ON THE DENSITY OF ULAM SEQUENCES

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Abstract. In this paper we show under some special conditions that the natural density of Ulam numbers is zero.

1. Introduction

Ulam numbers are special infinite sequence of the natural numbers of the form $1, 2, \ldots, \mathbb{U}_n, \ldots$, where the n^{th} term in the sequence for $n \geq 3$ is the smallest number expressible as the sum of two earlier terms in exactly one way and larger in magnitude than the preceding terms. This sequence has several well-known properties and for many more of these, see [3]. The Ulam numbers and their properties including many of its statistics had also been studied by many authors (see [2]), not least of which are their density. The density is one of the most fundamental problems regarding this sequence, and Ulam is known to have conjectured the density is zeros [1]. In this paper we exploit the notion of the density of points on a combinatorial structure, which we choose to call **circles of partition**, to show that the natural density of the Ulam sequence $\mathbb{U}$ is zero. That is, we show that

Theorem 1.1. *Let $\mathbb{U}$ denotes the set of all Ulam numbers and $\mathcal{D}(\mathbb{U})$ denotes the natural density. If $\mathcal{D}(\mathbb{U}_{\mathcal{C}(\infty)})$ exists and*

$$\#\{(x,y)|\ x \in \mathbb{U},\ y \notin \mathbb{U},\ m \leq n,\ m \in \mathbb{U},\ m = x+y\}$$
$$\leq \#\{(x,y)|\ x \in \mathbb{U},\ y \notin \mathbb{U},\ m \leq n,\ m \notin \mathbb{U},\ m = x+y\}$$

with

$$\#\{(x,y)|\ \{x,y\} \cap \mathbb{U} \neq \emptyset,\ m \leq n,\ m \notin \mathbb{U},\ m = x+y\} \ll \frac{n^{1-\epsilon}}{2}$$

for some $\epsilon > 0$ then

$$\mathcal{D}(\mathbb{U}) = 0.$$

2. The Circle of Partition

In this section we introduce the notion of the circle of partition. We study this notion in-depth and explore some potential applications in the following sequel.

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Definition 2.1. Let $n \in \mathbb{N}$ and $\mathbb{M} \subset \mathbb{N}$. We denote with

$$\mathcal{C}(n, \mathbb{M}) = \{[x] \mid x, y \in \mathbb{M}, n = x + y\}$$

the Circle of Partition generated by n with respect to the subset $\mathbb{M}$. We will abbreviate this in the further text as CoP. We call members of $\mathcal{C}(n, \mathbb{M})$ as points and denote them by $[x]$. For the special case $\mathbb{M} = \mathbb{N}$ we denote the CoP shortly as $\mathcal{C}(n)$.

Definition 2.2. We denote the line $\mathbb{L}_{[x],[y]}$ joining the point $[x]$ and $[y]$ as an axis of the CoP $\mathcal{C}(n, \mathbb{M})$ if and only if $x + y = n$. We say the axis point $[y]$ is an axis partner of the axis point $[x]$ and vice versa. We do not distinguish between $\mathbb{L}_{[x],[y]}$ and $\mathbb{L}_{[y],[x]}$, since it is essentially the the same axis. The point $[x] \in \mathcal{C}(n, \mathbb{M})$ such that $2x = n$ is the **center** of the CoP. If it exists then it is their only point which is not an axis point. The line joining any two arbitrary point which are not axes partners on the CoP will be referred to as a **chord** of the CoP. The length of the chord joining the points $[x], [y] \in \mathcal{C}(n, \mathbb{M})$, denoted as $\mathcal{D}([x], [y])$ is given by

$$\mathcal{D}([x], [y]) = |x - y|.$$

Notation. We let

$$\mathbb{N}_n = \{m \in \mathbb{N} \mid \ m \leq n\}$$

be the **sequence** of the first n natural numbers. Further we will denote

$$||[x]|| := x$$

as the **weight** of the point $[x]$ and correspondingly the weight set of points in the CoP $\mathcal{C}(n, \mathbb{M})$ as $||\mathcal{C}(n, \mathbb{M})||$. Let us denote the assignment of an axis $\mathbb{L}_{[x],[y]}$ to a CoP $\mathcal{C}(n, \mathbb{M})$ as

$$\mathbb{L}_{[x],[y]} \ \hat{\in}\ \mathcal{C}(n, \mathbb{M}) \text{ which means } [x], [y] \in \mathcal{C}(n, \mathbb{M}) \text{ and } x + y = n$$

and the number of axes of a CoP as

$$\nu(n, \mathbb{M}) := \#\{\mathbb{L}_{[x],[y]} \ \hat{\in}\ \mathcal{C}(n, \mathbb{M}) \mid x < y\}.$$

Obviously holds

$$\nu(n, \mathbb{M}) = \left\lfloor \frac{k}{2} \right\rfloor, \text{ if } |\mathcal{C}(n, \mathbb{M})| = k.$$

Remark 2.3. It is important to notice that a typical CoP need not have a center. In the case of an absence of a center then we say the circle has a deleted center. However all CoPs $\mathcal{C}(n)$ with even generators have a center. It is easy to see that the CoP $\mathcal{C}(n)$ contains all points whose weights are positive integers from 1 to $n-1$ inclusive:

$$\mathcal{C}(n) = \{[x] \mid \ x \in \mathbb{N}, x < n\}.$$

Therefore the CoP $\mathcal{C}(n)$ has $\left\lfloor \frac{n-1}{2} \right\rfloor$ different axes.

Proposition 2.4. *Each axis is uniquely determined by points $[x] \in \mathcal{C}(n, \mathbb{M})$.*

Proof. Let $\mathbb{L}_{[x],[y]}$ be an axis of the CoP $\mathcal{C}(n, \mathbb{M})$. Suppose as well that $\mathbb{L}_{[x],[z]}$ is also an axis with $z \neq y$. Then it follows by Definition 2.2 that we must have $n = x + y = x + z$ and therefore $y = z$. This cannot be and the claim follows immediately. $\square$

3. **The Density of Points on the Circle of Partition**

In this section we introduce the notion of density of points on CoP $\mathcal{C}(n,\mathbb{M})$ for $\mathbb{M}\subseteq\mathbb{N}$. We launch the following language in that regard.

Definition 3.1. Let be $\mathbb{H}\subset\mathbb{N}$. Then the quantity

$$\mathcal{D}\left(\mathbb{H}\right)=\lim_{n\to\infty}\frac{|\mathbb{H}\cap\mathbb{N}_n|}{n}$$

denotes the density of $\mathbb{H}$.

Definition 3.2. Let $\mathcal{C}(n,\mathbb{M})$ be CoP with $\mathbb{M}\subset\mathbb{N}$ and $n\in\mathbb{N}$. Suppose $\mathbb{H}\subset\mathbb{M}$ then by the density of points $[x]\in\mathcal{C}(n,\mathbb{M})$ such that $x\in\mathbb{H}$, denoted $\mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty,\mathbb{M})})$, we mean the quantity

$$\mathcal{D}\left(\mathbb{H}_{\mathcal{C}(\infty,\mathbb{M})}\right)=\lim_{n\longrightarrow\infty}\frac{\#\{\mathbb{L}_{[x],[y]}\hat{\in}\,\mathcal{C}(n,\mathbb{M})|\ \{x,y\}\cap\mathbb{H}\neq\emptyset\}}{\nu(n,\mathbb{M})}.$$

Proposition 3.3. *Let $\mathcal{C}(n)$ with $n\in\mathbb{N}$ be a CoP and $\mathbb{H}\subset\mathbb{N}$. Then the following inequality holds*

$$\mathcal{D}(\mathbb{H})=\lim_{n\longrightarrow\infty}\frac{\left\lfloor\frac{|\mathbb{H}\cap\mathbb{N}_n|}{2}\right\rfloor}{\left\lfloor\frac{n-1}{2}\right\rfloor}\leq\mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty)})\leq\lim_{n\longrightarrow\infty}\frac{|\mathbb{H}\cap\mathbb{N}_n|}{\left\lfloor\frac{n-1}{2}\right\rfloor}=2\mathcal{D}(\mathbb{H}).$$

Proof. The upper bound is obtained from a configuration where no two points $[x],[y]\in\mathcal{C}(n)$ such that $x,y\in\mathbb{H}$ lie on the same axis of the CoP. That is, by the uniqueness of the axes of CoPs with $\nu(n,\mathbb{H})=0$, we can write

$$\begin{aligned}\#\left\{\mathbb{L}_{[x],[y]}\in\mathcal{C}(n)|\ \{x,y\}\cap\mathbb{H}\neq\emptyset\right\}&=\nu(n,\mathbb{H})+\#\left\{\mathbb{L}_{[x],[y]}\in\mathcal{C}(n)|\ x\in\mathbb{H},\ y\in\mathbb{N}\setminus\mathbb{H}\right\}\\&=\#\left\{\mathbb{L}_{[x],[y]}\in\mathcal{C}(n)|\ x\in\mathbb{H},\ y\in\mathbb{N}\setminus\mathbb{H}\right\}\\&=|\mathbb{H}\cap\mathbb{N}_n|.\end{aligned}$$

The lower bound however follows from a configuration where any two points $[x],[y]\in\mathcal{C}(n)$ with $x,y\in\mathbb{H}$ are joined by an axis of the CoP. That is, by the uniqueness of the axis of CoPs with $\#\left\{\mathbb{L}_{[x],[y]}\in\mathcal{C}(n)|\ x\in\mathbb{H},\ y\in\mathbb{N}\setminus\mathbb{H}\right\}=0$, then we can write

$$\begin{aligned}\#\left\{\mathbb{L}_{[x],[y]}\in\mathcal{C}(n)|\ \{x,y\}\cap\mathbb{H}\neq\emptyset\right\}&=\nu(n,\mathbb{H})\\&=\left\lfloor\frac{|\mathbb{H}\cap\mathbb{N}_n|}{2}\right\rfloor\end{aligned}$$

□

Theorem 3.4. *Let $\mathbb{U}$ denotes the set of all Ulam numbers and $\mathcal{D}(\mathbb{U})$ denotes the natural density. If $\mathcal{D}(\mathbb{U}_{\mathcal{C}(\infty)})$ exists and*

$$\begin{aligned}&\#\{(x,y)|\ x\in\mathbb{U},\ y\notin\mathbb{U},\ m\leq n,\ m\in\mathbb{U},\ m=x+y\}\\\leq\ &\#\{(x,y)|\ x\in\mathbb{U},\ y\notin\mathbb{U},\ m\leq n,\ m\notin\mathbb{U},\ m=x+y\}\end{aligned}$$

with

$$\#\{(x,y)|\ \{x,y\}\cap\mathbb{U}\neq\emptyset,\ m\leq n,\ m\notin\mathbb{U},\ m=x+y\}\ll\frac{n^{1-\epsilon}}{2}$$

for some $\epsilon>0$ then

$$\mathcal{D}(\mathbb{U})=0.$$

Proof. First let $\mathbb{U} \subset \mathbb{N}$ denotes the sequence of Ulam numbers. Then by appealing to Proposition 3.3, we obtain the lower bound

$$\lim_{n\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n)|\ \{x,y\} \cap \mathbb{U} \neq \emptyset\}}{\nu(n,\mathbb{N})} \geq \mathcal{D}(\mathbb{U}).$$

Since $\nu(n,\mathbb{N}) = \left\lfloor \frac{n-1}{2} \right\rfloor$, it follows by the uniqueness of the axes of CoPs the decomposition

$$\begin{aligned}\lim_{n\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n)|\ \{x,y\} \cap \mathbb{U} \neq \emptyset\}}{\nu(n,\mathbb{N})} &= \lim_{n\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{U})\}}{\left\lfloor \frac{n-1}{2} \right\rfloor} \\ &+ \lim_{n\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n)|\ x \in \mathbb{U}, y \in \mathbb{N} \setminus \mathbb{U}\}}{\left\lfloor \frac{n-1}{2} \right\rfloor}\end{aligned} \blacksquare$$

since $\mathcal{D}(\mathbb{U}_{\mathcal{C}(\infty)})$ exists. We can now further write the following decomposition of the generators

$$\begin{aligned}\lim_{n\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{U})\}}{\left\lfloor \frac{n-1}{2} \right\rfloor} &= \lim_{n\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m,\mathbb{U})|\ m \leq n,\ m \in \mathbb{U}\}}{\left\lfloor \frac{n-1}{2} \right\rfloor} \\ &+ \lim_{n\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m,\mathbb{U})|\ m \leq n,\ m \notin \mathbb{U}\}}{\left\lfloor \frac{n-1}{2} \right\rfloor}. \end{aligned} \tag{3.1}$$

It follows by virtue of the Ulam sequence the following reduction

$$\lim_{n\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m,\mathbb{U})|\ m \leq n,\ m \in \mathbb{U}\}}{\left\lfloor \frac{n-1}{2} \right\rfloor} = \lim_{n\longrightarrow\infty} \frac{1}{\left\lfloor \frac{n-1}{2} \right\rfloor} = 0$$

since $\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m,\mathbb{U})|\ m \leq n,\ m \in \mathbb{U}\} = 1$. Similarly, we have the decomposition

$$\begin{aligned}&\lim_{n\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n)|\ x \in \mathbb{U}, y \in \mathbb{N} \setminus \mathbb{U}\}}{\left\lfloor \frac{n-1}{2} \right\rfloor} \\ &= \lim_{n\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m)|\ x \in \mathbb{U}, y \in \mathbb{N} \setminus \mathbb{U},\ m \leq n,\ m \in \mathbb{U}\}}{\left\lfloor \frac{n-1}{2} \right\rfloor} \\ &+ \lim_{n\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m)|\ x \in \mathbb{U}, y \in \mathbb{N} \setminus \mathbb{U}, m \leq n,\ m \notin \mathbb{U}\}}{\left\lfloor \frac{n-1}{2} \right\rfloor} \\ &\leq 2 \lim_{n\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m)|\ x \in \mathbb{U}, y \in \mathbb{N} \setminus \mathbb{U}, m \leq n,\ m \notin \mathbb{U}\}}{\left\lfloor \frac{n-1}{2} \right\rfloor}\end{aligned}$$

by exploiting the condition

$$\begin{aligned}&\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m)|\ x \in \mathbb{U}, y \in \mathbb{N} \setminus \mathbb{U},\ m \in \mathbb{U},\ m \leq n\} \\ &\leq \#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m)|\ x \in \mathbb{U}, y \in \mathbb{N} \setminus \mathbb{U},\ m \notin \mathbb{U},\ m \leq n\}\end{aligned}$$

so that

$$\begin{aligned}\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n)|\ \{x,y\} \cap \mathbb{U} \neq \emptyset\} &= \#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m,\mathbb{U})|\ m \leq n,\ m \notin \mathbb{U}\} \\ &+ \#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m)|\ x \in \mathbb{U}, y \in \mathbb{N} \setminus \mathbb{U}, m \leq n,\ m \notin \mathbb{U}\}\end{aligned} \blacksquare$$

and

$$\mathcal{D}(\mathbb{U}) \leq \lim_{m\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(m)|\ \{x,y\}\cap\mathbb{U}\neq\emptyset,\ m\leq n,\ m\notin\mathbb{U}\}}{\left\lfloor\frac{n-1}{2}\right\rfloor} + \lim_{m\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(m)|\ x\in\mathbb{U},\ y\notin\mathbb{U},\ m\leq n,\ m\notin\mathbb{U}\}}{\left\lfloor\frac{n-1}{2}\right\rfloor}.$$

By exploiting the condition

$$\#\{(x,y)|\ \{x,y\}\cap\mathbb{U}\neq\emptyset,\ m\leq n,\ m\notin\mathbb{U},\ m=x+y,\ x<y\} \ll \frac{n^{1-\epsilon}}{2}$$

for some $\epsilon>0$, we have

$$\begin{aligned}\mathcal{D}(\mathbb{U}) &\leq \lim_{m\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(m)|\ \{x,y\}\cap\mathbb{U}\neq\emptyset,\ m\leq n,\ m\notin\mathbb{U}\}}{\left\lfloor\frac{n-1}{2}\right\rfloor} \\ &\quad + \lim_{m\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(m)|\ x\in\mathbb{U},\ y\notin\mathbb{U},\ m\leq n,\ m\notin\mathbb{U}\}}{\left\lfloor\frac{n-1}{2}\right\rfloor} \\ &\ll \lim_{n\longrightarrow\infty}\frac{1}{n^{\epsilon}} = 0.\end{aligned}$$

□

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