# Lower bound for arbitrarily aligned Minimal Enclosing Rectangle

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**Abstract**

We determine the lower bound for arbitrarily aligned perimeter and area Minimal Enclosing Rectangle (MER) problems to be $\Omega(n \log n)$ by using a reduction technique for a problem with this known lower bound.

**Definitions:**

**Minimum Enclosing Rectangle (MER):** (also minimum bounding rectangle, minimum bounding box) for a point set $M$ in 2 dimensions is the box with the smallest perimeter within which all the points lie. Second flavor of the problem concerns area vs perimeter. In the discussion below we consider both flavors.

**Maximum Gap:** given a set $S$ of N real numbers $x_1, x_2, ..., x_N$, find the maximum difference between two consecutive members of $S$. (Two numbers $x_i$ and $x_j$ of $S$ are said to be consecutive if they are such in any permutation of $(x_1, ..., x_N)$ that achieves natural ordering.)

**Procedure:** Let's reduce Maximum Gap problem to our target Minimal Enclosing Rectangle problem.

1. We determine $x_{min}$ and $x_{max}$ of set $S$. ($O(n)$)

2. We will uniformly (meaning — preserving sequencing and relative distances of the natural ordering) transform set $S$ onto a quarter of a unit circle, such that $x_{min}$ maps to (-1,0) and $x_{max}$ to (0,1) Cartesian coordinates. An example of such transformation: for $x_i$ member of $S$:

$$(-\cos(\frac{\pi(x_i - x_{min})}{2(x_{max} - x_{min})}), \sin(\frac{\pi(x_i - x_{min})}{2(x_{max} - x_{min})}))$$

After transforming $S$ into the upper left quadrant, then we will repeat it for the 3 other quadrants – rotating clockwise $\frac{\pi}{2}$ each time. ($O(n)$)

3. Now, we can make a call to an MER algorithm on the resulting set $M$.

4. The output of MER is a rectangle $R$. Let's take any side of it. This will be an equation of a line. Now we can plug into it all of the points of $M$ one by one, taking care to exclude all $x_{max}$ vertices. There should be 2 and only 2 which will satisfy from the same quadrant (see *Observation 4* below). If only one point satisfies, we determine the quadrant it belongs to, then plug in $x_{max}$ from that quadrant to get the second point. [We do this because by construction $x_{max}$ from one quadrant equals to $x_{min}$ of the next quadrant clockwise.] ($O(n)$)

5. For any one of these points let's determine which quadrant it belongs to (they both must be in the same one, since both ends of the quadrant arc are vertices of M by construction). Then apply the inverse transformation to get the original values back. ($O(1)$)

6. Take the absolute value of the difference – this will be the Maximum Gap. ($O(1)$)

Total cost of the transformation is $O(n)$ (we are not including computational cost of MER in this tally).

**Analysis:**

**Lemma 1** *The MER R in step 4 above is a square, having its sides aligned with the same chord (i.e. chord representing the same original gap from S) in each quadrant.*

*Observation 1:* by construction, all of the points of $M$ are extreme and are vertices of CH(M).

*Observation 2:* chords representing the same gap in adjacent quadrants are orthogonal to each other. Figure 1 demonstrates this situation. Chords $AB$ and $CD$ each correspond to the gap between some original consecutive elements $x_i$ and $x_j$ from $S$. Let's show why $\angle AED = \frac{\pi}{2}$. By construction, $\angle AOC = \angle BOD = \frac{\pi}{2}$ as well as $\angle AOB = \angle COD = \alpha$, where $0 < \alpha \leq \frac{\pi}{2}$. Therefore $\beta = \frac{\pi}{2} + \alpha$. $\gamma = \frac{\pi - \alpha}{2}$, since $\triangle AOB$ is isosceles [because $AO = BO = 1$, where 1 is the radius of our unit circle]. From all of this follows: $\delta = \gamma - \frac{\pi - \beta}{2} = \frac{\pi}{4}$. By applying same logic to the other quadrant, we get $\angle ADE = \frac{\pi}{4}$ as well. Finally, looking at $\triangle AED$: $\angle AED = \pi - \delta - \angle ADE = \frac{\pi}{2}$.

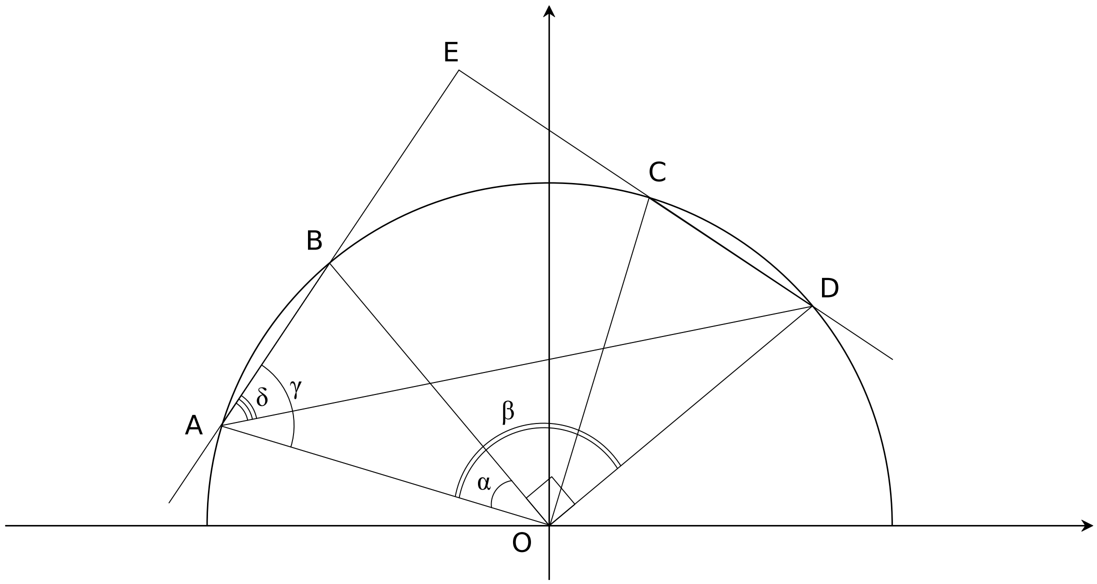


Figure 1: Orthogonal chords

*Observation 3:* each side of $R$ has to contain either a single vertex or a side of a $CH(M)$. Assume opposite, then there is some distance between the side of $R$ and a closest vertex (side) of $CH(M)$. This means either our rectangle cuts into a $CH(M)$ or we can slide the side parallel to itself towards its closest vertex(side) of $CH(M)$ until it coincides with the closest vertex (side) of $CH(M)$ to achieve a smaller perimeter and area enclosing rectangle. In either of these cases, we did not have an MER to begin with. Thus we achieve contradiction.

*Observation 4:* each side of $R$ has to contain one and only side of a $CH(M)$ (eliminating single vertex as an option). Assume opposite, then at least one side of $R$ only contains one vertex from $CH(M)$. Let's drop a line from this vertex to the center of the circle $O$. By construction, it's length is going to be 1. Let's continue this line through the center of the circle, until it hits the opposite quadrant's circle's circumference. Straight line has angle of $\pi$, which gives us another vertex of $CH(M)$ by two turns of $\frac{\pi}{2}$ based on *Observation 2*, so this means we found the opposite side of $R$. By similar logic if we drop a perpendicular line via $O$, we hit two other vertices of $CH(M)$ and thus perimeter of our $R$ is 8 and area is 4.
Now let's construct another enclosing rectangle $L$ as follows: for the original vertex above, let's find the nearest neighbor, we know there has to be at least one (given a set $S$ at least has to have 2 elements for Maximum Gap to be valid). Let's draw a straight line through these two vertices. This line will contain one side of an enclosing rectangle we are building. [To prove that it is, we need to show that this line does not bisect $CH(M)$. Assume the opposite, first we observe, since this line bisects a circle which fully encloses $CH(M)$ which means this line bisects $CH(M)$ within the chord of this

circle (see *Observation 1*). This means there is another vertex belonging to $M$ between the two nearest neighbors within it, which is a contradiction.] We continue with our construction by using the same method as before - from each vertex we drop a straight line through the center of the circle. We make the same observation that where these lines intersect the circle in the opposite quadrant ($\pi$ turn), we find mirror image vertices (*Observation 2*). The line through those newly identified vertices contains the opposite side of $L$. By making $\frac{\pi}{2}$ turns and drawing straight lines we find perpendicular sides of $L$. We observe that all four sides of $L$ bisect the unit circle by construction. This means that its perimeter must be $< 8$ and area $< 4$.
So we just built an enclosing rectangle $L$ for set $M$ that has a smaller perimeter or area (depending on the MER flavor used) than MER, which is a contradiction. Thus a side of MER for $M$ cannot contain just a single vertex, it has to contain an entire side of $CH(M)$

*Conclsuion 1:* From the *Observations 2 & 4* and since there could be only one perpendicular dropped on a straight line from a single point, all sides of $R$ will be aligned with the same original element (gap) [mapped to all four quadrants].

*Conclusion 2:* An interval $d$ from the center of the circle to a midpoint of the chord lying on a side of $R$ will be orthogonal to that cord (side) [median to the base of an isosceles triangle]. This interval will also be orthogonal and equal to its "sibling" in the adjacent quadrant. This means that R must be a square. Also, $d$ is half the length of the side of the enclosing square.

*Conclusions 1 & 2* satisfy **Lemma 1** conditions. $\square$

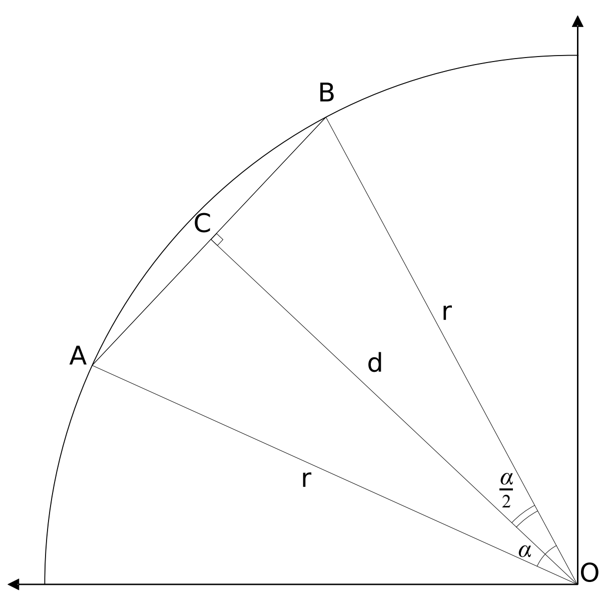


Figure 2: Arc length to MER side length relationship

**Lemma 2** *Two vertices of $M$ identified in step 5 of the Procedure section the largest arc of any adjacent vertices defined in the Procedure.*

As shown in **Lemma 1**, MER $R$ will align with the chord of a unit circle of two neighboring vertices of $M$ in the same quadrant. This satisfies vertex adjacency. As depicted on Figure 2, for upper left quadrant, we now need to derive a formula for the arc $\widehat{\mathrm{AB}}$ expressed solely as a function of $d$, which is a half of the side of MER $R$ in order to see the type of relationship between the length of the side of $R$ (and thus its perimeter and area) and the length of the arc it sections in each quadrant.

$$\widehat{\mathrm{AB}} = \alpha$$

$$r = \sqrt{d^2 + \frac{AB^2}{4}} = 1 \Rightarrow AB = 2\sqrt{1-d^2}$$

$$AB = 2r\sin(\frac{\alpha}{2}) = 2\sin(\frac{\alpha}{2})$$

$$2\sqrt{1-d^2} = 2\sin(\frac{\alpha}{2})$$

$$d = \sqrt{1-\sin^2(\frac{\alpha}{2})} = \sqrt{\cos^2(\frac{\alpha}{2})} = \cos(\frac{\alpha}{2})$$

And after final substitution, we get:

$$d = \cos(\frac{\widehat{\mathrm{AB}}}{2}) \Rightarrow \left| \begin{array}{l} Perimeter(R) = 8\cos(\frac{\widehat{\mathrm{AB}}}{2}) \\ Area(R) = 4\cos^2(\frac{\widehat{\mathrm{AB}}}{2}) \end{array} \right.$$

Based on the original construction of set $M$ and looking at only upper left quadrant, we can derive the possible range of $\widehat{\mathrm{AB}}$ values as $[0, \frac{\pi}{2}]$ with 0 when original $S$ has repeating numbers to $\frac{\pi}{2}$ when the $S$ contains only two elements. On this range, both functions $8\cos(\frac{\widehat{\mathrm{AB}}}{2})$ and $4\cos^2(\frac{\widehat{\mathrm{AB}}}{2})$ are smoothly declining, thus perimeter and area of $R$ have inverse relationship to the length of $\widehat{\mathrm{AB}}$. Since we know $R$ has a minimum possible perimeter and area (MER and being a square), $\widehat{\mathrm{AB}}$ is the maximum possible arc between adjacent vertices of $M$. $\square$

**Theorem 3** *Maximum Gap $\propto_{O(n)}$ Minimal Enclosing Rectangle (both perimeter and area)*

From **Lemma 2**, we know that running MER algorithm on $M$ produces the longest arc between nearest vertices of $M$ in the same quadrant. Based on how $M$ is constructed in the Procedure from $S$, we can derive the original elements of $S$ (steps 4 & 5) $x_i$ and $x_j$ which due to the uniformity of the transformation in step 2 have to be the naturally sequenced members of $S$ with the largest gap of all neighboring pairs among all the consecutively ordered members of $S$. Cost of the transformation is $O(n)$. $\square$

**Theorem 4** *MER problem is* $\Theta(n \log n)$ *(both perimeter and area)*

We have shown that Maximum Gap $\propto_{O(n)}$ Minimal Enclosing Rectangle. Maximum Gap was shown to be $\Omega(n \log n)$ [1] (Corollary 6.2) in algebraic decision tree model [2], [3], [4]. Therefore MER is $\Omega(n \log n)$.
It was also shown [5](Theorem 3) that MER is $O(n \log n)$. $\square$

# References


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