Combined theory of Special Relativity and Quantum Mechanics

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Abstract

Lorentz transformation plays a key role in Special Relativity by relating the spacetime distance between events being observed in a pair of inertial frames of reference. Depending on the relative velocity of the inertial frames, the magnitude of Lorentz transformation varies between the limits 0 and 1. The upper limit 1 represents a case where the pair of inertial frames of reference are stationary *relative* to each other. The lower limit 0 represents the other extreme case where the relative velocity of the frames is at the speed of light c. Similar numerical limits, on the other hand, appear in Quantum Mechanics but in the context of the summation of the probability density distribution of a particle over a region of space. The upper limit 1 represents a case where the probability of finding a particle in a region of space is certain. The lower limit, represents the opposite case where the probability of finding a particle in a region of space is not likely. The range of the limits being between 0 and 1 in both theories is not a numerical coincidence. In this paper, a combined theory is introduced which relates the Lorentz transformation of Special Relativity to the wavefunction of Quantum Mechanics. The combined theory offers a new insight to the physical reality. For instance, it is found that the inherent quantum uncertainties in the spacetime coordinate of a quantum particle in vacuum constitute a timelike four-vector whose length A is invariant. It is also found that *local acceleration*, like *velocity* itself, has an upper limit; such that no physical object can undergo a local acceleration higher than c^2/A. The latter, in turn, constrains the mass of the smallest possible black hole - called Unit Black Hole (UBH) - to $Ac^2/4G$ and its event horizon diameter to the invariant A. The diameter of the event horizon, the mass and the Hawking temperature of more massive black holes are subsequently quantized starting from those of the UBH.

Keywords — Special relativity, Quantum mechanics, Black hole, Acceleration limit

1 Introduction

Quantum Mechanics (QM) and Special Relativity (SR) are the two most triumphant theories in physics. The theories of SR and QM, however, have a fundamentally incompatible approach in describing the physical reality. For instance, while QM provides an inherently *probabilistic* and *quantized* picture of the natural world, SR is based on the *deterministic* and *continuous* description of the physical particles position in space and time. In this paper an attempt is made to combine these theories through establishing a relationship between the *time dilations* of SR and the *spatial* *probability density distributions* of QM. The time dilation of SR is associated with the kinematics of a particle; such that the faster the velocity of a particle relative to an inertial frame of reference, the higher its coordinate time dilation in that frame. More specifically, the time dilation of SR *relates the proper time interval in the rest frame of a particle to that of any inertial frame in which the particle is found to be in motion.*

Since SR is a deterministic and continuous theory, the definition of the *rest frame* in which both the position and momentum of a particle are known with certainty is compatible with the basic assumptions of the theory. Unlike SR, in the theory of QM, the rest condition is not permitted in its absolute sense. Therefore, in the combined SR-QM theory, the notion of *rest* frame of SR is revised into *a frame in which a quantum particle has a minimum inherent uncertainty in its position and momentum.* The quantum particle in such frame is at the nearest state to being stationary, as such the classical rest frame in the combined theory is called the *nearest frame*; or equivalently textitnear-rest frame. The higher the relative velocity between the nearest frame and a given inertial frame of reference, the higher the spacetime coordinate uncertainties of the particle in that frame.

The spatial probability density distribution of QM quantifies the probability of presence of a quantum particle over a span of space in which the distribution assumes a non-zero value. According to the revised definition given before, the probability density distribution in the nearest frame of a quantum particle is therefore expected to have the shortest span. In any other given inertial frame of reference, while the probability density distribution still integrates to the probability of 1, the distribution, nonetheless, covers a wider length interval compared to that of the nearest frame.

In this paper, we begin with the abstract motion of a quantum particle in vacuum and show that the seemingly irreconcilable theories of SR and QM are fundamentally interconnected through a relationship between the coordinate time dilation and the spatial probability density distribution. From the combined SR-QM theory, we then learn that the uncertainties in the relativistic spacetime coordinates of a quantum particle constitute a time-like four-vector whose length interval is invariant. The latter leads to the invariant cosmological time interval which the age of universe is always an integer multiple of. This invariant time interval plus the invariancy of speed of light necessitates an upper limit in the local acceleration. The necessity arises from the realization that if a particle could accelerate any higher than the acceleration limit it would have attained a velocity higher than that of a light beam. The local acceleration limit, then allows to define the smallest black hole in nature and subsequently arrive at the quantized equations for the mass, size and temperature of black holes.

2 Lorentz transformation

Consider a quantum particle with mass m and the nearest frame $I(x,t)$, moving undisturbed in empty space on a rectilinear path under a constant momentum p'. From *undisturbed* we mean the particle trajectory or momentum is not influenced by any external factors, such as gravity, radiation or interaction with other particles. The particle is assumed to be in a non-stationary state along the x'-axis of another inertial frame of reference, $I'(x',t')$. To limit ourselves to one a dimensional analysis, we further assume the spatial axes x and x' of the frames I and I' are parallel. From Special Relativity, the spacetime coordinates of an *event* registered in these two inertial frames of reference, I and I', are related to each other through the Lorentz transformation [1]. One such event is when the origins of the frames of reference I and I' are both *simultaneous* and *coincident.* As shown in the Lorentz diagram, Fig 1, the spacetime coordinate of this event is marked by the event E_0 where the spatial and time coordinates of both frames are set at zero, hence, meeting the simultaneity and coincidence condition of the event. Another event is when the particle might be at a distance x' from the origin of I', when the coordinate clock in I' reads time t'. This corresponds

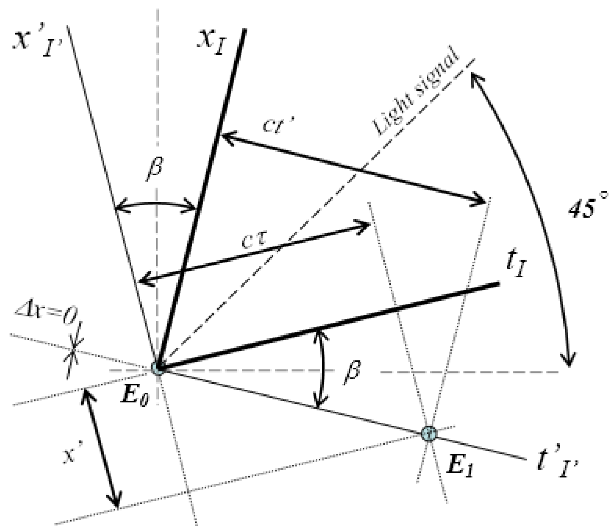

Figure 1: Lorentz diagram between two inertial frames of reference I and I'

to the coordinate clock in the nearest frame I reading the particle's *proper time* τ. The spacetime coordinate of that event is marked by the event E_1 in Fig 1. The *spatial distance* between the events E_0 and E_1 in the frame I' is $\Delta x' = x'$; while in the particle's nearest frame I, the same distance is obviously $\Delta x \approx 0$. The angle β between the inertial frames of reference I and I' is given by $\sin(\beta) = \frac{v'}{c} = \frac{x'}{ct'}$, where c is the speed of light in vacuum. From the relationship, it is evident that as $v' \to c$ the angle $\beta \to \frac{\pi}{2}$. At the other extreme, when the objects are relatively stationary to each other, i.e. as $v' \to 0$ the angle $\beta \to 0$, which results in overlapping the coordinate frames.

Since Special Relativity is a *deterministic* theory, there are no uncertainties $\delta x'$ and $c\delta t'$ associated with the spacetime coordinate of the event E_1. Moreover, Special Relativity being a *continuous* theory, the particle velocity v' (or its momentum p') are considered as *non-quantized* and continuous variables. In the following sections, we attempt to introduce the quantum uncertainties in the context of Special Relativity.

3 Quantum Wavefunction

Limiting ourselves to a special case in which the particle's momentum is constant, next we define the wavefunction $\psi(x)$ of the particle such that the square of its magnitude $|\psi(x)|^2$ to have the following character:

$$|\psi(x')|^2 = \begin{cases} 1/\epsilon & |x'| < \epsilon/2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

By definition, the function $|\psi(x')|^2$ could be interpreted as the *probability density distribution* of the particle's position along x'-axis. The dimensionality of the probability density is that of the inversed-length. Hence, by definition, the numeric 1 in the numerator of Eqn 1 must have no unit and simply represent a *probability.* The parameter ϵ, in the denominator of the equation, on the other hand, must have the length dimensionality and simply represent a *length interval* on x' where $|\psi(x')|^2$ is non-zero and constant. From Quantum Mechanics, the latter condition corresponds to a case that the particle has a definite momentum and hence has a *uniform probability* of being anywhere on that interval. The length interval ϵ, therefore, could be interpreted as the uncertainty in the spatial coordinate of the particle on x'.

We begin with a postulation that, like any length interval in SR, *the length interval corresponding to the spacetime coordinate uncertainty of a quantum particle obey Lorentz transformation.* On that basis, it is postulated that the particle velocity v' and its spatial uncertainty ϵ along the path x' are related as follows:

$$\epsilon = \frac{A}{\sqrt{c^2/v'^2 - 1}} \tag{2}$$

As will be shown later, the length interval A has two important features. First, it represents the *the invariant interval of the spacetime coordinate uncertainty four-vector in spacetime.* Second, it represents the *diameter of the event horizon of the smallest possible black hole in nature.*

The functions $|\psi(x')|^2$ and ϵ, as defined in Eqn's 1 and 2, are shown in Fig 2. By examining Eqn

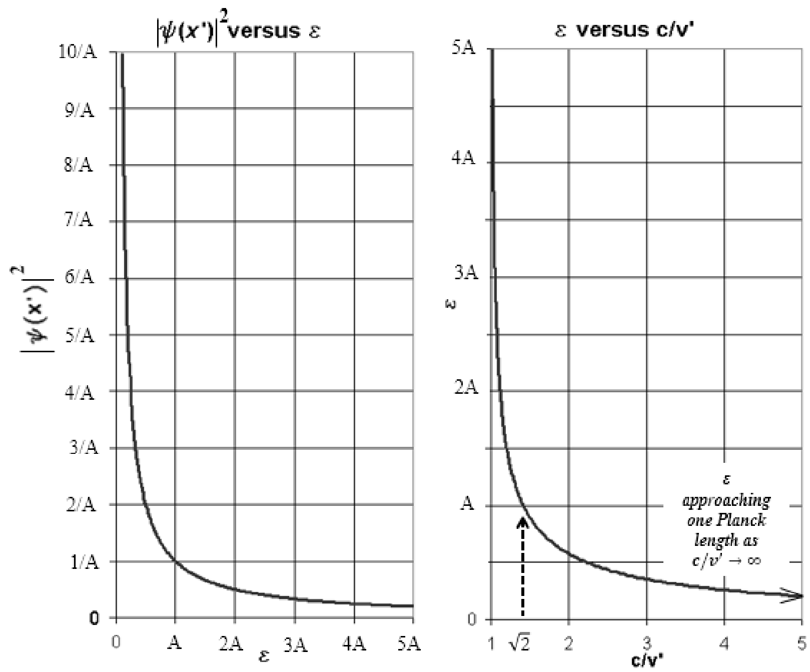

Figure 2: Probability density distribution $|\psi(x')|^2$ versus ϵ and ϵ versus $\frac{c}{v'}$

2, it clear that as the particle velocity $v' \to 0$, i.e. as it gets closer and closer to a stationary condition in I', the uncertainty in the position of the particle in that frame also approaches to zero, i.e. $\epsilon \to 0$; and subsequently, its spatial probability density distribution in that frame approaches to infinity, i.e. $|\psi(x')|^2 \to \infty$. Hence, *as expected for a stationary particle, the positional probability density distribution peaks where the particle is located and vanishes anywhere else on x'.* This is represented by the vertical axis in Fig 2a. Inversely, as the speed of the particle increases, the length interval ϵ, representing the uncertainty in spatial coordinate of the particle increases progressively in length, such that at the limit velocity $v' = c$, the spatial uncertainty becomes infinite, i.e. $\epsilon \to \infty$, and subsequently, $|\psi(x')|^2 \to 0$. This condition is shown by the horizontal axis in Fig 3a, representing a full uncertainty in the spacetime coordinate of a free photon prior to its observation by an inertial observer in I' . For all other conditions, where velocity of the particle is between two extreme limits, $0 < v' < c$, the spatial uncertainty ϵ and the magnitude of the wavefunction $|\psi(x')|^2$ assume some non-extreme values, as shown in Fig 3b. Finally, note that the area under the probability density distribution defined in Eqn (1) integrates to the probability of 1:

$$\int_{-\infty}^{\infty} dx' |\psi(x')|^2 = \int dx' \frac{1}{\epsilon} = 1 \tag{3}$$

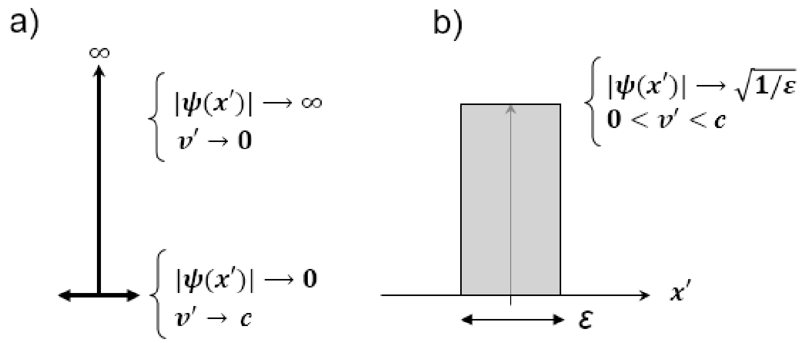

Figure 3: Spatial probability density distribution of a particle versus its velocity

4 Relativistic time dilation and quantum wavefunction

From Special Relativity, the intervals of proper-time $d\tau$ and coordinate-time dt' are related by the Lorentz transformation [1] as follows:

$$\frac{d\tau}{dt'} = \sqrt{1 - \frac{v'^2}{c^2}} \tag{4}$$

where the proper time τ is measured at the origin of the particle's nearest frame of reference I; and the coordinate time t' is measured at the origin of the inertial frame of reference I'. The numerical value of the time dilation from Eqn 4, when v' is between two extreme cases of $v' = c$ and $v' = 0$ is always between 0 and 1, respectively. As the absolute probabilities associated with the uncertainty in the spacetime coordinate of a particle also vary between 0 and 1, it would not be surprising to show that the time dilations of SR and wavefunction of QM are physically related. To demonstrate this, we now re-arrange Eqn 4 as follows:

$$\frac{d\tau}{dt'} = \frac{v'}{c}\sqrt{\frac{c^2}{v'^2} - 1} \tag{5}$$

substituting for $\sqrt{c^2/v'^2 - 1}$ from Eqn 2 we get:

$$\frac{d\tau}{dt'} = \frac{v'}{c}\frac{A}{\epsilon} \tag{6}$$

and further by substituting for $1/\epsilon$ from Eqn 1 we arrive at a fundamental relationship between the *time dilation* of theory of Relativity and the *wavefunction* of Quantum Mechanics as follows:

$$\frac{d\tau}{dt'} = \frac{v'}{c}A|\psi(x')|^2 \tag{7}$$

Eqn 7, in a sense, relates *the relativistic time dilations corresponding to the kinematics of a particle with the probability density distribution of its position in space*. Multiplying both sides by $\frac{c}{v'}$ and replacing $cd\tau = ds$, and $v'dt' = dx'$ we get:

$$A|\psi(x')|^2 dx' = ds \tag{8}$$

By integrating Eqn 8 over coordinate space x' we then arrive at:

$$A\int dx'|\psi(x')|^2 = \delta s \tag{9}$$

In reference to Eqn 3, the integral of Eqn 9 is equal to the probability of 1, therefore we finally arrive at the following relationship for the uncertainty interval invariant:

$$\delta s = A \tag{10}$$

and subsequently;

$$\delta\tau = \frac{A}{c} \tag{11}$$

5 Four-vector of spacetime coordinate uncertainties

The components of the four-vector of the spacetime coordinate uncertainties are the $c\delta t'$ plus three spatial uncertainties $\delta x'$, $\delta y'$ and $\delta z'$. In this paper, however, without a loss in generality, we have limited our analysis to the situation in which the spatial axes of the inertial frames of reference I and I' are all parallel and their relative motion is along their x, x'-axes. As shown in Fig 4, the invariant base of the uncertainty triangle in this case is *always* A, the hypotenuse is $c\delta t'$ and perpendicular to the base is $\delta x'$. Therefore, the quantum uncertainties in spacetime coordinate of a particle constitute a *timelike four-vector* with invariant length of $\delta s = A$. For an observer that travels with the particle, the hypotenuse would be parallel to the base. For that case, the space-like uncertainty $\delta x' = 0$ (as the particle would look stationary to the observer), but the time-like uncertainty of the particle would be the *constant* $\delta t' = \delta\tau = A/c$. The time interval agreeable by all observers, could be used as the *cosmological time interval.* At any given epic, the age of the universe is therefore an *integer multiple* of this constant time interval.

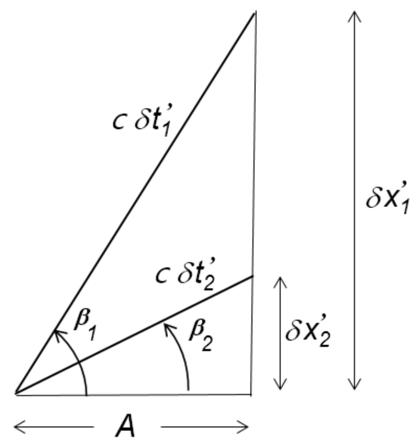

Figure 4: Invariant length A of uncertainty four vector of two particles with different velocities

6 The locus of uncertainties

Since the spacetime uncertainty interval $\delta s = A$ is invariant under the Lorentz transformation, it could be used to define the uncertainties $c\delta t'$ and $\delta x'$ in the spacetime coordinate of the particle as follows:

$$\delta s^2 = (c\delta t')^2 - (\delta x')^2 = A^2 \tag{12}$$

Eqn 12 represents a north-opening hyperbola as shown in Fig 5. The higher the coordinate velocity v' relative to the inertial frame of reference I', the higher the inherent uncertainty in the spacetime coordinates $c\delta t'$ and $\delta x'$ of it in that frame. Eqn 12 can be equivalently written as:

$$\cosh^2\alpha - \sinh^2\alpha = 1 \tag{13}$$

where the hyperbola parameter α is twice the area under the locus and the intersecting ray from the origin. In reference to Fig 5, for two particles with velocities of $v'/c = 0.6$ and $v'/c = 0.83$ for example, the difference in the inherent spacetime uncertainties are $0.75A$ (space-like) and $0.55A$ (time-like). Alternatively, as shown in Fig 6, the spacetime coordinate uncertainty of the particle can be determined by a complex number b as follows:

$$b = \rho_b\, e^{i\theta} \tag{14}$$

where ρ_b is the magnitude and θ is the phase angle of b . The real and imaginary components of

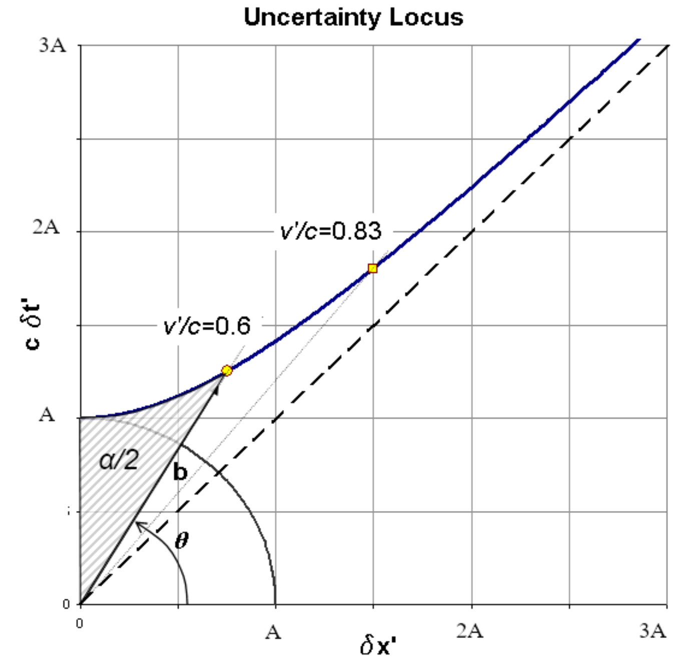

Figure 5: Locus of uncertainties of a particle in state of definite momentum

the complex number b will represent the inherent coordinate uncertainties. As shown in Fig 6, as the momentum of the particle varies due to its acceleration (or deceleration), the complex number b traces the uncertainty locus up towards the asymptote (or down towards the ordinate). The phase angle θ, represents the *instantaneous slope the world line* of the particle in spacetime and is given by:

$$\tan\theta = \frac{cdt'}{dx'} = \frac{c}{v'} \tag{15}$$

As shown in Fig 6, the phase angle θ varies between the limits $\pi/4 < \theta < \pi/2$, where the upper limit $\pi/2$ corresponds to the condition of a stationary particle where the complex number b is closely (but not entirely) aligned with the lateral (imaginary) axis. As we shall see later, due to the inherent quantum uncertainties, the phase angle θ of a stationary particle must always be less than $\pi/2$. The lower limit $\pi/4$ corresponds to that of the light particle where the complex number b is closely (but again not entirely) aligned with the asymptote $\theta = \pi/4$.

Now it is worth to note that in Special Relativity the *light-like* (or null) interval has the invariant interval $c^2\delta t'^2 - \delta x'^2 = 0$, i.e $\theta = \pi/4$. In the combined SR-QM theory, however, the light-like signal is only *asymptotically* tangent to the line of 45 degree; hence, according to Eqn 12, the invariant interval in the case of a light particle is also $c^2\delta t'^2 - \delta x'^2 = A^2$, i.e. not zero.

The spacetime coordinate uncertainties, therefore, could be found by intersecting the uncertainty locus with the ray $c\delta t' = (c/v')\delta x'$ passing through the origin of the coordinate system. From Eqn 15, the slope of this intersecting ray is the instantaneous slope θ of the world line of the particle. Hence, the spacetime uncertainties $c\delta t'$ and $\delta x'$ can be obtained by solving the system of equations:

$$\begin{aligned} (c\delta t')^2 - (\delta x')^2 &= A^2 \\ c\delta t' - (c/v')\delta x' &= 0 \end{aligned} \tag{16}$$

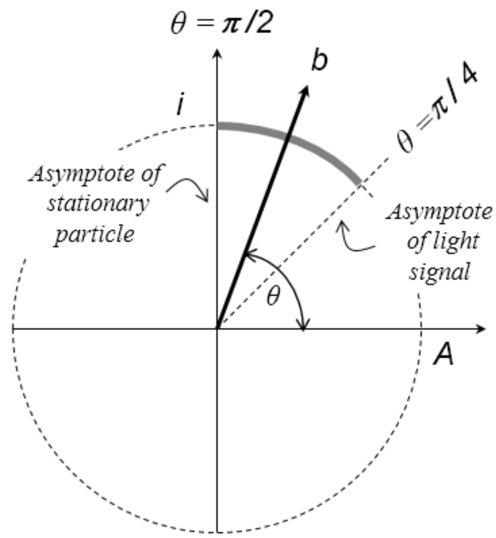

Figure 6: The real and imaginary components of b are uncertainties in space and time

which results in:

$$\delta x' = \rho_b \cos(\theta) = A\sinh(\alpha) = \frac{Av'}{\sqrt{c^2 - v'^2}} \tag{17}$$
$$c\delta t' = \rho_b \sin(\theta) = A\cosh(\alpha) = \frac{Ac}{\sqrt{c^2 - v'^2}}$$

Using Eqn 17, we finally arrive at the following for the magnitude ρ_b of b :

$$\rho_b = A\sqrt{\frac{c^2 + v'^2}{c^2 - v'^2}} \tag{18}$$

The intersection of b with the unit circle on the complex plane is given by the coordinates:

$$\cos(\theta) = \frac{v'}{\sqrt{c^2 + v'^2}} \tag{19}$$
$$\sin(\theta) = \frac{c}{\sqrt{c^2 + v'^2}}$$

where θ, as discussed before, is the slope of particle's world line. The relativistic equations developed for the spacetime uncertainties will be quantized in the following section.

7 Quantization of spatial uncertainties

Having $|\psi(x')|^2$ as defined in Eqn 1 in the previous section, we are now ready to write for the wavefunction of a particle under a definite momentum p as follows:

$$\psi(x') = \sqrt{\frac{1}{\epsilon}} e^{ipx'/\hbar} \tag{20}$$

where $\hbar = h/2\pi$ is the reduced Planck constant with $h = 6.626E - 34$ (Jule.sec). Since the function $|\psi(x')|^2$ has to be single valued function on domain x', we demand for the periodicity of the wavefunction for *any* spatial uncertainty ϵ, i.e $\psi(x') = \psi(x' + \epsilon)$, hence:

$$\sqrt{\frac{1}{\epsilon}} e^{ipx'/\hbar} = \sqrt{\frac{1}{\epsilon}} e^{ipx'/\hbar} e^{ip\epsilon/\hbar} \tag{21}$$

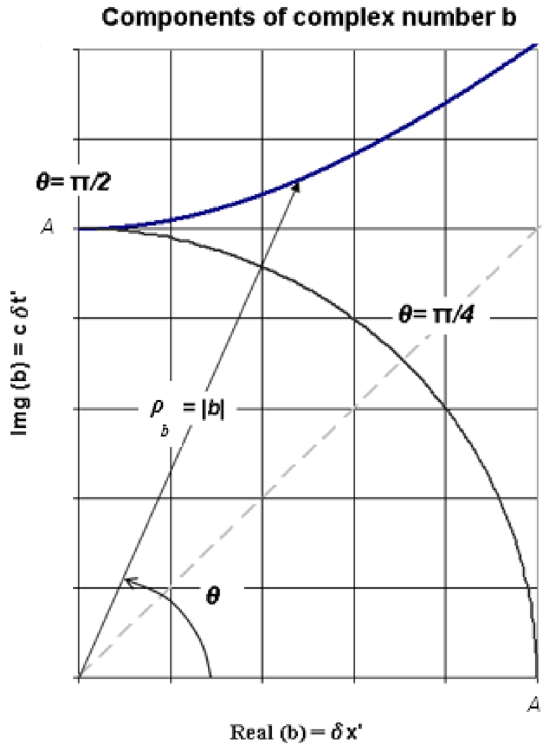

Figure 7: Complex number b tracing uncertainty locus in spacetime

From the last equation, we then have $e^{ip\epsilon/\hbar} = 1$, hence:

$$\frac{p_n \epsilon_n}{\hbar} = 2\pi n^2 \tag{22}$$

The reason for choosing a square of the quantum index n in the RHS is because each parameter in the product $p_n \epsilon_n$ in the LHS contributes one quantum index n. Isolating p_n in Eqn 22 we have:

$$p_n = \frac{h}{\epsilon_n} n^2 \tag{23}$$

From the theory of Special Relativity, however, for the momentum of the particle we have:

$$p = \frac{mv'}{\sqrt{1 - \frac{v'^2}{c^2}}} \tag{24}$$

multiplying both the numerator and denominator by c/v' we get:

$$p = \frac{mc}{\sqrt{\frac{c^2}{v'^2} - 1}} \tag{25}$$

substituting for $\sqrt{c^2/v'^2 - 1}$ by ϵ/A in the previous equation we get:

$$p_n = mc\frac{\epsilon_n}{A} \tag{26}$$

and now by substituting for p_n from Eqn 23 and solving for ϵ_n we get:

$${\epsilon_n}^2 = A\frac{h}{mc} n^2 \tag{27}$$

By definition, Compton wavelength [2] of a given particle represents *the wavelength of a photon whose energy is equal to the rest energy of the particle*; therefore, for a particle with rest mass m we have:

$$mc^2 = \frac{hc}{\hat{\lambda}} \tag{28}$$

which in turn gives the following for the Compton wavelength of a particle with rest mass m:

$$\hat{\lambda} = \frac{h}{mc} \tag{29}$$

Now, let's define a *reference mass* $\bar{m}$ whose Compton wavelength $\hat{\lambda} = A$ as follows:

$$\bar{m} = \frac{h}{Ac} \tag{30}$$

Note that $\bar{m}$ represents *a particle whose rest mass is the smallest none-zero mass physically possible*; therefore, any particle with rest mass less than $\bar{m}$ is considered as a massless particle. Substituting for the term h/Ac in Eqn 27 from Eqn 30, gives an explicit equation for the spatial uncertainty ϵ_n in terms of particle mass m and reference mass $\bar{m}$ as follows:

$$\epsilon_n = A\sqrt{\frac{\bar{m}}{m}}\,n \qquad\qquad n = 1, 2, ... \tag{31}$$

Accordingly, *the spatial uncertainty of a particle is inversely proportional to square root of its rest mass* m. It is also evident that the minimum spatial uncertainty corresponds to that of a stationary particle with the minimum quantum index $n = 1$. The higher the quantum index n the higher the particle velocity and therefore the spatial uncertainty.

8 Quantization of momentum

The quantized form of momentum p_n, could simply obtained from Eqn 26 by substituting for ϵ_n from Eqn 31 as follows:

$$p_n = mc\sqrt{\frac{\bar{m}}{m}}\,n \qquad\qquad n = 1, 2, ... \tag{32}$$

According to Eqn 32 *the momentum of a particle is directly proportional to the square root of its rest mass* m.

9 Quantization of timelike uncertainties

The equation of the coordinate time uncertainty $\delta t'$ will be obtained from Eqn 12 by substituting for $\delta x'$ from Eqn 31 as follows:

$$\delta t'_n = \frac{A}{c}\sqrt{1 + \frac{\bar{m}}{m}n^2} \qquad\qquad n = 1, 2, ... \tag{33}$$

10 Quantization of particle velocity

The relativistic coordinate velocity v' can be obtained from Eqn 25 by substituting for momentum p from Eqn 32 and solving for c/v' as follows:

$$\frac{c}{v'_n} = \sqrt{1 + \frac{m}{\bar{m}n^2}} \qquad\qquad n = 1, 2, ... \tag{34}$$

From Eqn 34, the coordinate velocity of a particle with rest mass m at the quantum index $n = 1$ is given by:

$$v'_1 = c\sqrt{\frac{\bar{m}}{m + \bar{m}}} \tag{35}$$

Coordinate velocity v'_1 represents the particle velocity at its nearest frame of reference, i.e. it represents the *smallest* none-zero velocity that the particle could attain. From Eqn 35, it is evident that the higher the rest mass m of a particle the smaller is its near rest velocity. The near rest velocity of the reference mass $\bar{m}$ is then given by $\bar{v}'_1 = c/\sqrt{2}$.

11 Quantization of phase angle

The quantized equation of the phase angle θ of the b vector, could be obtained from Eqn 25 as follows:

$$\frac{c^2}{v'^2} = 1 + \frac{m^2c^2}{p^2} \tag{36}$$

substituting for c/v' and p from Eqn's 15 and 32, respectively, and solving for the phase angle θ we will have:

$$\theta_n = \tan^{-1}\sqrt{1 + \frac{m}{\bar{m}n^2}} \qquad n = 1, 2, ... \tag{37}$$

From Eqn 37, for a stationary particle with the quantum index $n = 1$, we have:

$$\theta_1 = \tan^{-1}\sqrt{1 + \frac{m}{\bar{m}}} \tag{38}$$

From above, for particles of increasing smaller mass, as $m \to 0$ the term $\sqrt{1 + \frac{m}{\bar{m}}} \to 1$ and with that $\theta_1 \to \pi/4$. Hence, *the quantized equation of phase angle confirms that the mass-less $m = 0$ particles travel with the speed of light.* Inversely, for particles of increasing larger mass, the term $\sqrt{1 + \frac{m}{\bar{m}}}$ increases and with that the phase angle $\theta_1 \to \pi/2$, reducing the coordinate uncertainties at near rest condition. Therefore, *the higher the rest mass of a particle the less the spacetime coordinate uncertainties in its near rest condition.*

12 Quantization of energy

According to the theory of Special Relativity, the total energy E of a particle in terms of its mass m and momentum p is given by:

$$E^2 = p^2c^2 + m^2c^4 \tag{39}$$

substituting for the momentum p from Eqn 32, we then find particle's total energy in discrete values n as follows:

$$E_n = mc^2\sqrt{1 + \frac{\bar{m}}{m}n^2} \qquad n = 1, 2, ... \tag{40}$$

According to Eqn 40, the *relativistic rest energy* $E = mc^2$ requires a correction for quantum particles for which the absolute rest condition is prohibited. Therefore, for the lowest quantum index $n = 1$ we have:

$$E_1 = mc^2\sqrt{1 + \frac{\bar{m}}{m}} \tag{41}$$

It is event that the quantum multiplier $\sqrt{1 + \frac{\bar{m}}{m}}$ in front of mc^2 is greater than one. As mentioned above, this is to account for the fact that in quantum mechanics, unlike relativity, the rest condition in its absolute sense is prohibited. Hence, *the total energy of a particle near rest is equal to its relativistic rest energy $E = mc^2$ augmented by the quantum rest kinetic energy.* Moreover, for particles of increasing larger mass, the term $\sqrt{1+\frac{\bar{m}}{m}} \to 1$ and with that $E_1 \to mc^2$. This means *the kinetic energy corresponding to the rest condition of a massive particle is near zero* - as expected from the classical physics.

13 Four-vector of momentum

According to the theory of Special Relativity, the invariant length of *momentum four-vector* of a particle of mass m is the constant m^2c^2. Hence, using the equations developed earlier, the latter *must turn out to be independent of the quantum index* n. To verify this, we begin with the definition of the momentum four-vector in Special Relativity [Ref 3]:

$$P = (\frac{E_n}{c}, p_n) \tag{42}$$

Substituting for the energy and momentum terms from Eqns 40 and 32 we arrive at:

$$P = (mc\sqrt{1+\frac{\bar{m}}{m}n^2},\ mc\sqrt{\frac{\bar{m}}{m}}n) \tag{43}$$

Subsequently, for the magnitude of the four-vector P we then have:

$$P.P = m^2c^2(1+\frac{\bar{m}}{m}n^2) - m^2c^2\,\frac{\bar{m}}{m}n^2 = m^2c^2 \tag{44}$$

which is found to be independent of quantum index n - as required.

14 Non-relativistic approximations

We begin by recalling that the approximation $\sqrt{1+z} \approx (1+\frac{z}{2})$ is mathematically valid if $z \approx 0$. Using this approximation, in the non-relativistic conditions where the term $\frac{\bar{m}}{m}n^2$ is sufficiently small, from Eqn 40 the total energy of a particle with rest mass m can be approximated as follows:

$$E_n = mc^2 + \frac{1}{2}m(c^2n^2\frac{\bar{m}}{m}) \qquad n = 1, 2, ..., N \tag{45}$$

As discussed before, for the non-relativistic condition to be valid the maximum quantum N in Eqn 45 must be sufficiently small. The first term mc^2 is the rest energy, and hence, the second term in the equation is expected to be the kinetic energy KE of the particle. To realize the second term, we note that in the non-relativistic conditions the kinetic energy KE in terms of rest mass m and momentum p is also given by:

$$KE = \frac{p^2}{2m} \tag{46}$$

Again substituting for the momentum p from Eqn 32 we get:

$$KE_n = \frac{1}{2}m(c^2n^2\frac{\bar{m}}{m}) \qquad n = 1, 2, ..., N \tag{47}$$

Comparing with the classical equation $KE = \frac{1}{2}mv'^2$, the quantum equation of non-relativistic velocities will be as follows:

$$v'_n = c\sqrt{\frac{\bar{m}}{m}}\,n \qquad n = 1, 2, ..., N \tag{48}$$

15 Upper limit of acceleration

In addition to the *four-vector of coordinate intervals* that obey Lorentz transformation and have an invariant interval, velocities and accelerations in the theory of relativity also constitute four-vectors. Derivative of the coordinate intervals with respect to the proper-time τ are called *proper-velocities.* They constitute a *time-like* four-vector with the invariant length c. Derivative of proper-velocities with respect to the proper-time generate a *space-like* four-vector of proper-accelerations. By definition, the invariant length of proper-acceleration four-vector is called local-acceleration a, which physically is understood to be the magnitude of the acceleration relative to an inertial frame which is instantaneously at rest with the accelerating particle. Local acceleration is measurable by an accelerometer carried with the particle.

Next we discuss the limit local acceleration a_u which is a direct consequence of the lower limit velocities v'_1, upper limit velocity c and the uncertainty time interval $\delta t'_1$. The limit acceleration a_u corresponds to a case that the coordinate velocity of a particle, varies from the lower limit of the stationary value v'_1, given by Eqn 35, to the upper limit c, within the invariant time interval $\delta\tau = A/c$. Hence,

$$a = \frac{c^2}{A}(1 - \sqrt{\frac{\bar{m}}{m + \bar{m}}}) \tag{49}$$

It is evident that for particles of large mass where $m \gg \bar{m}$ the term $\sqrt{\frac{\bar{m}}{m+\bar{m}}} \to 0$ and with that the local acceleration $a \to a_u$ given by:

$$a_u = \frac{c^2}{A} \tag{50}$$

The latter represents the maximum acceleration a particle can attain in the physical world.

16 Unit Black Hole

The limit acceleration a_u, considered as the gravitational pull at event horizon of black holes, can be used to constrain the mass and size of the smallest possible black hole in nature; herein, called a Unit Black Hole (UBH). The reason for such naming will become apparent when the quantization of the black holes mass and event horizon diameter will be discussed at the subsequent sections. Accordingly, for the Event Horizon radius R_E of a black hole with mass M_B using Schwarzchild's equation [4] we have:

$$R_E = \frac{2GM_B}{c^2} \tag{51}$$

To arrive at the limit acceleration a_u, we now define UBH as a black hole with the event horizon diameter A and mass M_0, and re-write the Schwarzchilds Eqn 51 for the UBH as follows:

$$\frac{A}{2} = \frac{2GM_0}{\frac{c^2}{A}A} \tag{52}$$

substituting for $\frac{c^2}{A}$ from Eqn 50 and re-arranging we arrive at:

$$a_u = \frac{GM_0}{(\frac{A}{2})^2} \tag{53}$$

From Eqn 53, we conclude that the limit acceleration a_u should be interpreted as the local acceleration of a free falling particle when located at the distance $A/2$ from the geometric center of the UBH. At that distance, the gravitational pull is at limit and cannot physically increase any further. Diameter A could be considered as the boundary of *physical singularity.* Re-arranging Eqn 53 further and substituting for a_u from Eqn 50 we then arrive at the mass of UBH as follows:

$$M_0 = \frac{Ac^2}{4G} \tag{54}$$

The mass M_0 represents the minimum mass of black holes in cosmos.

17 Physical Units in SR-QM theory

Unlike the units of the SI or imperial systems, which have their origins in the human experience of the physical world, the units of the combined SR-QM theory must be based on the *natural* units; free from any *anthropocentric bias*. One such natural system of units that is entirely based on the most fundamental constants of nature, G, c and h is that of Max Planck, where the units of length, mass and time are defined as follows:

$$l_p = \sqrt{\frac{Gh}{c^3}} \qquad (55)$$
$$m_p = \sqrt{\frac{ch}{G}}$$
$$\tau_p = \sqrt{\frac{Gh}{c^5}}$$

As mentioned earlier, the reference mass $\bar{m}$ is considered the smallest none-zero rest mass such that particles lighter than $\bar{m}$ are physically massless. We now express the reference mass $\bar{m}$ as a perfect square fraction of Planck mass m_p as follows:

$$\bar{m} = \frac{1}{j^2}\, m_p \qquad (56)$$

Note that in [5] the discrepancy in escape rates of CH4 and N2 molecules from exosphere of Pluto is used to constrain the reference mass to the range $3.1979E-45 < \bar{m} < 3.2039E-45$. Once the precise value of $\bar{m}$ is determined - either by additional observational data or by the experimental set up proposed in [5] - the *reference index* j is then trivially obtained from Eqn 56. Subsequently, the invariant length A can be obtained in terms of Planck length l_p by substituting for $\bar{m}$ in Eqn 30 as follows:

$$A = j^2\, l_p \qquad (57)$$

Using the same structure of Eqn 56, we then similarly define the rest mass m of a given particle as a perfect square fraction of Planck mass m_p as follows:

$$m_z = \frac{1}{z^2}\, m_p \qquad z = 1, 2, ..., j \qquad (58)$$

Note that the quantum index z for the particle mass begins from the index $z = 1$ all the way to the reference index $z = j$ to satisfy $\bar{m} \leq m \leq m_p$. Moreover, note that by defining the *gravitational coupling constant* $\bar{\alpha}_g$ using a pair of $\bar{m}$'s attracting each other, then the relation between the index j and the gravitational coupling constant $\bar{\alpha}_g$ would be as follows:

$$\bar{\alpha}_g = \frac{G\bar{m}^2}{hc} = (\frac{\bar{m}}{m_p})^2 = j^{-4} \qquad (59)$$

Substituting for m, A and $\bar{m}$ from Eqn's 58, 57 and 56 in Eqn 31, the fully quantized equation of the spacetime coordinate uncertainty ϵ of a quantum particle in terms of Planck length l_p is then given as follows:

$$\epsilon_{n,z} = jzn\, l_p \qquad z = 1, 2, ..., j \quad n = 1, 2, ... \qquad (60)$$

The cosmological time interval $\delta\tau = A/c$ yields to the following in terms of Planck time τ_p :

$$\delta\tau = \frac{A}{c} = \frac{j^2\sqrt{\frac{Gh}{c^3}}}{c} = j^2\sqrt{\frac{Gh}{c^5}} = j^2\, \tau_p \qquad (61)$$

Subsequently, quantized coordinate time uncertainty $\delta t'$ would be given by:

$$\delta t'_{n,z} = j\sqrt{j^2 + z^2 n^2}\,\tau_p \tag{62}$$

In addition, for the upper limit of local acceleration a_u we then have the following:

$$a_u = \frac{c^2}{A} = \frac{c^2}{j^2\sqrt{\frac{Gh}{c^3}}} = \frac{1}{j^2}\sqrt{\frac{c^7}{Gh}} = \frac{1}{j^2}\,a_p \tag{63}$$

where the term $a_p = \sqrt{c^7/Gh}$ is *Planck acceleration.* Further, the fully quantized equation for the momentum of the quantum particle in terms of Planck momentum $p_p = m_p c$ is given as follows:

$$p_{n,z} = \frac{n}{jz}\,p_p \qquad z = 1, 2, ..., j \quad n = 1, 2, ... \tag{64}$$

Similarly, the fully quantized equation for the energy of a quantum particle in terms Planck energy $E_p = m_p c^2$ is given as follows:

$$E_{n,z} = \frac{1}{jz^2}\sqrt{j^2 + n^2 z^2}\,E_p \qquad z = 1, 2, ..., j \quad n = 1, 2, ... \tag{65}$$

18 The Uncertainty Principle

In this section, we will verify that the equations developed earlier for the uncertainty in position, momentum, time and energy of a particle with rest mass m are in agreement with the *uncertainty principle* [3] given by the inequalities $\Delta x'\,\Delta p \geq \hbar/2$ and $\Delta t'\,\Delta E \geq \hbar/2$. From Eqn 64, the uncertainty in momentum for a unit variation of the quantum index $\Delta n = 1$ is given by the difference:

$$\Delta p = \frac{1}{2}(p_{n+1,z} - p_{n-1,z}) = \frac{1}{jz} m_p c \tag{66}$$

The product of the uncertainty in momentum Δp from Eqn 66 and uncertainty in position $\Delta x'$ from Eqn 60 would then be as follows:

$$\Delta x'\,\Delta p = jzn\,l_p \times \frac{1}{jz} m_p c = nh > \hbar/2 \qquad n = 1, 2, ... \tag{67}$$

The inequality is evidently satisfied for every quantum index n, including in the lowest quantum state $n = 1$ where the particle is closest to its stationary state. Similarly, by knowing that $h = E_p \tau_p$ and substituting for the time uncertainty $\Delta t' = \delta t'_{n,z}$ from Eqn 62 and energy uncertainty $\Delta E = \frac{1}{2}(E_{n+1,z} - E_{n-1,z})$ from Eqn 65 results in:

$$\Delta t'\,\Delta E = \frac{1}{z^2}\sqrt{j^2 + z^2 n^2}\,\left(\sqrt{j^2 + z^2(n+1)^2} - \sqrt{j^2 + z^2(n-1)^2}\,\right) h > \hbar \tag{68}$$

The inequality is evidently satisfied for all combination of quantum indexes (n, z), including the limiting case $(1, 1)$ for any value of j.

19 Quantized black hole mass and size

Substituting for the invariant A from Eqn 57 in the Eqn 54, the mass of the unit black hole M_0 in terms of Planck mass m_p will be as follows:

$$M_0 = \frac{Ac^2}{4G} = \frac{c^2 j^2\sqrt{\frac{Gh}{c^3}}}{4G} = \frac{1}{4}\,j^2 m_p \tag{69}$$

Now note that, as discussed earlier, the diameter of the event horizon of the UBH is $A = j^2 l_p$ precisely. It is evident the next physically possible diameter for a black hole minutely more massive than UBH is when the event horizon diameter increases by one Planck length l_p only, as shown in Fig 8. According to Schwarzchilds Eqn 51, the mass of such black hole is equal to the mass of UBH plus a quarter of Planck mass m_p, i.e. $M_1 = M_0 + \frac{1}{4}m_p$. Note that one quarter of Planck mass corresponds to the quantum mass index $z = 2$ in Eqn 58. Therefore, for every $m_2 = \frac{1}{4}m_p$ mass added to a given black hole the event horizon increases by one Planck length in *diameter*. Hence, the quantized equation for the black holes mass M_b in terms of Planck mass is given as follows:

$$M_b = \frac{1}{4}(j^2 + b)m_p \qquad b = 0, 1, 2, ... \tag{70}$$

Similarly, the quantized equation for the event horizon diameter D_b of black holes is given in terms of Planck length as follows:

$$D_b = (j^2 + b)l_p \qquad b = 0, 1, 2, ... \tag{71}$$

where the quantum index $b = 0$, in both equations, corresponds to the UBH. As discussed before, the reference index j will be constrained once the reference mass $\bar{m}$ is accurately determined. Knowing the mass and size of the UBH, its mass density ρ_u is trivially obtained in terms of Planck density $\rho_p = c^5/G^2h$ as follows:

$$\rho_u = \frac{3}{2\pi j^4}\rho_p \tag{72}$$

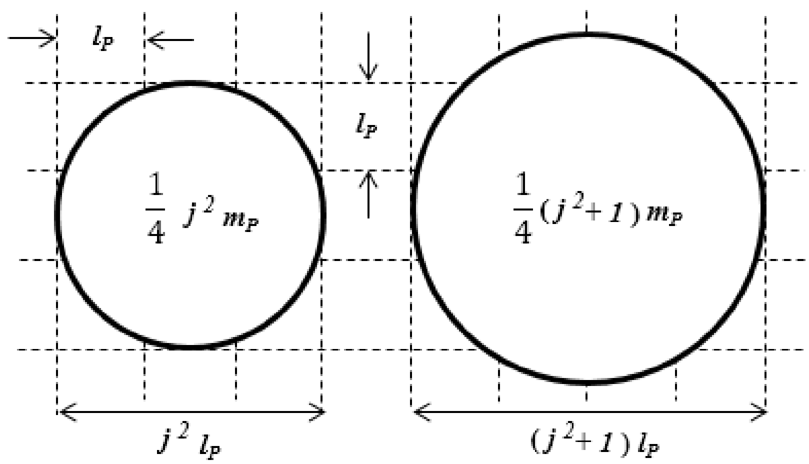

Figure 8: Quantized mass and event horizon of black holes (j not to scale)

20 Quantized Hawking temperature

According to the Bekenstein-Hawking the black hole entropy S_b, is given by [6]:

$$S = \frac{\kappa}{4l_p^2}H \tag{73}$$

where κ is the Boltzmann constant and H is the *surface area of the event horizon of black holes*, given by $H = \pi D^2$. Taking the quantized diameter $D_b = \pi(j^2 + b)^2\, l_p^2$ from Eqn 71, the quantum of entropy variation ΔS_b from Eqn 73 for a unit variation of the quantum index $\Delta b = 1$ is then given as follows:

$$\Delta S_b = \frac{1}{2}(S_{b+1} - S_{b-1}) = \frac{1}{2}\kappa\pi(j^2 + b) \qquad b = 0, 1, 2, ... \tag{74}$$

Similarly, the quantum of energy variation ΔE for a unit variation of the quantum index $\Delta b = 1$ is simply given by:

$$\Delta E = \frac{1}{4} m_p c^2 \tag{75}$$

From Eqn's 75 and 74 we then have:

$$\Delta S_b = \frac{\Delta E}{T_b} = \frac{1}{2} \kappa \pi (j^2 + b) \qquad b = 0, 1, 2, ... \tag{76}$$

From above, the quantized equation of the black hole temperature T_b is then given by:

$$T_b = \frac{1}{2\pi(j^2 + b)} T_p \qquad b = 0, 1, 2, ... \tag{77}$$

where $T_p = m_p c^2/\kappa$ is Planck temperature. Under the quantum index $b = 0$, the UBH has therefore the highest temperature $T_0 = T_p/2\pi j^2$ among black holes. Finally, note that Hawking's equation for black hole temperature is given by [7]:

$$T = \frac{\hbar c^3}{8\pi G M \kappa} \tag{78}$$

It can be trivially verified that by substituting for the black hole mass M in above with its quantized form $M_b = \frac{1}{4}(j^2 + b) m_p$ we arrive at Eqn 77; indicating the latter is indeed the quantized form of Hawking's radiation. From both equations is evident that black hole temperature is inversely proportional to its mass. From Eqn 77, the temperature drop ΔT_b of a black hole for a single quantum increase in its mass is given by:

$$\Delta T_b = -\frac{1}{2\pi} \frac{T_p}{(j^2 + b + 1)(j^2 + b)} \qquad b = 0, 1, 2, ... \tag{79}$$

21 Conclusion

By combining the theories of Special Relativity and Quantum Mechanics it was shown that the inherent spacetime coordinate uncertainties of quantum particles constitute a time-like four-vector with an invariant length interval A. It was found that determining the reference mass limit $\bar{m}$ (a minimum none-zero mass limit below which a particle is physically treated as massless) plays a key role in determining the magnitude of the invariant length interval A. The reference mass limit was best defined as a perfect square *fraction* of Planck mass. The invariant length interval, in turn, was found to be a perfect square *multiple* of Planck length. Quantum equations for particles inherent spatial uncertainty, momentum, coordinate time and energy were expressed in terms of Planck length l_p , Planck momentum $m_p c$, Planck time τ_p and Planck energy $m_p c^2$, respectively. Moreover, it was found that the local-acceleration has a physical upper limit a_u, expressed as a perfect square fraction of Planck acceleration $\sqrt{c^7/Gh}$. The limit acceleration was then used to constrain the mass of the smallest possible black hole in nature. Finally, the mass, diameter and temperature of black holes were quantized in terms of Planck mass, Planck length and Planck temperature, respectively.

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