Wave packets of relaxation type in boundary problems of quantum mechanics

Igor B. Krasnyuk †
†Institute Physics and Technology NAN Ukraine

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Abstract

An initial value boundary problem for the linear Schrȯdinger equation with nonlinear functional boundary conditions is considered. It is shown that attractor of problem contains periodic piecewise constant functions on the complex plane with finite points of discontinuities on a period. The method of reduction of the problem to a system of integro-difference equations has been applied. Applications to optical resonators with feedback has been considered. The elements of the attractor can be interpreted as white and black solitons in nonlinear optics.

Keywords: The Schrȯdinger equation • The functional two points boundary conditions • asymptotic periodic piecewise constant distributions of relaxation type

1 Introduction

In this paper it will be considered an initial value boundary problem (IVBP) which describes, for example, the dynamics of a kicked charged particle moving in a confined 'quantum box'. We assume that interaction between particles take place only at flat walls, where there is surface potential $W(u_1, u_2, S_1, S_2)$ depending both on surface amplitudes of particles and their phases (see, [12]. The problem can be described by two linear quantum equations coupled by nonlinear differential or functional boundary conditions. Such problem arise in optical waveguide technologies. The boundary conditions used as 'high-speed switches' [15], and ones plays the role of 'frequency conversion' which is used to produce laser fields at wavelengths that are inaccessible to materials [16], and 'can also be used to interface individual parts of a network or to transfer information from one field to another at a different wavelength [19]' [14]. Moreover, the mathematical model below can be used to describe ridge waveguide. The known example of this aim is $MgO : LiNbO_3$Sakai.

The IVBP boundary describes the dynamics of a kicked charged particle moving in a double-well, or more complex potential, and a time-dependent magnetic field. In certain cases the stroboscopic dynamics reduces to the complex logistic map, thus providing physical meaning for the Mandelbrot set [12]. The logistic map has bounded and stable trajectories for the control parameter $0 < a < 4$. Thus, the same is true for solutions of the quantum problem for small parameter $h > 0$, or in the quasiclassic or WKB − approximation [22]. In other cases, we obtain iterated function systems consisting of the inverse complex logistic map, thus providing physical meaning for Julia sets. Remind

Our approach can be generalized to complex mappings with a maximum of order q.ales with some surdouble-well potential and a time-dependent magnetic field. In certain cases the stroboscopic dynamics reduces to the complex logistic map, thus providing physical meaning for the Mandelbrot set. In other cases we obtain iterated function systems consisting of the inverse complex logistic map, thus providing physical meaning for Julia sets. Our approach can be generalized to complex mappings with a maximum of order q. which describes the dynamics of the two free particles with opposite impulses that are places into confined quantum box. Thus, we consider the dynamics of a kicked charged particle moving in a double-well or more complex potential which is placed at flat walls if the box. In [12], it has been studied the deterministic version of a classical Langevin problem, where it has been looked the movement of a charged particle in a double-well potential. It is shown that the Langevin problem can be reduced to the study of a family of iterated function systems, containing the complex logistic map. This result provides physical meaning for the Julia set. Similar approach has been used in [11] to the study of an initial value boundary problem for the Liouville equation with nonlinear dynamic boundary conditions. The problem describes a velocity of changing on time of the probability of particles at walls that confines the particles. Note that these velocities are nonlinear functions of the density of the probability of particles to occupied the flat walls. The attractor of the problem has been constructed. This attractor contains periodic piecewise constant functions with finite, countable or uncountable (homeomorphic to the Cantor set) lines of discontinuities on a period, which propagate along characteristics of the Liouville equation. We call such elements of the attractor by the limit generalized distributions of relaxation, pre-turbulent and turbulent type, correspondingly with respect to the classification Sharkovsky classification [6]. In the present paper, we generalise the results [12, 11] on the IVBP, where the motion of free particles with different impulses will be described by the generalised Shrödinger type equations. Corresponding operators are linear with a small parameters, and with symbols that are polynomial functions

$$P_n(p) = \sum_{j=0}^{n} a_j p^j, \quad n = 2, 3, ... \tag{1}$$

Here, $p \in R$, p corresponds to the operator $\hat{p} = -ih\frac{d}{dx}$, where $h > 0$ is a small parameter. If $n - 2$ then we have deal with the Shrödinger equation. Let us define

$$\hat{E} = -ih\frac{\partial}{\partial t}, \quad \hat{p} = -ih\frac{d}{dx} \tag{2}$$

and consider the uncoupled system of equations

$$\left(-\hat{E} + P_n^1(\hat{p})\right) y_k(x,t) = 0, \quad k = 1, 2. \tag{3}$$

Let initial conditions are a special form

$$y_k(x, 0) = e^{\frac{i}{h}\alpha_k x}. \tag{4}$$

Then we can find a solution in the form

$$y_1(x,t) = e^{\frac{i}{h}\lambda_1^1 x + \lambda_1^2 t}, \quad y_2(x,t) = e^{\frac{i}{h}\lambda_2^1 x + \lambda_2^2 t}, \tag{5}$$

where $\lambda_i^j \in R, i, j = 1, 2$.

A corresponding initial problem has been solved in [9], where is shown that ones may be reduced to Hamilton-Jacobi equations

$$\lambda_2^1 + P_n^1(\lambda_1^1) = 0, \quad \lambda_2^2 + P_n^1(\lambda_1^2) = 0, \tag{6}$$

and, respectively, to a system of transport equations

$$\frac{\partial \varphi_1}{\partial t} + \frac{\partial P_n^1(\lambda_1^1)}{\partial p}\frac{\partial \varphi_1}{\partial x} = 0, \tag{7}$$

$$\frac{\partial \varphi_2}{\partial t} + \frac{\partial P_n^2(\lambda_2^1)}{\partial p}\frac{\partial \varphi_2}{\partial x} = 0. \tag{8}$$

Let us define

$$\frac{\partial P_n^1(\lambda_1^1)}{\partial p} = \lambda_1, \quad \frac{\partial P_n^2(\lambda_2^1)}{\partial p} = \lambda_2. \tag{9}$$

Then

$$\frac{\partial \varphi_1}{\partial t} + \lambda_1 \frac{\partial \varphi_1}{\partial x} = 0, \tag{10}$$

$$\frac{\partial \varphi_2}{\partial t} + \lambda_2 \frac{\partial \varphi_2}{\partial x} = 0, \tag{11}$$

where we assume that $\lambda_1 > 0$ and $\lambda_2 < 0$.

Now we consider the functional boundary conditions

$$\varphi_1(0, t) = \varphi_2(0, t), \quad \varphi_2(l, t) = \Phi(\varphi_1(l, t)). \tag{12}$$

Then integration of these ODE along characteristics with help of boundary conditions (12) leads to the relations:

$$\varphi_1(l, t) = \varphi_1(0, t - l/\lambda_1) = \varphi_2(0, t - l/\lambda_1) = \varphi_2(l, t - l/\lambda_1 - l/\lambda_2) = \Phi(\varphi_1(l, t - l/\lambda_1) - 1/\lambda_2)). \tag{13}$$

Define $\Delta = l/\lambda_1 + l/\lambda_2$. Then from (15) we arrive at

$$\varphi_1(l, t) = \Phi(\varphi_1(l, t - \Delta)). \tag{14}$$

Solutions of this equation can be find, step by step, iterating an initial function $h_1(t)$, which is given on interval $[-\Delta, 0)$. Let us define $y(t) = \varphi_1(l, t)$. Then $h_1(t)$ can be determined by method of characteristic so that $y(t) = \varphi(t) = \varphi(t) = \varphi_1(t)$ for $t \in [-1/\lambda_2, 0)$, and $\varphi(t) = \varphi_2(t) = \varphi_1(t)$ for $t \in [0, 1/\lambda_1)$ (see, Figure 85, [23]).

2 Hamilton-Jacobi equations

The Hamilton-Jacobi equations have solutions

$$\lambda_1^1 = \alpha_1,\ \lambda_1^2 = -P_n^1(\alpha_2), \quad \lambda_2^1 = \alpha_1,\ \lambda_2^2 = -P_n^2(\alpha_2). \tag{15}$$

Thus,

$$S_k(x,t) = \alpha_k x - P_n^k(\alpha_k)t, \quad k = 1,2. \tag{16}$$

For the solvability of IVBP we must assume that

$$\frac{\partial P_n^1(\lambda_1^1)}{\partial p}\frac{\partial P_n^1(\lambda_1^1)}{\partial p} < 0. \tag{17}$$

Now for the Hamilton-Jacobi equations

$$\frac{\partial S_k}{\partial t} + P_n^k\left(\frac{\partial S_1}{\partial t}\right) = 0, \quad k = 1,2, \tag{18}$$

we postulate the periodic boundary conditions

$$S_1(0,t) = S_1(l,t), \quad S_2(0,t) = S_2(l,t),\ t > 0, \tag{19}$$

$$S_1(x,0) = S_1^0(x), \quad S_2(x,0) = S_1^l(x),\ 0 < x < l. \tag{20}$$

Next, we prolonged the initial conditions on $x \in R$ l - periodically. In this case, solutions of initial problem for a phase will be a solution of the boundary problem with help of prolongation of linear phases $S_1(\zeta)$, $S_1(\eta)$ periodically, where

$$S_k(x,t) = \alpha_1 x - P_n^k(\alpha_1)t, \quad k = 1,2; n = 0,1,.... \tag{21}$$

3 WKB - approximation with a complex phase

Thus, we consider IVBP for two linear PDE with symbols, which are polynomial of order [1] $n = 2,3,...$, and with nonlinear functional or dynamic boundary conditions. For example, for $n = 2$ we have the two uncouples Shrödinger equations. The boundary conditions reflect the connection between amplitudes and phases of (in) and (out) waves at walls of quantum box. We consider $1D$ case, but results may be generalized on $3D$ case. It must be noted that the boundary conditions include an exponential factor that depends of a phase. Initial conditions have the form

$$u(x,t,h) = A(\omega)[\varphi_0(x,t)e^{i\omega S_1(x,t)}e^{-i\omega S_2(x,t)} + O(1/\omega)] \tag{22}$$

where $S_1, S_2 \geq 0$, φ_0 are smooth functions. If

$$S(x,t) = S_1(x,t) + iS_1(x,t), \quad \omega = 1/h, \tag{23}$$

then a solution (22) has the form

$$u(x,t,h) = A(1/h)\left(\varphi_0(x,t)e^{\frac{i}{h}S(x,t)} + O(h)\right). \tag{24}$$

We call such solutions by WKB - solutions. Here, $h > 0$ is a small parameter. It means that we consider high-frequency approximation or approximation of geometric optics. In some cases, this approximation is called by the approximation 'of thin laser beams': that is, for each fixed $t > 0$ a solution is 'localized' at a neighbourhood of some curve (see, [22],c.33)). The motivation of the introduction of a small parameter $h > 0$ or 'inner Planck constant' is explanted by the fact that for the quantization are used asymptotic on $h \to 0$ solutions of this equations ([22],.31). The construction of the asymptotic solutions can be provided by the method of reduction of the problem to a system of equations of quantum mechanics to a system of equations of classic mechanics: to the Hamilton-Jacobi equations for phases and transport equations or Liouville equations for amplitudes.

The construction of complex solutions of these equations of quantum mechanics for infinitly thin laser beams allows to find WKB - solutions in the form

$$u_k(x,t,h) = e^{iS_k/h} \sum_{j=0}^{m} \varphi_j^k(x,t) h^j, \; k = 1,2, \tag{25}$$

where S_k and $\varphi_j^k(x,t)$ are solutions of the Hamilton-Jacobi equations and Liouville equations. Note that the Hamilton-Jacobi equations can be solved exactly. A zero approximation can be determined with accuracy $O(h^2)$ and one is a real function, but another functions admit imaginary corrections to a phase, that is for each next $\pi/2$. In applications to the boundary problem, it is similar to the famous that the

In the present paper, this method of reduction will be applied to the boundary problems of quantum mechanics. The results can be applied to problems of nonlinear optics, to the Ginzburg-Landau equations for a two-component order parameter: example is the system of the Gor'kov equations, which describe a density of Cooper pairs in superconductors of type 2 [7], and so on.

$$u^1(0,t) = \lambda_1 u^2(0,t), \quad u^2(l,t) = \lambda_2 u^2(l,t), \quad t > 0. \tag{26}$$

Let us consider a system of partial differential equations with constant coefficients which have polynomial symbols

$$P_n(p) = \sum_{j=0}^{n} a_j p^j, \tag{27}$$

where $P_n(p)$ is a polynomial of variable $p \in R^1$ of power $n = 1, 2, ...$ Formally, the transformation of variable p on operator $\hat{p} = -h\frac{d}{dx}$ leads to the differential operator (27)

$$P_n(\hat{p}) = P_n(-ih\frac{d}{dx}) = \sum_{j=0}^{n} a_j (-ih)^j \frac{d^j}{(dx)^j} \tag{28}$$

with constant coefficients.

$$\left(-\hat{E}_k + P_n^k(\hat{p})\right) \phi(x,t) = 0, \; k = 1,2, \tag{29}$$

where $\hat{E} = ih\frac{\partial}{\partial t}$.

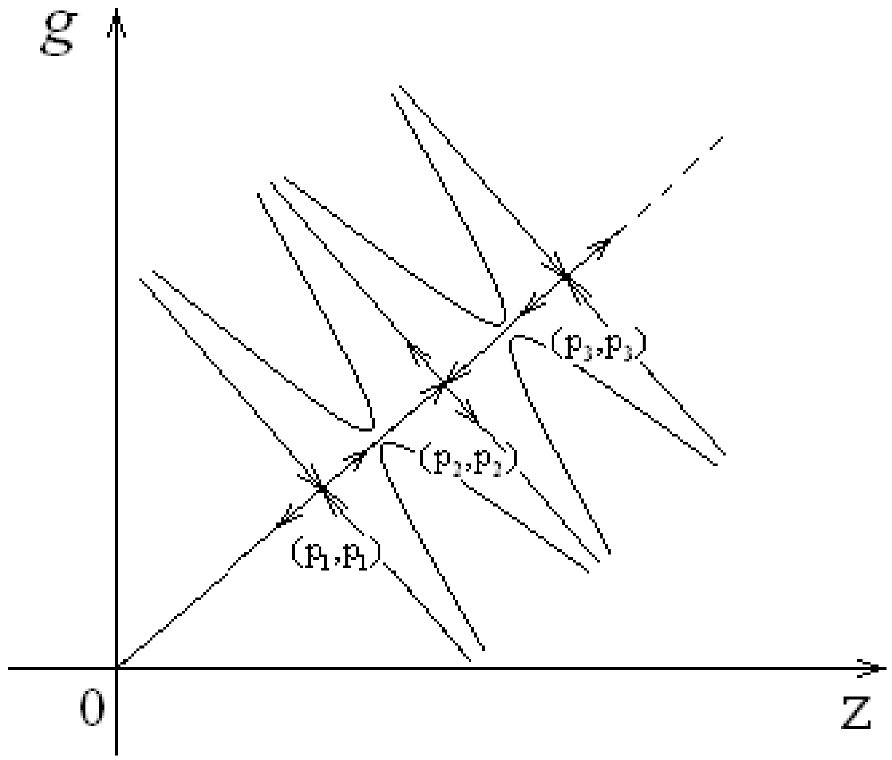

Figure 1: The trajectories of hyperbolic dynamical systems with attractive and saddle points in a plane.

4 Beck's type boundary conditions

We consider the functional boundary conditions

$$\phi_1(0,t,h) = S_1(\phi_2(0,t,h)), \quad \phi_1(0,t,h) = S_2(\phi_2(0,t,h)) \tag{30}$$

and the initial conditions

$$\phi_1(x,0,h) = h_1(x,h), \quad \phi_2(x,0,h) = h_2(x,h), \tag{31}$$

where $S_1, S_2 : R \to R$ are given functions. As follows from [12], such boundary conditions can describe the dynamics of a kicked free particle moving in quantum box with a double-well surface potentials. A corresponding classical case it has been considered in [8]. Indeed, as noted by Beck [12], we 'Though we will usually call the dynamical variable in our equations the velocity of a particle, our approach is much more general. Double-well potentials have many applications in physics, in subject areas as diverse as chemical kinetics, non-equilibrium thermodynamics, elementary particle physics and cosmology'.

Indeed, at time t the free particle gets a strength $c = a + ib$ in x - direction. Consider the velocity $v^-(t) = (u^-(t), w^-(t))$ and $v^+(t) = (u^+(t), w^+(t))$ before and after the kick. Then we have

$$u^+ = u^- + a, \quad w^+ = w^- + b \tag{32}$$

that is equivalent

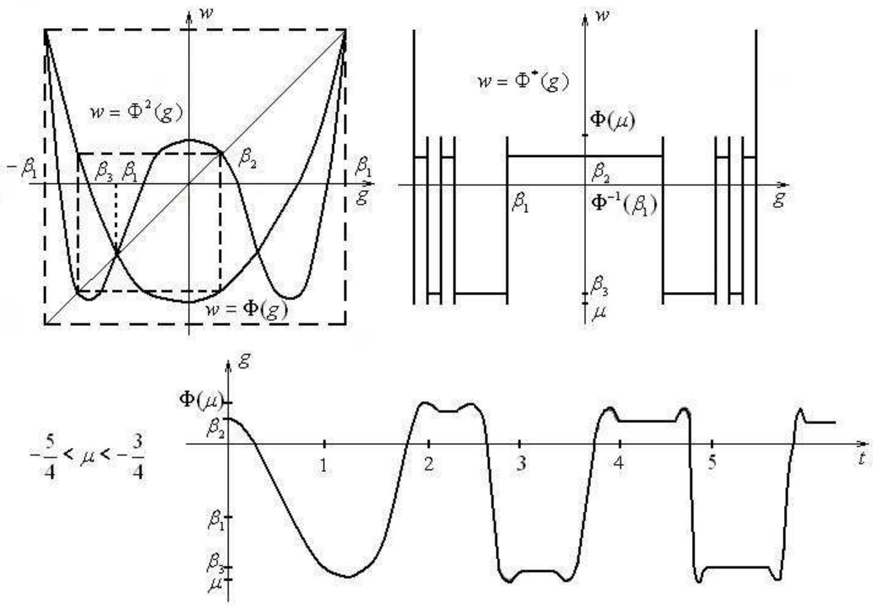

Figure 2: Limit solutions of relaxation type.

$$z^{+} = z^{-} + c, \tag{33}$$

where $c = a + ib$ ia a complex constant. Next, we assume that the strength is acting at each of two flat walls in a quantum box. Then we can consider a generalization of (33) on nonlinear case so that

$$\psi_1(0,t) = \Phi(\psi_2(0,t)), \quad \psi_2(l,t) = \psi_1(l,t)) \tag{34}$$

where $\Phi : I \to I$ is a given function, I is an open bounded interval. Here, $\psi_1 = z^{+}$ and $\psi_2 = z^{-}$. The index $(\pm)$ labels quantities before $(-)$ and after $(+)$ the kick. If $\Phi := Id$, where Id is identical map, then we obtain linear boundary conditions of type (33).

In [12] it is shown that in unbounded homogeneous space the complex nonlinear mappings Φ arise as stroboscopic mappings of certain classical particle dynamics. In a sense, that it has been studied the deterministic version of a typical Langevin problem. Generalization of [12] it has been considered in [11] on example on $2D$ - dimensional initial value boundary problem for the Liouville equation with nonlinear dynamic boundary conditions which describes velocity of changing on time of the probability of particles at walls that confines the particles. These velocities are nonlinear functions of the density of the probability of particles to occupied the flat walls. The attractor of the problem has been constructed. This attractor contains periodic piecewise constant functions with finite, countable or uncountable points of discontinuities on a period, which propagates along characteristics of the Liouville equation. We call such elements of the attractor by the distributions of relaxation, pre-turbulent and turbulent type, respectively. There are also random distributions of particles, which can be produced by the nonlinear feedback on the walls. The results has been obtained by the reduction of the problem to dynamical system which is described by system of difference equations, depending on coordinates and momenta as of parameters. It is shown that the changing of these parameters leads to period doubling bifurcations of elements of the attractor on 4 - dimensional torus. The problem is solved in class of quasi-periodic functions.

The main contribution in behaviour of solutions of IVBP include boundary conditions in complex space because we assume that equations of quantum mechanics are linear. next, these equations can be reduced in WKB - approximation to a canonical system, which represents coupled system of the Hamilton-Jacobi and transport equations fore phases and amplitudes, respectively. equations and the system of transport equation. The problem is to reduce boundary conditions for quantum equation to the boundary conditions for classical canonical equations. The corresponding example has been done in [12] for the problem which describes the dynamics of a charged particle moving in some arbitrary potential and a magnetic field under the influence of kicks.

5 Decomposition on amplitudes and phases

In this section, it will be discussed a problem of decomposition of density u as

$$u = u_1 e^{i\tau S_1} + ... + u_k e^{i\tau S_k}, \tag{35}$$

where $\tau = 1/h$. The problem is to fined phases S_j. If $j = 1$ then we have the known WKB - decomposition. If in the series is unique term then appears only phase factor. But for many terms the choice of relative phases for the Cauchy problem is important. Below it will be shown a special procedure for the determination of phases.

We begin with the Cauchy problem. Let a solution is

$$u(x,t) = \sum_{l=0}^{\infty} h^l \phi_l(x,t). \tag{36}$$

As example, consider the initial problem

$$\left(\hat{E} + P_n(\hat{)}\right) \psi(x, t = 0, \tag{37}$$

$$\psi(x,0) = e^{ih(\lambda_1 x + \lambda_2 t)} \phi_0 x. \tag{38}$$

Then solutions are

$$y(x,t) = e^{ih(\lambda_1 x + \lambda_2 t)} \phi(x,t). \tag{39}$$

Substituting (39) into equation

$$(-ih\frac{\partial}{\partial t} + P_n(-ih\frac{\partial}{\partial x}))y(x,t) = 0, \tag{40}$$

we arrive at

$$\left[(\lambda_2 - ih\frac{\partial}{\partial t}) + P_n(\lambda_1 - ih\frac{\partial}{\partial x})\right]\phi(x,t) = 0. \tag{41}$$

Initial conditions give

$$y(x,0) = e^{\frac{i}{h}\lambda_1 x}\phi(x,0) = e^{\frac{i}{h}\alpha x}\phi_0(x). \tag{42}$$

From (42) it follows that

$$\lambda_1 = \alpha, \quad \phi(x,0) = \phi_0(x). \tag{43}$$

Now we note that the term

$$(\lambda_2 - ih\frac{\partial}{\partial t}) + P_n(\lambda_1 - ih\frac{\partial}{\partial x}) \tag{44}$$

can be obtained from the function

$$(\lambda_2 - ihE') + P_n(\lambda_1 - ihp') \tag{45}$$

by the formal transformation $E' \to \partial/\partial x$. Now we decompose functional (45) in the Taylor series on h so that

$$F(h) = \sum_{k=0}^{n} \frac{h^k}{k!}\frac{d^k}{dh^k}F(h)\bigg|_{h=0} = (\lambda_2 + P_n(\lambda_1)) + h(-iE' - \frac{\partial P_n}{\partial p}(\lambda_1)p') + \sum_{k=2}^{n}\frac{h^k}{k!}(-i)^k\frac{\partial^k P_n}{\partial p^k}(\lambda_1)(P')^k. \tag{46}$$

Then the transformation $E' \to \frac{\partial}{\partial t}$ and $p' \to \frac{\partial}{\partial x}$ leads to

$$(\lambda_2 - ih\frac{\partial}{\partial t} + P_n(\lambda_1 ih\frac{\partial}{\partial x}) = (\lambda_2 + p_n(\lambda_)) - ih(\frac{\partial}{\partial t} + \frac{\partial P_n}{\partial p}(\lambda_1)\frac{\partial}{\partial x} + \sum_{k=2}^{n}\frac{(-ih)^k}{k!}\frac{\partial^k P_n}{\partial p^k}(\lambda_1)\frac{\partial^k}{\partial x^k}. \tag{47}$$

From (47) it follows that relation (47) can be rewritten as

$$(\lambda_2 + P_n(\lambda_1))\phi(x,t) - ih\left(\frac{\partial\phi}{\partial t} + \frac{\partial P_n}{\partial p}(\lambda_1)\frac{\partial\phi(x,t)}{\partial x}\right) + \sum_{k=2}^{n}\frac{(-ih)^k}{k!}\frac{\partial^k P_n}{\partial p^k}(\lambda_1)\frac{\partial^k\phi(x,t)}{\partial x^k} = 0. \tag{48}$$

From (277) with help of choice of constant λ_1, λ_2 and a function $\phi(x,t)$ it is impossible to obtain an exact solution of the problem. But equating to zero more number of terms in expansion (48) we can get asymptotic solution on $h\,(h \to 0)$.

Thus we have the equation

$$\lambda_2 + P_n(\lambda_1) = 0 \tag{49}$$

and the equation

$$\frac{\partial\phi}{\partial t} + \frac{\partial P_n}{\partial p}(\lambda_1) + \frac{\partial\phi}{\partial x} = 0. \tag{50}$$

Equation (49) has a solution

$$S(x,t) = \alpha x - P_n(\alpha)t, \quad \lambda_1 = \alpha, \quad \lambda_2 = -P_n(\alpha). \tag{51}$$

Equation (49) can be rewritten in more clear form which is equivalent (50). Indeed, consider at plane (x,t) a vector-field v with coordinates which are independent from x,t. Such vector-field is

$$v=\left(\frac{\partial P_n^k}{\partial p_k}(\lambda_1),1\right),\ k=1,2. \tag{52}$$

It means that in left part of equation (50) the derivative along trajectories of vector-field v exists. Then the transport equation is ODE

$$\frac{d\phi}{dt}=0 \tag{53}$$

where $\frac{d}{dt}$ is a derivative along trajectories of vector-field. From (53) it follows that ϕ must be constant along trajectories. The transport equation allows to obtain a solution with accuracy $O(h^2)$. To obtain following terms of asymptotic series, we must find $\phi(x,t)$ as formal power series on h. For ϕ_0 we again obtain the transport equation. Then the right part of ODE is order $O(h^s)$ for each integer $s>0$.

6 Complex transport equations in the first approximation

Next, we consider a function $\phi_1(x,t)$. Then, with accuracy of order $O(h^2)$, we get the equation:

$$\frac{\partial\phi_1}{\partial t}+\frac{\partial P_n}{\partial p}(\lambda_1)\frac{\partial\phi_1}{\partial x}=-\frac{i}{2}\frac{\partial^2P_n}{\partial p^2}(\lambda_1)\frac{\partial^2\phi_0}{\partial x^2} \tag{54}$$

which with help of the determination $\frac{d}{dt}$ can be written as

$$\frac{d\phi_1}{dt}=-\frac{i}{2}\frac{\partial^2P_n}{\partial p^2}(\lambda_1)\frac{\partial^2\phi_0}{\partial x^2}. \tag{55}$$

If $\phi_0(x,t)$ has been determined earlier then the integration of equation (55) allows to obtain a function $\phi_1(x,t)$.

Further, we consider terms of the equation which have orders $h^3,h^4,...,$. Then we obtain a recurrent system of equations, which determine functions $\phi_s(x,t)$, where each successive function can be obtained from the previous function with help of integration along vector-field v.

7 Systems of linear quantum equations with nonlinear boundary conditions

Consider the following system of equations

$$-h\frac{\partial\psi_k}{\partial t}+H_k(x,p,t)\psi_k=0,\ k=1,2, \tag{56}$$

with the initial conditions

$$\psi_k(x,0) = e^{\frac{i}{h}} S_0^k(x)\psi_0(x), \tag{57}$$

and the boundary conditions

$$\psi_1\bar{\psi}_1 = \Phi_1(\psi_2\bar{\psi}_2)\Big|_{x=0}, \quad \psi_2\bar{\psi}_2 = \Phi_2(\psi_1\bar{\psi}_1)\Big|_{x=1} \tag{58}$$

where Φ_1, Φ_2 are given functions. Here, $\bar{\psi}$ is conjugate quantity to ψ. Hamiltonian of a problem $H_k(x,p,t)$ satisfies to the estimation

$$\Big|D_x^\alpha D_p^\beta H_k(x,p,t)\Big| \le C_{\alpha,\beta}(1+|x|+|p|)^m \tag{59}$$

where $m > 0$ is a fixed number, α, β are multi indexes, $C_{\alpha\beta}$ are constants.

If $H_k = P_n^{(k)}$ then with accuracy $O(h^2)$ the problem can be reduced to the system of equations

$$\frac{\partial \phi_1^0}{\partial t} + \frac{\partial P_n^1}{\partial p}(\lambda_1)\frac{\partial \phi_1^0}{\partial x} = 0, \tag{60}$$

$$\frac{\partial \phi_2^0}{\partial t} + \frac{\partial P_n^2}{\partial p}(\lambda_2)\frac{\partial \phi_2^0}{\partial x} = 0 \tag{61}$$

with the boundary conditions

$$|\phi_1^0|^2 = \Phi_1(|\phi_2^0)|^2|_{x=0} \quad |\phi_2^0|^2 = \Phi_2(|\phi_1^0)|^2|_{x=1}, \tag{62}$$

and the initial conditions

$$\phi_k^0(x,0) = h_k(x),\ k = 1,2. \tag{63}$$

A solution has the form

$$u_1^0(x,t) = y(t - x/\lambda_1), \quad u_2^0(x,t) = y(t + x/\lambda_2), \tag{64}$$

where $\lambda_{1,2} \to \frac{\partial P_n^{1,2}}{\partial p}(\lambda_{1,2})$,$n = 0,1,2,...$ are coefficients in the corresponding hyperbolic equations. We assume that $\lambda_1\lambda_2 < 0$. Then from [8, 23] we arrive at

$$y(t+2\Delta) = \Phi(y(t)),\ t \in [-1,\infty), \quad \Delta = l/V_1 + l/V_2, \tag{65}$$

with an initial condition

$$y(t)|_{[-1,1)} = h(t), \tag{66}$$

where $h(t) = \phi_1(-t)$ for $t \in [-1,0)$ and $h(t) = \phi_2(t)$ for $t \in [0,1)$. Difference equation can be obtained by simple substitution of a solution in form (64) into the boundary conditions. Here, Φ belongs to a class $C^2(I,I)$, the map is structural stable. Particularly, we can consider well-known unimodal maps [3], for example, the quadratic map $u \mapsto u^2+\mu$. For some $\mu \in R$, such maps have infinite number of periodic points. Note that a point u and a trajectory $\mathcal{O}(u)$ is called periodic of a period m if $f^{(m)}(u) = u,\ f^{(j)}(u) \ne u,\ 0 < j < m$. For example, a periodic trajectory of period 2 contains two points $u_0, u_1 = f(u_0)\ f^{(2)}(u_0) = u_0,\ f^{(2)}(u_1) = u_1)$. For

$\mu = -2$ the map has invariant measure which is absolutely continuous with respect to the Lebesgue measure. It means trajectories of corresponding dynamical system are 'stochastic'.

In structural stable case, we define the separator D of Φ as a set $D = \bigcup_{n\geq 0} h^{-n}\bar{P}^{-}$. Here, $\bar{P}^{-}$ is closer of a set P^{-} where P^{-} is a set of repelling points of the map. Separator represents nowhere dense on interval I closed set of the Lebesque measure zero, which is finite, countable or uncountable. Particularly, there is the following theorem [23]: D is uncountable if and only if Φ has circles with periods which are different from 2^i, $i = 0, 1, ...$. Using D, we can construct a set $\Gamma = \tilde{h}^{-1}(D)$ where $\tilde{h}$ depends on initial data of the boundary problem. In structural stable case, $h(t)$ satisfies to the condition $h(t) \neq 0, \quad t \in \Gamma$. Then topological properties of Γ are identical to the topological properties of the separator D. Γ is closed and nowhere dense in $[0, 1]$ and measure $meas\,(\Gamma) = 0$. Γ determines a set a point of 'discontinuities' for solutions of canonical system of equations in the zero approximation (as $h = 0$).

The main statement of the present paper is that solutions of IVBP for canonical system of equations are asymptotically stable in Skorohod or Hausdorff metrics if the small parameter $h < h_0$, where h_0 determines by parameters of the quantum problem. The Hausdorff metric is well-known. It is distance in corresponding topology between graphics of solutions. This metric is applied to deterministic solutions. The Skorohod metric can be applied to the random solutions which represent an attractor of the problem. The Skorohod metric is [23]

$$s(v, \tilde{v}) = \sup_{\alpha\in\Lambda}\{||v \circ \alpha - \tilde{v}||_{C^0(\Pi, I\times I)} + ||\alpha - Id||_{C^0(\Pi,\Pi)}\} \tag{67}$$

where Λ is a set of homeomorphisms, Id is identical homeomorphism. Below it will be shown that in zero approximation solutions of the canonical problem are stable with respect to perturbations of initial and boundary conditions in Skorohod and Hausdorff metrics. It must be noted that there exist specific 'stability', and for specific initial conditions, which determine 'solitons'. Indeed, initial functions must be from a region of attraction in the zero approximation. Then it can be proved that all solutions from an attractive region tend for all following approximation to a limit solution in zero approximation as $t \mapsto \infty$ with accuracy $O(h^2)$ for first approximation, with accuracy $O(h^3)$ for second approximation, and so on. In this case, we have deal with approximated attractor for the origin IVBP. We can confined itself by the approximation with accuracy $O(h)$.

A limit solution can be found, step by step, by the formula

$$p(t) = \Phi^{4m-1} \circ \Phi^{\Delta} \circ h(t - 2(2m - 1)),\ t \in [4m - 3, 4m - 1),\ m = 1, 2, ..., \tag{68}$$

where m is least common multiple of periods of attractive circles of the map $\Phi := \Phi_1 \circ \Phi_2$. A set of points of 'discontinuities' is determined by the formula

$$\Upsilon_{R^+} = \bigcup_{n=1}^{\infty}\{t : t - 2n \in \Gamma\}. \tag{69}$$

8 The first approximation

Consider one of the components $\phi_1(x, t)$ for the system of transport equations. Initially, we selected in the decomposition on small parameter h of the origin quantum equations terms of order h^2. As a result, we obtain a system of uncoupled linear equations, which determine perturbations for a zero approximation with accuracy $O(h^2)$:

$$\frac{\partial \phi_1^1}{\partial t} + \frac{\partial P_n^1}{\partial p}(\lambda_1)\frac{\partial \phi_1^1}{\partial x} = -\frac{i}{2}\frac{\partial^2 P_n^1}{\partial p^2}(\lambda_1)\frac{\partial^2 \phi_0^1}{\partial x^2}, \tag{70}$$

$$\frac{\partial \phi_2^1}{\partial t} - \frac{\partial P_n^2}{\partial p}(\lambda_2)\frac{\partial \phi_2^1}{\partial x} = -\frac{i}{2}\frac{\partial^2 P_n^2}{\partial p^2}(\lambda_2)\frac{\partial^2 \phi_0^2}{\partial x^2}. \tag{71}$$

The complex functions ϕ_1^1, ϕ_2^1 arise because we used 'incorrect' decomposition ([9], formula (19)):

$$\varphi(x,t) \equiv \sum_{j=0}^{\infty} h^j \varphi_j(x,t). \tag{72}$$

Correct decomposition is

$$\varphi(x,t) \equiv \sum_{j=0}^{\infty} (ih)^j \varphi_j(x,t). \tag{73}$$

Indeed, from strong theory it follows that common representation of a solutions (on characteristics) is ([1], p.79):

$$u = u_0 e^{i\tau S_0} + u_1 e^{i\tau S_1} + ... + u_k e^{i\tau S_k} \tag{74}$$

where $\tau = 1/h$. The difficulty lies in the fact that this expression is not one-to-one for the choice of phases S_j. If there were only one term in the sum, then this arbitrariness would lead only to (a rather harmless) phase factor. However, if in the sum of several terms, then the choice of relative phases is essential. The correct terms for the oscillating terms in (74) are obtained from the projections of semi-density ϱ that is a solution of a transport equation by multiplication on a constant phase factor. These factors are different from each to other by degree i.

This problem can be studied on the lagrange manifolder Λ (see,[1], p.79). For solving of the problem, the method of stationary phase has been applied.

Indeed, the boundary conditions are

$$\phi_1^1 = \Phi_1(\phi_2^1)|_{x=0}, \quad \phi_2^1 = \Phi_2(\phi_1^1)|_{x=l}. \tag{75}$$

The main observation is that in (70),(71) the right parts tend to zero as $t \to \infty$ for almost all characteristic of the difference equations. Then we may think that, as $t \to \infty$ solutions of the boundary problem tend to solutions for the non-perturbed equations:

$$\frac{\partial \phi_1^1}{\partial t} + \frac{\partial P_n^1}{\partial p}(\lambda_1)\frac{\partial \phi_1^1}{\partial x} = 0, \tag{76}$$

$$\frac{\partial \phi_2^1}{\partial t} - \frac{\partial P_n^2}{\partial p}(\lambda_2)\frac{\partial \phi_2^1}{\partial x} = 0. \tag{77}$$

Then the problem is reduced to the Sharkovsky problem ([23], p.247) (without right parts in the hyperbolic equations) with nonlinear boundary conditions:

$$\phi_1^0 + h\phi_1^1 = \Phi_1(\phi_2^0 + h\phi_2^1)|_{x=0} \quad \phi_2^0 + h\phi_2^1 = \Phi_2(\phi_1^0 + h\phi_1^1)|_{x=1}. \tag{78}$$

Then from (78) we arrive at

$$\phi_1^0 + h\phi_1^1 = \Phi_1(\phi_2^0) + h\Phi_1'(\phi_2^0)\phi_2^0|_{x=0}, \quad \phi_2^0 + h\phi_2^1 = \Phi_2(\phi_1^0) + h\Phi_2'(\phi_1^0)\phi_2^0|_{x=l}. \tag{79}$$

9 Asymptotic for quasi-invariant initial data

If $h = 0$, we obtain the well-known IVBP with typical attractors which represent piecewise constant periodic function with finite or infinite lines of discontinuities that lie on characteristics of hyperbolic equations. Define, for simplicity, $\phi_1^0 = u_1$, $\phi_2^0 = u_2$. Then we find that

$$\Phi_1 \circ \Phi_2(u_1) = u_1, \quad u_2 = \Phi_2(u_1), \quad \Phi_1(u_2) = u_1. \tag{80}$$

Next we define $u_1(x, 0) \equiv a_1$, where a_1 is a single attractive fixed point on interval I of the map $f := \Phi_1 \circ \Phi_2 : I \to I$. Put $u_2(x, 0) = \Phi_2(u_1(x, 0))$. Then the problem can be reduced to the difference equation [23]

$$u_1(t) = f(u_1(t - \Delta), \quad \Delta = l/V_1 + l/V_2, \tag{81}$$

where V_1, V_2 coefficients in the hyperbolic equations, $V_1, V_2 > 0$. Since $f \in C^2(I, I)$ has a single point $a_1 \in P^+$, where P^+ is a set of attractive fixed points, from (81) it follows that $(u_1 \to a_1, u_2 \to \Phi_2(a_1))$ as $t \to \infty$. Further, from structural stability of the map f it follows that the same is true if $(u_1(x, 0), u_2(x, 0)) \in (O_\delta(P^+, \Phi_1(P^+))$, where O_δ are some neighbourhoods of these points. Next, it is known [23] that if f is monotone (without extremum) then a set $P^+ = (a_1, ..., a_n)$ is finite. Values of piecewise constant limit function $p \in P^+$ almost all points, excluding finite number of 'jumps', where value of p is 'vertical interval'. In this case, we have deal with solutions of relaxation type.

10 Asymptotic of limit solutions

As a result, solutions of the transport equations $(41''), (42'')$ can be represented as

$$\phi_k(x, t) = \phi_0^1(t - (P_n^k(\lambda^k))' x), \quad k = 1, 2. \tag{82}$$

Then from (82) it follows that solutions of equations with perturbations can be represented as

$$\phi_k(x, t) = \phi_0^k(t - (P_n^k(\lambda^k))' x) + ih\phi_1^k(t - (P_n^k)'(\lambda_k)x), \quad k = 1, 2. \tag{83}$$

Asymptotic of these solutions are

$$\phi_k(x, t) = e^{i\pi/2} h\phi_1^k(t - (P_n^1)'(\lambda_k)x), \quad k = 1, 2. \tag{84}$$

11 The second approximation

In this case, with accuracy $O(h^3)$, we obtain a similar system of equations

$$\frac{\partial \phi_1^2}{\partial t} + \frac{\partial P_n^1}{\partial p}(\lambda_1)\frac{\partial \phi_1^2}{\partial x} = -\frac{i}{2!}\frac{\partial^2 P_n^1}{\partial p^2}(\lambda_1)\frac{\partial^2 \phi_1^1}{\partial x^2} - \frac{i}{3!}\frac{\partial^3 P_n^1}{\partial p^3}(\lambda_1)\frac{\partial^3 \phi_0^1}{\partial x^3}, \tag{85}$$

$$\frac{\partial \phi_2^2}{\partial t} + \frac{\partial P_n^1}{\partial p}(\lambda_1)\frac{\partial \phi_2^2}{\partial x} = -\frac{i}{2!}\frac{\partial^2 P_n^2}{\partial p^2}(\lambda_2) - \frac{\partial^2 \phi_2^2}{\partial x^2} - \frac{i}{3!}\frac{\partial^3 P_n^2}{\partial p^3}(\lambda_2)\frac{\partial^3 \phi_0^2}{\partial x^3}. \tag{86}$$

Not that the second derivatives of zero and first approximations tend to zero as time tends to infinity for almost all points on characteristics. It means that limit asymptotic can be described by the limit equations:

$$\frac{\partial \phi_1^2}{\partial t} + \frac{\partial P_n^1}{\partial p}(\lambda_1)\frac{\partial \phi_1^2}{\partial x} = \frac{1}{2!}\frac{\partial^2 P_n^1}{\partial p^2}(\lambda_1)\frac{\partial^2 \phi_0^1}{\partial x^2}, \tag{87}$$

$$\frac{\partial \phi_2^2}{\partial t} + \frac{\partial P_n^1}{\partial p}(\lambda_1)\frac{\partial \phi_2^2}{\partial x} = \frac{1}{2!}\frac{\partial^2 P_n^2}{\partial p^2}(\lambda_2)\frac{\partial^2 \phi_0^2}{\partial x^2}. \tag{88}$$

Remind that for next approximations boundary conditions have the form

$$\phi_1^0 + h\phi_1^1 + h^2\phi_2^1 = \Phi_1(\phi_2^0 + h\phi_1^1 + h^2\phi_2^1)|_{x=0}, \tag{89}$$

$$\phi_2^0 + h\phi_2^1 + h^2\phi_2^2 = \Phi_2(\phi_1^0 + h\phi_1^1 + h^2\phi_1^2)|_{x=0}. \tag{90}$$

Then, as above, on limit solution (p_1, p_2), where $p_1 \in P^+$, where P^+ belongs to a set of attractive points of a map $\Phi_1 \circ \Phi_2$, $p_2 = \Phi_2(p_1)$, we obtain the linearised boundary conditions (89),(90)

$$\phi_2^1 = \Phi_1'(p_1)(\phi_2^1|_{x=0}, \quad \phi_2^1 = \Phi_2'(p_2)(\phi_1^1|_{x=1}. \tag{91}$$

Define $\phi_1^2 \mapsto i\phi_1^2$ $\phi_2^2 \mapsto i\phi_2^2$. As a result, for the non perturbed system (85),(86) we obtain the difference equation

$$\phi_1^2(1, t+\Delta) = \Phi_1'(p)\Phi_2'(p)(\phi_1^2(1,t)), \quad \Delta = l/V_1 + l/V_2. \tag{92}$$

Since $|\Phi_1'(p)\Phi_2'(p)| < 1$, then $\phi_1^2(1,t) \to 0$ as $t \to \infty$. For non perturbed system, the function Φ_1^1 has the same properties.

Below it will be shown that for perturbed system the functions $\phi_1^k(x,t)$, $k = 1, 2$ have the same property. Formally, it is possible because in the right part of perturbed system there is the factor $\partial^2\phi_0^k(x,t)\partial x^2$, $k = 1, 2$, which tend to zero as $t \to \infty$.

Then, with accuracy $O(h^2)$, we obtain the following system

$$\frac{\partial u_1}{\partial t} + \frac{\partial P_n^1}{\partial p}(\lambda_1)\frac{\partial u_1}{\partial x} = F_1(\phi_0^1), \tag{93}$$

$$\frac{\partial u_2}{\partial t} - \frac{\partial P_n^2}{\partial p}(\lambda_2)\frac{\partial u_2}{\partial x} = F_2(\phi_0^1), \tag{94}$$

where $u_1 = \phi_1^1$, $u_2 = \phi_2^1$.

The boundary conditions are

$$u_1 = \Phi_1'(p)u_2|_{x=0}, \quad u_2 = \Phi_2'(p)u_1|_{x=1}. \tag{95}$$

The problem of existence and uniqueness of solutions has been considered in [4]. Next, by integration along characteristics we can show that this solution satisfies to a system of integro-difference equations:

$$\begin{aligned} u_1(l,t_0) = u_1(0,t_0-l/V_1) + V_1\int_{t_0-l/V_1}^{t_0}\frac{\partial^2\phi_0^1}{\partial x^2}(V_1t - V_1(t_0-l/V_1),t)dt = \\ \Phi_1(u_2(0,t_0-l/V_1) + V_1\int_{t_0-l/V_1}^{t_0}\frac{\partial^2\phi_0^1}{\partial x^2}(V_1t - V_1(t_0-l/V_1),t)dt = \\ \Phi_1\left(\Phi_2(u_1(0,t_0-l/V_1-l/V_2)) + V_2\int_{t_0-l/V_1-l/V_2}^{t_0-l/V_1}\frac{\partial^2\phi_0^2}{\partial x^2}(V_2t - V_2(t_0-l/V_1),t)dt\right) + \\ V_1\int_{t_0-l/V_1}^{t_0}\frac{\partial^2\phi_0^1}{\partial x^2}(V_1t - V_1(t_0-l/V_1),t)dt, \end{aligned} \tag{96}$$

$$\begin{aligned} u_2(l,t_0) = u_2(l,t_0-l/V_2) + V_2\int_{t_0-l/V_2}^{t_0}\frac{\partial^2\phi_0^2}{\partial x^2}(V_2t + l/V_2 - t_0,t)dt = \\ \Phi_2\left(\Phi_1(u_2(l,t_0-l/V_2-l/V_1)) + V_1\int_{t_0-l/V_2-l/V_1}^{t_0-l/V_2}\frac{\partial^2\phi_0^1}{\partial x^2}(V_1(t-t_0+l/V_2+l/V_1),t)dt\right) + \\ V_2\int_{t_0-l/V_2}^{t_0}\frac{\partial^2\phi_0^2}{\partial x^2}(V_2t + l/V_2 - t_0,t)dt. \end{aligned} \tag{97}$$

From these equations we arrive at

$$\begin{aligned} u_1(l,t_0) = u_1(0,t_0-l/V_1) + l\frac{\partial^2\phi_0^1}{\partial x^2}(t_0-l/V_1) = \Phi_1(u_2(0,t_0-l/V_1) + l\frac{\partial^2\phi_0^1}{\partial x^2}(t_0-l/V_2) = \\ \Phi_1\left(\Phi_2(u_1(0,t_0-l/V_1-l/V_2)) + l\frac{\partial^2\phi_0^1}{\partial x^2}(t_0-l/V_1)\right) + l\frac{\partial^2\phi_0^2}{\partial x^2}(t_0-l/V_2), \end{aligned} \tag{98}$$

$$\begin{aligned} u_2(l,t_0) = u_2(l,t_0-l/V_2) + l\frac{\partial^2\phi_0^1}{\partial x^2}(t_0-l/V_1) = \\ \Phi_2\left(\Phi_1(u_2(l,t_0-l/V_2-l/V_1)) + l\frac{\partial^2\phi_0^1}{\partial x^2}(t_0-l/V_1)\right) + l\frac{\partial^2\phi_0^1}{\partial x^2}(t_0-l/V_2). \end{aligned} \tag{99}$$

Note that one of components $\phi_0^{1,2}$ satisfies to the difference equation

$$u(\zeta) = G(u(\zeta-\Delta) \tag{100}$$

where $\Delta = l/V_1 + l/V_2$ and $G := \Phi_1\Phi_2$, or $G := \Phi_2\Phi_1$. Since G is hyperbolic, from (100) it follows that

$$u'(\zeta) = G'(u(\zeta - \Delta))u'(\zeta - \Delta) \tag{101}$$

where $u'(\zeta) \in O_\gamma(P^+)$ and P^+ is a set of attractive points of the map. Then

$$u''(\zeta) = G''(u(\zeta - \Delta))(u'(\zeta - \Delta))^2 + G'(u(\zeta - \Delta))u''(\zeta - \Delta)). \tag{102}$$

From (102) it follows that

$$|u'(\zeta)| \leq \lambda |u'(\zeta - \Delta)|, \tag{103}$$

where $\lambda < 1$. Hence, $|u'(\zeta)| \to 0$ as $t \to +\infty$. Then from (102 we obtain that $|u''(\zeta)| \to 0$ as $t \to +\infty$. At each fixed point a linearised equation

$$u(\zeta) = \lambda u(\zeta - \Delta) \tag{104}$$

has a positive solution $u(\zeta) = u(\zeta_0)e^{k(t-t_0)}$, where $k = \frac{1}{\Delta}\ln\lambda$ and $\lambda < 1$. Thus

$$\frac{\partial^2 \phi_0^j}{\partial x^2}(t_0 - l/V_j) = \left(\frac{k_j}{V_j}\right)^2 e^{k_j(t_0 - l/V_j)}, \quad j = 1, 2, \quad k_j < 0. \tag{105}$$

From (105),(98), (99) we arrive at

$$u_1(l, t_0) = u_1(0, t_0 - l/V_1) + l\left(\frac{k_1}{V_1}\right)^2 e^{k_1(t_0 - l/V_1)} = \Phi_1(u_2(0, t_0 - l/V_1) + l\left(\frac{k_2}{V_2}\right)^2 e^{k_2(t_0 - l/V_2)} = \tag{106}$$
$$\Phi_1\left(\Phi_2(u_1(0, t_0 - l/V_1 - l/V_2)) + \left(\frac{k_1}{V_1}\right)^2 e^{k_1(t_0 - l/V_1)}\right) + l\left(\frac{k_2}{V_2}\right)^2 e^{k_2(t_0 - l/V_2)},$$

$$u_2(l, t_0) = u_2(l, t_0 - l/V_2) + l\left(\frac{k_1}{V_1}\right)^2 e^{k_1(t_0 - l/V_1)} = \tag{107}$$
$$\Phi_2\left(\Phi_1(u_2(l, t_0 - l/V_2 - l/V_1)) + l\left(\frac{k_1}{V_1}\right)^2 e^{k_1(t_0 - l/V_1)}\right) + l\left(\frac{k_1}{V_1}\right)^2 e^{k_1(t_0 - l/V_1)}.$$

Without loss of generality, we assume that $\Phi_2 := Id$, where Id is identical map. Then from the above equations we obtain that

$$u_1(l, t_0) = \Phi_1\left(u_1(l, t_0 - l/V_1 - l/V_2) + \left(\frac{k_2}{V_1}\right)^2 e^{k_1(t_0 - l/V_1)}\right) + l\left(\frac{k_2}{V_2}\right)^2 e^{k_1(t_0 - l/V_2)}, \tag{108}$$

$$u_2(l, t_0) = \Phi_1(u_2(l, t_0 - l/V_2 - l/V_1)) + l\left(\frac{k_1}{V_1}\right)^2 e^{k_1(t_0 - l/V_1)} + l\left(\frac{k_1}{V_1}\right)^2 e^{k_2(t_0 - l/V_1)}. \tag{109}$$

Since $k_{1,2}$ are negative it is easy to see that exponential factors in these difference equations with non-autonomic perturbations tend to zero as $t \to +\infty$. Then it can be shown that asymptotic of solutions can be determined as asymptotic of limit difference equations

$$u_1(l, t_0) = \Phi_1 \left(u_1(l, t_0 - \Delta)\right) + l \left(\frac{k_2}{V_2}\right)^2 e^{k_1(t_0 - l/V_2)}, \tag{110}$$

$$u_2(l, t_0) = \Phi_1(u_2(l, t_0 - \Delta)), \quad \Delta = l/V_1 - l/V_2. \tag{111}$$

But we know that for unimodal map (with one extremum) solutions of this equations tend to piecewise constant periodic functions with finite or infinite points of discontinuities on a period.

12 Applications to non-coherent optical solitons

The phenomenon of appearing optical solutions determines by the dynamical balance between the concurrence of the two factors: (1) by the detention of the optical beam to expand your own media which is produced by the diffraction;(2) by the detention of the beam to restrict your own media due to self-focusing [21]. Experiments (see, [21]) show the possibility for existence of solitons which are spatial non-coherent and quasi monochromatic; (3) non-coherent together on spatial-temporal variables. By these experiments it has been initiated a set of theoretical works which concern to the non-coherent solitons (see,[20, 21]) . However, these works has been confined by the research of case (3). It means that the corresponding theory could not model can model, for example, non-coherent white light that is to study spatial-temporal coherent properties of the solitons and the evolution of the spectral density. In the present paper it has been studied this problem, and in this section it will be considered a simplest clear example of such situation. Indeed, below we consider the spatial-temporal on (x, t) light. We assum that: (4) spatial profile of light belongs to the interval of frequencies $[\omega, \omega + d\omega]$; (5) spatial correlation length (across of soliton) is always larger for low frequencies and smaller for high frequencies (see, [21]).

We begin the research from the following equation:

$$i\left(\frac{\partial f^\omega}{\partial z} + \theta \frac{\partial f^\omega}{\partial x}\right) + \frac{1}{2k_\omega}\frac{\partial^2 f^\omega}{\partial x^2} + \frac{k_\omega}{n_0}\delta n(I) f^\omega(x, z, \theta) = 0. \tag{112}$$

Here, f^ω is the coherent density (on the given frequency) of the optical beam; $k_\omega = n_0\omega/c$, where n_0 is refractive index, ω is frequency, c is velocity of light; θ determines angle between a direction of light (at plane (z, x)) and axes Oz.

Spatial-temporal cogerent properties of a beam may be researched in terms of the spectral density

$$B_\omega(x_1, x_2, z) = \int_{-\infty}^{+\infty} d\theta \exp\left[ik_\omega(x_1 - x_2)\right] f^\omega(x_1, z, \theta) f^\omega(x_2, z, \theta). \tag{113}$$

Note that equation (112) is equivalent to the corresponding equation (113).

We suppose that optical medium is dispersive. If we assume that $\partial \delta n(I)/\partial t \equiv 0$ then the dispersion may be included in the consideration with help of the dependence $n_0 = n_0(\omega)$. Then instead of the classical equation (112) we consider the Shrödinger equation (in laboratory system of coordinate) with optical source, we confined by $1D$ - dimensional case. Note that in [11] it has been considered semiconductor lasers or laser diodes. The laser is inverted carrier density system. There is the generation and recombination of 'solitons' which are recombine. The energy released can be produced by thermal recombination or optical photon recombination, which is used in semiconductor lasers. Note also that electronic oscillator is an electronic circuit that produces a periodic signal. Oscillators convert direct current to an alternative current signal. If we use the feedback oscillator, which increase which can increase amplitudes of signal then we obtain different boundary conditions for phases and amplitudes in the canonical equations.

For example, let us consider the region $0 < x < l$, $z \geq 0$ which is occupied by the resonator. The equations have the form

$$ih\left(\frac{\partial f}{\partial t} + \theta\frac{\partial f}{\partial x}\right) + \frac{h^2}{2k}\frac{\partial^2 f}{\partial x^2} + \frac{k}{n_0}\delta n(I) f = 0 \tag{114}$$

where index ω will be omitted.

Solutions of equation (114) will be find as $f := \varphi(\zeta, t)$, where $\zeta = t - x/V$. Then from (114)it follows the equation

$$-ih\frac{\partial \varphi}{\partial t} + +\frac{h^2}{2k}\frac{\partial^2 \varphi}{\partial \zeta^2} - \frac{k}{n_0}\delta n(I) f = 0, \tag{115}$$

Let $\bar{x} = \sqrt{k}x$. Then this equation can be written as

$$-ih\frac{\partial \varphi}{\partial t} + \frac{h^2}{2}\frac{\partial^2 \varphi}{\partial \zeta^2} - \frac{k}{n_0}\delta n(I) f = 0. \tag{116}$$

The Shrödinger equation has the form:

$$-ih\frac{\partial \varphi}{\partial t} + +\frac{h^2}{2}\frac{\partial^2 \varphi}{\partial \zeta^2} = 0 \tag{117}$$

with the special initial conditions

$$\varphi(x, 0) = \varphi_0 e^{iS_0(x)/h} \tag{118}$$

where V, S_0, φ_0 are smooth real functions. The Hamiltonian is

$$H(p, q) = \frac{p^2}{2} + V(q). \tag{119}$$

Asymptotic solutions of initial problem (117,118) have the form

$$\varphi(x, t) = e^{iS_0(x,t)/h}\varphi(x, t) \tag{120}$$

where unknown functions $S(x, t)$ and $\varphi(x, t)$ are smooth. Substituting (120) into (117) we obtain the equation:

$$[S_t + V(x) + \frac{1}{2}(S_x)^2]\varphi + (-ih)\big[S_x\varphi_x + \varphi_t + \frac{1}{2}\varphi S_{xx}] + \left(-\frac{ih}{2}\right)\varphi_{xx} = 0. \tag{121}$$

From (121) with accuracy $O(h^2)$ we arrive at

$$S_t + V(x(\zeta,t)) + \frac{1}{2}S_\zeta^2 = 0, \quad S(\zeta,o) = S_0(\zeta) \tag{122}$$

where $\varphi(\zeta,t)$ satisfies to the initial problem

$$\varphi + \varphi_\zeta S_\zeta + \frac{1}{2}\varphi S_{\zeta\zeta} = 0, \quad \varphi(\zeta,0) = \varphi_0(\zeta). \tag{123}$$

The result is

$$S_t + \frac{1}{2}S_\zeta^2 = 0. \tag{124}$$

A solution (124) has a form $S(\zeta,t) = \lambda_1 t + \lambda_2\zeta$ that leads to the algebraic relation

$$\lambda_1 + \lambda_2^2 = 0. \tag{125}$$

From (125) it follows that

$$S(\zeta,t) = \lambda_2\left(\zeta - \frac{1}{2}\lambda_2 t\right) \tag{126}$$

where λ_2 can be find from the relation

$$S(\zeta_{t=0},0) = \lambda_2\zeta_{t=0} = -\frac{\lambda_2}{\theta}x = S_0(x). \tag{127}$$

Thus, a phase has the form

$$S(x,t) = \lambda_1 t + \lambda_2\left(t - \frac{x}{\theta}\right). \tag{128}$$

Since the phase is linear, we have $_{\zeta\zeta} \equiv 0$ and $S_\zeta = 0$, and the equation has the form

$$\varphi(\zeta,t) := \varphi\left(t - \frac{\zeta}{\lambda_2}\right). \tag{129}$$

Now, we consider the boundary conditions

$$\varphi_t(0,t) = F_1[\varphi(0,t)], \quad \varphi_t(l,t) = F_2[\varphi(l,t)], \quad t > 0, \tag{130}$$

where F_1 and F_2 are given functions.

Assume that the system of ODE is integrable, so that there is an integral

$$W[\varphi(0,t),\varphi(l,t)] = \mu, \quad \mu \in R. \tag{131}$$

Suppose that there is an open bounded interval $I \subset R^+$ such that for all $\varphi(0,t),\varphi(l,t) \in I$ for each fixed $t > 0$ relation (131) is solvable so that

$$\varphi(l,t) = \Phi_\mu[\varphi(0,t)] \tag{132}$$

where $\Phi_\mu : I \mapsto I$ is unimodal map C^2 - class. Then

$$\varphi(l,t) = \varphi\left[\left(1 - \frac{1}{\lambda_2}\right)t + \frac{l}{\lambda_2\theta}\right], \tag{133}$$

$$\varphi(0,t) = \varphi\left[\left(1 - \frac{1}{\lambda_2}\right)t\right]. \tag{134}$$

Let $\lambda = 1 - \lambda_2^{-1}$, $t \mapsto \lambda t$ and $L = \frac{l}{\lambda_2\theta}$. Then functional quality (133) can be written as

$$\varphi(t + l) = \Phi_\mu\varphi(t), \quad -l < t < +\infty. \tag{135}$$

As a result, we obtain a difference equation [5, 6]. If Φ_μ is unimodal, structural stable and hyperbolic then a set of fixed points of this map is finite. Then there is a set of initial functions $h(t), t \in [-L, 0)$ such that solutions of the difference equation can be find, step by step, by iterations of the initial function $h(t)$ with help of Φ_μ. As a result, for $t \to \infty$ the iterations of $h(t)$ tend to a periodic piecewise constant function with finite or infinite 'points' of discontinuities Γ on a period. If Γ is finite then we say about oscillations of relaxation type. If Γ is countable then we have oscillations of pre-turbulent type. If Γ is uncountable then we have oscillations of turbulent type.

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