phi\}\,.\,|\mathbf{x}> + |\mathbf{z}>$$

$$=$$

$$_{[J]}\Phi(d\mathbf{x})\,.\,|\mathbf{x}> + \frac{G\,.\,m}{2\,.\,r^3}\,.\,[T]^{-1}\,.\,\{Id_3 - \frac{3}{r^2}\,.\,\phi\}\,.\,|\mathbf{x}> + |\mathbf{z}>$$

$$=$$

$$_{[A]}\Phi(d\mathbf{x})\,.\,|\mathbf{x}> - \frac{G\,.\,m}{r^3}\,.\,[T]^{-1}\,.\,|\mathbf{x}> + |\mathbf{z}>$$

There exists a difference between this generic result and the most simple decomposition without residual part:

$$|[d\mathbf{x},\,\mathbf{x}]_{[A]}> = {}_{[A]}\Phi(d\mathbf{x})\,.\,|\mathbf{x}>$$

The difference writes:

$$-\frac{G\,.\,m}{r^3}\,.\,[T]^{-1}\,.\,|\mathbf{x}> + |\mathbf{z}>$$

Let now specialize for a while to the case [A] = [J] and get:

$$|d\mathbf{x} \wedge \mathbf{x}>$$

$$=$$

$$_{[J]}\Phi(d\mathbf{x}) + \frac{G\,.\,m}{2\,.\,r^3}\,.\,\{Id_3 - \frac{3}{r^2}\,.\,\phi\}\,.\,|\mathbf{x}> + |\mathbf{z}>$$

There is a difference between this result and the most simple decomposition without residual part:

$$|d\mathbf{x} \wedge \mathbf{x}> = {}_{[J]}\Phi(d\mathbf{x})\,.\,|\mathbf{x}>$$

The difference writes:

$$\frac{G\,.\,m}{2\,.\,r^3}\,.\,\{Id_3 - \frac{3}{r^2}\,.\,\phi\}\,.\,|\mathbf{x}> + |\mathbf{z}> = -\frac{G\,.\,m}{r^3}\,.\,|\mathbf{x}> + |\mathbf{z}>$$

The important point here is that even a classical angular momentum does not systematically have a simple decomposition without residual part within a three-dimensional Euclidean context. At a first glance, this is a surprising fact.

In examining the formalism of the first part of the difference, one states that it looks like a very classical Newtonian acceleration ... although its origin seemed to be a tetra-polar polarization!

Anyway, looking for an explanation for the existence of this difference is the first priority. A long time ago, the physics was facing a problem concerning the preservation of the angular momentum in presence of an electron surrounding the atom. It turned out that one had to consider the sum of two sub-angular momentum: (i) one for the motion of the electron around the atom (in some way: the classical one) and (ii) another one for the intrinsic spin of that electron: the effective specificity in quantum physics. This led to writing [12; p. 15, (42)]:

$$\underbrace{\hbar \, . \, \mathbf{J}}_{total\,angular\,momentum} = \underbrace{\mathbf{L}}_{(i)} + \underbrace{\frac{1}{2} \, . \, \hbar \, \sigma}_{(ii)}$$

This usual discussion concerning the angular momentum in quantum physics [12; p. 15 - German language] suggests the answer for the problematic one is studying here. For example, one may propose:

$$\underbrace{|d\mathbf{x} \times \mathbf{x} >}_{total\,angular\,momentum}$$
$$=$$
$$\underbrace{{}_{[J]}\Phi(d\mathbf{x})}_{usual\,angular\,momentum} + \underbrace{\frac{G \, . \, m}{2 \, . \, r^3} \, . \, \{Id_3 - \frac{3}{r^2} \, . \, \phi\} \, . \, |\mathbf{x} > + |\mathbf{z} >}_{angular\,momentum\,related\,to\,a\,quantified\,spin}$$

4.6 The asymptotic flatness boundary condition on X

As mentioned in [03(b); p.138], the Bowen-York solutions **X** do not in general vanish. But the asymptotic flatness boundary condition on **X** [03(b); p.132, (8.44)] remains:

$$Lim_{r \to \infty} \mathbf{X}_{LYB} = 0$$

One must verify if the Bowen-York solutions respect this boundary condition when they are written with the formalism which is obtained within the theory of the (E) question. For this purpose, let calculate:

$$Lim_{r \to \infty} \frac{r^2}{6 \, . \, G \, . \, m} \, . \, {}^{(3)}[P]_{|A|} \, . \, |^{(3)}\mathbf{P}^*_{LYB} >$$
$$=$$
$$Lim_{r \to \infty} - \frac{r^2}{6 \, . \, |A| \, . \, G \, . \, m} \, . \, [T]^{-1} \, . \, \{ \frac{1}{|A|} \, . \, {}_{[J]}\Phi(\mathbf{x}) + \frac{G \, . \, m}{2 \, . \, r^3} \, . \, \{Id_3 - \frac{3}{r^2} \, . \, \phi\}\} \, . \, |^{(3)}\mathbf{P}^*_{LYB} >$$

At infinity, one expects to find a flat Minkowski geometry of which the spatial part strongly mimics the three-dimensional Euclidean one. With different words, one is expecting that [A] = [J], |A| = -1 and $[T]^{-1}$ = Id_3 = -21.$[f]^{-1}$; hence:

$$Lim_{r \to \infty} \frac{r^2}{6 \, . \, G \, . \, m} \, . \, {}^{(3)}[P]_{|A|=-1} \, . \, |^{(3)}\mathbf{P}^*_{LYB} >$$
$$=$$
$$Lim_{r \to \infty} - \frac{r^2}{126 \, . \, G \, . \, m} \, . \, {}_{[J]}\Phi(\mathbf{x}^*) \, . \, |^{(3)}\mathbf{P}^*_{LYB} > + \frac{1}{12 \, . \, r} \, . \, \{Id_3 - \frac{3}{r^2} \, . \, \phi\} \, . \, |^{(3)}\mathbf{P}^*_{LYB} >$$

To go further, one must recall the condition of coherence linking the kinetic momentum and the angular momentum within these circumstances:

$$\mathbf{P}^*_{LYB} - \frac{2 \, . \, G \, . \, m}{147 \, . \, r^5} \, . \, \mathbf{S}^*_{LYB} = \lambda \, . \, \mathbf{x} = -\frac{\lambda}{21} \, . \, \mathbf{x}^*$$

Let now consider the two parts of the limit separately:

- One gets for the first part:

$$Lim_{r \to \infty} - \frac{r^2}{126\,.\,G\,.\,m} \cdot {}_{[J]}\Phi(\mathbf{x}^*)\,.\,|^{(3)}\mathbf{P}^*_{LYB} >$$

$$=$$

$$Lim_{r \to \infty} - \frac{1}{9261\,.\,r^3}\,.\,(\mathbf{x}^* \wedge {}^{(3)}\mathbf{S}^*_{LYB})$$

- One also gets for the second part:

$$Lim_{r \to \infty} \frac{1}{12\,.\,r}\,.\,\{Id_3 - \frac{3}{r^2}\,.\,\phi\}\,.\,|^{(3)}\mathbf{P}^*_{LYB} >$$

$$=$$

$$Lim_{r \to \infty} \frac{1}{12\,.\,r}\,.\,\{Id_3 - \frac{3}{r^2}\,.\,\phi\}\,.\,| - \frac{\lambda}{21}\,.\,\mathbf{x}^* + \frac{2\,.\,G\,.\,m}{147\,.\,r^5}\,.\,\mathbf{S}^*_{LYB} >$$

$$=$$

$$Lim_{r \to \infty}\{(-\frac{\lambda}{126\,.\,r} - \frac{6\,.\,G\,.\,m}{147,\,.\,r^7}\,.\, < \mathbf{x}^*,\, \mathbf{S}^*_{LYB} >_{Id_3})\,.\,\mathbf{x}^* + \frac{G\,.\,m}{882\,.\,r^6}\,.\,\mathbf{S}^*_{LYB}\}$$

The intensity (Euclidean norm) of the angular momentum and its eventual dependence by respect for the distance r play here a crucial role. If one believes that this intensity decreases with r, then - yes- the boundary condition is valid.

4.7 The asymptotic flatness boundary condition on Ψ

Another consequence of the non-systematic vanishing of the Bowen-York solutions is that the Hamiltonian constraint associated with the Einstein's equations for the scalar field Ψ (figuring in Lichnerowicz equation and being obliged to respect the asymptotic flatness boundary condition $\Psi = 1$ when r $\to \infty$) is no longer a simple Laplace equation [03(b); p.132, (8.43)].

This important equation writes in general [03(b); p.98, (6.103)]:

$$\underbrace{D_i D^i}_{=\Delta} \Psi - \frac{1}{8}\,.\,R\,.\,\Psi + \frac{1}{8}\,.\,(A_{ij}\,.\,A^{ij})\,.\,\Psi^{-7} + (2\,.\,\pi\,.\,E - \frac{1}{12}\,.\,K^2)\,.\,\Psi^5 = 0$$

The definition of the Laplace operator is given in [03(b); p.101, (6.132)]. The Bowen-York initial data [03(b); p.136, (8.67)] are:

$$R = 0,\ A^{ij}_{TT} = 0,\ E = 0,\ K = 0\, and\, \mathbf{P}_{LYB} = \mathbf{0}$$

They imply that the Lichnerowicz equation is a simple Laplace equation at t = 0:

$$\Delta\Psi(t = 0) = 0$$

The existence of non-trivial solutions imposes to reconsider this equation at t > 0. One cannot really predict the evolution, therefore it is perhaps pertinent to consider the generic formalism of Lichnerowicz equation. Assuming that the scalar field never vanish, one may rewrite the Lichnerowicz equation as:

$$(\frac{\Delta\Psi}{\Psi} - \frac{1}{8}\,.\,R)\,.\,\Psi^8 + \frac{1}{8}\,.\,(A_{ij}\,.\,A^{ij}) + (2\,.\,\pi\,.\,E - \frac{1}{12}\,.\,K^2)\,.\,\Psi^{12} = 0$$

At this stage, one may apply a unique constraint on the evolution and suppose that:

$$\forall t\,:\,\frac{\Delta\Psi}{\Psi} << \frac{1}{8}\,.\,R$$

This is transforming it into a polynomial of degree three after the change of variable:

$$Z = \Psi^4 \neq 0$$

Indeed, its new formulation is now:

$$(2\,.\,\pi\,.\,E - \frac{1}{12}\,.\,K^2)\,.\,Z^3 - \frac{1}{8}\,.\,R\,.\,Z^2 + \frac{1}{8}\,.\,(A_{ij}\,.\,A^{ij}) = 0$$

The coefficient of degree zero can be calculated because the Bowen-York extrinsic curvature is known [03(b); p.137, (8.71)]. The solutions of this polynomial can now be obtained with the old Tartaglia-Cardan [04; pp. 81-82] method and one must verify that:

$$Lim_{r\rightarrow\infty}\Psi = 1$$

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