Notes on Critical Zeros of an L-function

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Abstract

This is the summary report of "Trace Formula in Noncommutative Geometry and the Zeros of the Riemman Zeta Function" by A.Connes.

We set the following notation.

K a global field

K_ν a local field, completion of K at the place ν of K

$\mathbb{A}_K$ the adele ring of K

C_K the idele class group $K^*\backslash \mathrm{GL}_1(\mathbb{A}_K)$

$\hat{C}_K$ the dual group of C_K.

0.

We will summarize the spectral interpretation of critical zeros of $L(\chi, s)$ associated χ of C_K by Alain Connes. Let h be a test function. The Weil explicit formula says

$$\sum_{\nu} \int'_{K_\nu^*} \frac{h(u^{-1})}{|1-u|}\, d^*u = \hat{h}(0) + \hat{h}(1) - \sum_{L(\chi,\rho)=0} \hat{h}(\chi,\rho).$$

Suppose that there exists a representation U of C_K and that

$$\operatorname{tr} U(h) = \sum_{\nu} \int'_{K_\nu^*} \frac{h(\mu^{-1})}{|1-\mu|}\, d^*\mu$$

is satisfied. We see that

$$\operatorname{tr} U(h) = \hat{h}(0) + \hat{h}(1) - \sum_{L(\chi,\rho)=0} \hat{h}(\chi,\rho)$$

holds. We can say that critical zeros of $L(\chi, s)$ appear as the spectra of the operator U. It is just *the spectral interpretation of critical zeros of $L(\chi, s)$*.

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Let

$$X = K^{*}\backslash\mathbb{A}_K.$$

The left regular representation U of C_K on ${L^2}_\delta(X)$ which is a weighted L^2 space can be used to accomplish our task. Namely, it holds that

$$\mathrm{tr}U(h) = \hat{h}(0)+\hat{h}(1) - \sum_{\substack{L(\chi,\rho)=0 \\ \mathrm{Re}\rho = 1/2}} \hat{h}(\chi,\rho) + \infty\cdot h(1).$$

However we will not try to treat the representation $(U, {L^2}_\delta(X))$ directly. Instead of the representation $(U, {L^2}_\delta(X))$, we will think of the operator $\mathrm{Q}_\Lambda U$ where U is the left regular representation of C_K on $L^2(X)$. Because, firstly there is a possibility of using some results to compute $\mathrm{tr}\,\mathrm{Q}_\Lambda U$, secondly we can eliminate the parameter δ of ${L^2}_\delta$.

We try to compute $\mathrm{tr}(\mathrm{Q}_\Lambda U(h))$. This has the relationship to the validity of the Riemann Hypothesis. Suppose that we can compute as follows;

$$\mathrm{tr}(\mathrm{Q}_\Lambda U(h)) = 2\log'(\Lambda)h(1) + \sum_v \int'_{K_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u + o(1) \quad \Lambda \longrightarrow \infty$$

where $2\log'(\Lambda) = \int_{\lambda\in C_K,\, |\lambda|\le[\Lambda^{-1},\,\Lambda]} d^*\lambda$. It gives $\mathrm{tr}U(h)$ independently of δ since the cut-off Q_Λ can be performed directly on $L^2(X)$. Thus we obtain a δ-independent trace formula:

$$\hat{h}(0)+\hat{h}(1) - \sum_{\substack{L(\tilde{\chi}_0,\rho)=0 \\ \mathrm{Re}\rho = 1/2}} \hat{h}(\tilde{\chi}_0,\rho) + \infty\cdot h(1) = 2\log'(\Lambda)h(1) + \sum_v \int'_{K_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u + o(1) \quad \Lambda \longrightarrow \infty.$$

The left side is spectral and the right side is geometrical. From the Weil explicit formula, we have seen that

$$\sum_v \int'_{K_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u = \hat{h}(0)+\hat{h}(1) - \sum_{L(\tilde{\chi}_0,\rho)=0} \hat{h}(\tilde{\chi}_0,\rho).$$

Therefore, one obtains that

$$\sum_{L(\tilde{\chi}_0,\rho)=0} \hat{h}(\tilde{\chi}_0,\rho) = \sum_{\substack{L(\tilde{\chi}_0,\rho)=0 \\ \mathrm{Re}\rho = 1/2}} \hat{h}(\tilde{\chi}_0,\rho).$$

It means the validity of the Riemann Hypothesis. Conversely, the validity of the Riemann Hypothesis implies that

$$\mathrm{tr}(\mathrm{Q}_\Lambda U(h)) = 2\log'(\Lambda)h(1) + \sum_v \int'_{K_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u + o(1) \quad \Lambda \longrightarrow \infty.$$

Lastly we will mention trace formulae. The trace formula which is given by a zeta function:

$$\underbrace{\cdots}_{\text{Zero points}} = \underbrace{\cdots}_{\text{Geometrical side}}$$

is a prototype. Selberg's trace formula is that

$$\underbrace{\cdots}_{\text{Eigenvalues of Laplacian}} = \underbrace{\cdots}_{\text{Geometrical side}} .$$

There exists an operator M such that it is commutative with the Laplacian of H. The operator is the integral operator which has $k(z, w)$ as an integral kernel

$$M(f)(z) = \int_{\mathrm{H}} k(z, w) f(w) d\mu(w) .$$

The Selberg's trace formula gives the explicit formula of Selberg's zeta function.
The trace formula given by Connes is the same type as Selberg's. It is that

$$\underbrace{\cdots}_{\text{Characters}} = \underbrace{\cdots}_{\text{Geometrical side}} .$$

Here $U(h)$: $C_c^\infty(X) \longrightarrow C_c^\infty(X)$

$$(U(h)\xi)(x) = \int_{C_K} h(g)(U(g)\xi)(x) d^*g .$$

The operator $U(h)$ is the integral operator which has $k_h(x, y)$ as an integral kernel

$$(U(h)\xi)(x) = \int_{C_K} k_h(x, y)\xi(y) d^*y .$$

1. Zeta-Functions and L-Functions

We try to characterize L-functions from the view of the representation theory.

Definition 1.1. **(Bruhat-Schwartz space)** Denote the Bruhat-Schwartz space on the adeles $\mathbb{A}_K$ by $\mathcal{S}(\mathbb{A}_K)$. It is the products Πf_ν over each place ν of K; for each infinite place ∞ each f_∞ is the usual Schwartz function on $\mathbb{R}^n$, for each finite place ν each f_ν is a Schwartz function on a local field K_ν and $f_\nu = \mathbf{1}_{\mathcal{O}_\nu}$ for all but finite many ν. Here, $\mathbf{1}_{\mathrm{A}}$: $G \longrightarrow \{0, 1\}$ for $\mathrm{A} \subseteq G$

$$\mathbf{1}_{\mathrm{A}}(x) = \begin{cases} 1 & \cdots \quad x \in \mathrm{A} \\ 0 & \cdots \quad x \notin \mathrm{A} \end{cases} .$$

[An example]

$$\mathcal{S}(\mathbb{A}_{\mathbb{Q}}) = \prod_{p \leq \infty} f_p = f_\infty \times \prod_{p < \infty} f_p ,$$

where $f_\infty \in \mathcal{S}(\mathbb{R})$, $f_p \in \mathcal{S}(\mathbb{Q}_p)$ and $f_p = \mathbf{1}_{\mathbb{Z}_p}$ for all but finite many p.

We will begin with the local case. Denote the set of the irreducible representations of $K_\nu{}^*$ by $\mathrm{Irr}(K_\nu{}^*)$. Let (π_ν, V_{π_ν}) be an irreducible representation of $K_\nu{}^*$. Put

$$\pi_\nu(f)v = \int_{K_\nu{}^*} f(g)\pi_\nu(g)v\, d^*g , \qquad f \in \mathcal{S}(K_\nu).$$

Suppose that $\mathrm{tr}\pi_\nu(f)$ can be defined, namely $\pi_\nu(f)$ is a trace class operator. So we may think that there exists a character $\mathrm{tr}\pi_\nu$ of $K_\nu{}^*$, and

$$\mathrm{tr}\pi_\nu(f) = \int_{K_\nu{}^*} f(g)\mathrm{tr}\pi_\nu(g)\, d^*g .$$

Denote the character $\mathrm{tr}\pi_\nu$ by $\chi_{0,\nu}$. Put

$$\chi_\nu(g) = \chi_{0,\nu}(g)|g|^s \quad s \in \mathbb{C}.$$

χ_ν is a quasi character of $K_\nu{}^*$.

Definition 1.2. **(Local zeta-functions)** For an arbitrary $f_\nu \in \mathcal{S}(K_\nu)$, let

$$\Delta_{\chi_{0,\nu}{}^s}(f_\nu) = \langle f_\nu, \Delta_{\chi_{0,\nu}{}^s} \rangle = \int_{K_\nu^*} f_\nu(g)\chi_{0,\nu}(g)|g|^s \, d^*g \,.$$

This integral converges absolutely at $\mathrm{Re}(s) > 0$.

Let

$$\Delta'_{\chi_{0,\nu}{}^s}(f_\nu) = \langle f_\nu, \Delta'_{\chi_{0,\nu}{}^s} \rangle = \int_{K_\nu^*} (f_\nu(x) - f_\nu(\nu^{-1}x))\chi_{0,\nu}(g)|g|^s \, d^*g \,.$$

It is a holomorphic function of s at $\mathrm{Re}(s) > 0$.

The global case is as follows.

Definition 1.3. **(Global zeta-functions)** Put $\chi_0 = \prod_\nu \chi_{0,\nu}$. For an arbitrary $f \in \mathcal{S}(\mathbb{A}_K)$, let

$$\Delta_{\chi_0{}^s}(f) = \langle f, \Delta_{\chi_0{}^s} \rangle = \int_{\mathbb{A}_K^*} f(x)\chi_0(x)|x|^s \, d^*x \,.$$

This integral converges absolutely at $\mathrm{Re}(s) > 1$.

Let

$$\Delta'_{\chi_0{}^s} = \prod_\nu \Delta'_{\chi_{0,\nu}{}^s} \,.$$

By construction $\Delta'_{\chi_0{}^s}(f)$ makes sense whenever $\mathrm{Re}(s) > 0$.

Lemma 1.1. Put $\chi = \prod_\nu \chi_\nu$ where $\chi_\nu(g) = \chi_{0,\nu}(g)|g|^s$ $s \in \mathbb{C}$. For $\mathrm{Re}(s) > 1$, the following integral converges absolutely

$$\Delta_\chi(f) = \int_{\mathbb{A}_K^*} f(x)\chi(x) d^*x = \int_{\mathbb{A}_K^*} f(x)\chi_0(x)|x|^s \, d^*x \quad \forall f \in \mathcal{S}(\mathbb{A}_K),$$

and $\Delta_\chi(f) = L(\chi_0, s)\, \Delta'_\chi(f)$. Here $\Delta'_\chi(f)$ is a holomorphic function of s at $\mathrm{Re}(s) > 0$.

Proof. For $\mathrm{Re}(s)>1$, $f(x)\chi(x) = f(x)\chi_0(x)|x|^s$ is integrable. So $\int_{\mathbb{A}_K^*} f(x)\chi(x)d^*x$ converges absolutely at $\mathrm{Re}(s)>1$. We can compute as follows;

$$\begin{aligned}
\Delta'_{\chi_\nu}(f_\nu) &= \int_{K_\nu^*} (f_\nu(g) - f_\nu(\nu^{-1}g))\chi_\nu(g)d^*g \\
&= \int_{K_\nu^*} f_\nu(g)\chi_\nu(g)d^*g - \int_{K_\nu^*} f_\nu(\nu^{-1}g)\chi_\nu(g)d^*g \\
&= \int_{K_\nu^*} f_\nu(g)\chi_\nu(g)d^*g - \int_{K_\nu^*} f_\nu(g)\chi_\nu(\nu g)d^*g \\
&= \int_{K_\nu^*} f_\nu(g)\chi_\nu(g)d^*g - \chi_\nu(\nu)\int_{K_\nu^*} f_\nu(g)\chi_\nu(g)d^*g \\
&= (1 - \chi_\nu(\nu))\int_{K_\nu^*} f_\nu(g)\chi_\nu(g)d^*g \\
&= (1 - \chi_{0,\nu}(\nu)|\nu|^s)\int_{K_\nu^*} f_\nu(g)\chi_\nu(g)d^*g \,.
\end{aligned}$$

It holds that

$$\Delta_{\chi_\nu}(f_\nu) = (1 - \chi_{0,\nu}(\nu)|\nu|^s)^{-1}\, \Delta'_{\chi_\nu}(f_\nu).$$

Therefore,

$$\prod_{v} \Delta_{\chi_v}(f_v) = L(\chi_0, s)\,\Delta'_{\chi}(f).$$

Here, the left term equals $\Delta_{\chi}(f)$. By construction $\Delta'_{\chi}(f)$ makes sense whenever $\mathrm{Re}(s) > 0$.

□

2. $L^2(X)$ and $L^2(C_K)$

Let $f \in \mathcal{S}(\mathbb{A}_K)$. We will think of the sum $\sum_{r\in K^*} f(rx)$. It converges absolutely and gives a function on X. Since $r\cdot 0 = 0$, $\sum_{r\in K^*} f(r\cdot 0) = \sum_{r\in K^*} f(0)$. If $f(0) \neq 0$, $\sum_{r\in K^*} f(0) = \infty$. So we require $f(0) = 0$. Moreover, consider $\sum_{r\in K^*} f(rx) = \sum_{r\in K} f(rx)$ ($f(0) = 0$), the sum approximate $\int_K f(rx)\,dr$. Since $dx = |x|d^*x$, $dax = |ax|d^*ax = |ax|d^*x = |a|dx$. Thus we can compute as follows;

$$\int_K f(rx)\,dr = \int_K f(r)|x|^{-1}\,dr = |x|^{-1}\int_K f(r)\,dr .$$

When $|x| \longrightarrow 0$, $|x|^{-1}\int_K f(r)\,dr$ doesn't make sense unless $\int_K f(r)\,dr = 0$. Since $K \hookrightarrow \mathbb{A}_K$, we also require $\int_{\mathbb{A}_K} f(x)\,dx = 0$.

Definition 2.1.

$$\mathcal{S}(\mathbb{A}_K)_0 = \{ f \in \mathcal{S}(\mathbb{A}_K) |\ f(0) = 0,\ \int_{\mathbb{A}_K} f(x)\,dx = 0 \}.$$

There exists an exact sequence:

$$0 \rightarrow \mathcal{S}(\mathbb{A}_K)_0 \rightarrow \mathcal{S}(\mathbb{A}_K) \xrightarrow{L} \mathbb{C} \oplus \mathbb{C}(1) \rightarrow 0 .$$

$\mathbb{C}$ is a trivial C_K module such that $T(a)\lambda = \lambda$ for $a \in C_K$, $\lambda \in \mathbb{C}$. $\mathbb{C}(1)$ is Tate twist such that $T(a)\lambda = |a|\lambda$ for $g \in C_K$, $\lambda \in \mathbb{C}$. Considering $\mathrm{Ker}\, L = \mathcal{S}(\mathbb{A}_K)_0$, we will understand that $\mathbb{C}$ corresponds to $f(0)$ and $\mathbb{C}(1)$ comes from $\int_{\mathbb{A}_K} f(j^{-1}x)\,dx = |j|\int_{\mathbb{A}_K} f(x)\,dx$.

Definition 2.2. Let $L^2(X, dx)_0$ be the completion of $\mathcal{S}(\mathbb{A}_K)_0$ for the norm given by

$$\|f\|^2 = \int_{C_K} \left| \sum_{r\in K^*} f(rx) \right|^2 dx .$$

Similarly, let $L^2(X, dx)$ be the completion of $\mathcal{S}(\mathbb{A}_K)$ for the above norm.

We obtain an exact sequence:

$$0 \to L^2(X)_0 \to L^2(X) \to \mathbb{C} \oplus \mathbb{C}(1) \to 0\,.$$

For $f(x) \in L^2(X, dx)_0$, let $(\mathrm{T}f)(a)$ be the restriction of $f(x)$ to C_K:

$$(\mathrm{T}f)(a) = |a|^{1/2} \Sigma_{r\in K^*} f(ra) \qquad \forall a \in C_K\,.$$

Lemma 2.1. Let $f(x) \in \mathcal{S}(\mathbb{A}_K)_0$, then the series

$$(\mathrm{T}f)(a) = |a|^{1/2} \Sigma_{r\in K^*} f(ra) \qquad \forall a \in C_K$$

converges absolutely and one has

$$\forall n\ \exists c\,|\,(\mathrm{T}f)(a)\,| \leq\ c\, e^{-n|\log|a||}$$

and $(\mathrm{T}\,\hat{f}\,)(a^{-1}) = (\mathrm{T}f)(a)$.

Proof. The Poisson summation formula reads, for any $f(x) \in \mathcal{S}(\mathbb{A}_K)$,

$$|x|\Sigma_{r\in K} f(rx) = \Sigma_{r\in K} \hat{f}\,(rx^{-1}),$$

where $\hat{f}\,(\xi) = \int_{\mathbb{A}_K} f(g)\alpha(g\xi)\,dg$ for a basic character α of the additive group $\mathbb{A}_K$. We obtain

$$|x|\Sigma_{r\in K^*} f(rx) = \Sigma_{r\in K^*} \hat{f}\,(rx^{-1}) + (\,-|x|f(0) + \hat{f}\,(0)\,).$$

Here $\hat{f}\,(0) = \int_{\mathbb{A}_K} f(g)\,dg$. If $f(x) \in \mathcal{S}(\mathbb{A}_K)_0$ then $|x|\Sigma_{r\in K^*} f(rx) = \Sigma_{r\in K^*} \hat{f}\,(rx^{-1})$. One obtains

$$|x^{-1}|^{1/2}\Sigma_{r\in K^*} \hat{f}\,(rx^{-1}) = |x|^{1/2}\Sigma_{r\in K^*} f(rx).$$

By this formula, it is enough to estimate $(\mathrm{T}f)(a)$ for $|a| \longrightarrow \infty$. $(\mathrm{T}f)(a)$ decays faster than any power of $|a|$ for $|a| \longrightarrow \infty$. Thus, for any n, there exists a constant c and

$$|\,(\mathrm{T}f)(a)\,| \leq\ c\,|a|^{-n}$$

is satisfied.

□

Let $L^2(C_K, d^*x)$ be the Hilbert space using the norm:

$$||\xi||^2 = \int_{C_K} |\xi(a)|^2\, d^*a\,.$$

Proposition 2.1.

$$(\mathrm{T}f)(a) \in L^2(C_K).$$

Proof. Suppose that $f(x) \in L^2(X)_0$. Then

$$\int_{C_K} \left| \sum_{r\in K^*} f(rx) \right|^2 dx < \infty.$$

Since $dx = |x| d^*x$,

$$\int_{C_K} |(\mathrm{T}f)(a)|^2 \, d^*a = \int_{C_K} |a| \left| \sum_{r\in K^*} f(ra) \right|^2 \frac{da}{|a|} = \int_{C_K} \left| \sum_{r\in K^*} f(ra) \right|^2 da < \infty.$$

□

Thus we can also obtain the following exact sequence:

$$0 \to L^2(X)_0 \xrightarrow{\mathrm{T}} L^2(C_K) \to \mathcal{H} \to 0$$

where $\mathcal{H} \cong L^2(C_K)/\mathrm{Im(T)}$. Let U be a left regular representation of C_K on $L^2(X, dx)$ and V be a left regular representation of C_K on $L^2(C_K, d^*x)$. Set

$$(U(g)f)(x) = f(g^{-1}x) \quad \forall g \in C_K, \, x \in \mathbb{A}_K.$$

It turns out that

$$\begin{aligned} \mathrm{T}(U(g)f)(a) &= \text{the restriction of } f(g^{-1}x) \\ &= |g|^{1/2}(V(g)\mathrm{T}f)(a) \qquad \forall a, g \in C_K. \end{aligned}$$

From this equation, it is that $|g|^{-1/2}\,\mathrm{T}(U(g)f)(a) = V(g)(\mathrm{T}f)(a)$.

Proposition 2.2.

Im(T) is an invariant subspace for V.

Proof. Suppose that $f \in L^2(X)_0$. For $(\mathrm{T}f)(a)$,

$$V(g)(\mathrm{T}f)(a) = |a|^{1/2} |g|^{-1/2} \textstyle\sum_{r\in K^*} f(rg^{-1}a).$$

If $|g|^{-1/2} f(g^{-1}x) \in L^2(X)_0$ then $(\mathrm{T}\,|g|^{-1/2} f(g^{-1}x))(a) = |a|^{1/2} |g|^{-1/2} \sum_{r\in K^*} f(rg^{-1}a)$. Thus, $V(\mathrm{Im(T)}) \subseteq \mathrm{Im(T)}$ namely Im(T) is an invariant subspace for V.

Fix $g_0 \in C_K$ and put $f_{g_0^{-1}}(x) = f(g_0^{-1}x)$. We can compute as follows;

$$\int_{C_K} \left| \sum_{r\in K^*} f(rg_0^{-1}gx) \right|^2 dx = |g|^{-1} \int_{C_K} \left| \sum_{r\in K^*} f(rg_0^{-1}x) \right|^2 dx = |g_0||g|^{-1} \int_{C_K} \left| \sum_{r\in K^*} f(rx) \right|^2 dx.$$

Since $f \in L^2(X)_0$ and $g \in C_K$, we can say that $|g_0||g|^{-1} \int_{C_K} \left| \sum_{r \in K^*} f(rx) \right|^2 dx < \infty$ for almost all g_0. Therefore $f_{g_0^{-1}}(gx) \in L^2(X)_0$ for almost all g_0. Especially $f_{g^{-1}}(gx) \in L^2(X)_0$ for an arbitrary $g \in C_K$. Now

$$\int_{C_K} \left| \sum_{r \in K^*} f_{g_0^{-1}}(r;gx) \right|^2 dx = \int_{C_K} |g|^{-1} \left| \sum_{r \in K^*} f_{g_0^{-1}}(r;x) \right|^2 dx = \int_{C_K} \left| \sum_{r \in K^*} |g|^{-1/2} f_{g_0^{-1}}(r;x) \right|^2 dx .$$

Here $\sum_{r \in K^*} f_{g_0^{-1}}(r;x) = \sum_{r \in K^*} f(rg_0^{-1}x)$. When $g_0 = g$ then $\int_{C_K} \left| \sum_{r \in K^*} f_{g_0^{-1}}(r;gx) \right|^2 dx < \infty$, so $\int_{C_K} \left| \sum_{r \in K^*} |g|^{-1/2} f_{g_0^{-1}}(r;x) \right|^2 dx < \infty$. It means that $|g|^{-1/2} f(g^{-1}x) \in L^2(X)_0$.

□

Because C_K is abelian locally compact, its regular representation $(V, L^2(C_K))$ does not contain any finite dimensional subrepresentation. This fact is an obstacle to our attempt computing the trace of U. So we will replace $L^2(C_K)$ by $L^2{}_\delta(C_K)$ using the polynomial weight $(\log^2|a|)^{\delta/2}$, i.e. the norm $\|\xi\|_\delta^2 = \int_{C_K} |\xi(a)|^2 (1+\log^2|a|)^{\delta/2} d^*a$.

Definition 2.3. Let each Hilbert space $L^2{}_\delta(X)_0$ and $L^2{}_\delta(X)$ $(\delta > 1)$ be the completion of $\mathcal{S}(\mathbb{A}_K)_0$ and $\mathcal{S}(\mathbb{A}_K)$ respectively with the square norm

$$\|f\|_\delta^2 = \int_{C_K} \left| \sum_{r \in K^*} f(rx) \right|^2 (1+(\log|x|)^2)^{\delta/2} dx .$$

The Hilbert space $L^2{}_\delta(C_K)$ is obtained from the space of functions with the square norm

$$\|\xi\|_\delta^2 = \int_{C_K} |\xi(a)|^2 (1+(\log|a|)^2)^{\delta/2} d^*a$$

where we normalize the Haar measure of the multiplicative group C_K

$$\int_{|g| \in [1,\Lambda]} d^*g \sim \log\Lambda \qquad \Lambda \to +\infty .$$

These spaces are weighted L^2 spaces. The followings are basically.

Polynomials are dense in the $L^2{}_\delta$.

So,

the orthogonal polynomials in $L^2{}_\delta$ are a complete orthogonal set in $L^2{}_\delta$.

Therefore we can decompose each $L^2{}_\delta$ space in the direct sum of finite dimensional subspaces.

Proposition 2.3.

(a) The representation $(V,\ L^2{}_\delta(C_K))$ isn't unitary.

(b) $\|V(a)\|_\delta = O((\log|a|)^{\delta/2}) \qquad |a| \longrightarrow \infty.$

(c) $\|V(a)\|_\delta = O((\log|a|)^{\delta/2}) \qquad |a| \longrightarrow 0.$

Proof.

(a)
$$\begin{aligned}\|V(a)\xi\|_\delta^2 &= \int_{C_K}|\xi(a^{-1}g)|^2(1+(\log|g|)^2)^{\delta/2}d^*g\\ &= \int_{C_K}|\xi(a^{-1}g)|^2(1+(\log|aa^{-1}g|)^2)^{\delta/2}d^*g\\ &= \int_{C_K}|\xi(g)|^2(1+(\log|ag|)^2)^{\delta/2}d^*g\,.\end{aligned}$$

Thus it does't always hold that $\|V(a)\xi\|_\delta^2 = \|\xi\|_\delta^2$.

(b) (c) Let $\rho(u) = (1+u^2)^{\delta/2}$. It is satisfied that

$$\frac{\rho(\log xy)}{\rho(\log x)} = \frac{\rho(\log x+\log y)}{\rho(\log x)} \le \mathrm{c}\cdot\rho(\log y), \qquad \mathrm{c} = 2^{\delta/2}.$$

We compute as follows;

$$\begin{aligned}\|V(a)\xi\|_\delta^2 &= \int_{C_K}|\xi(a^{-1}g)|^2\rho(\log|g|)d^*g\\ &= \int_{C_K}|\xi(g)|^2\rho(\log|ag|)d^*g\\ &\le \mathrm{c}\cdot\int_{C_K}|\xi(g)|^2\rho(\log|a|)\rho(\log|g|)d^*g = \mathrm{c}\cdot\rho(\log|a|)\int_{C_K}|\xi(g)|^2\rho(\log|g|)d^*g\,.\end{aligned}$$

Therefore

$$\|V(a)\|_\delta^2 \le \mathrm{c}\cdot(1+(\log|a|)^2)^{\delta/2}.$$

Then

$$(\|V(a)\|_\delta^2)^{2/\delta} \le (\mathrm{c}\cdot(1+(\log|a|)^2)^{\delta/2})^{2/\delta}.$$

We can say that

$$\|V(a)\|_\delta^{4/\delta} \le \mathrm{c}^{2/\delta}\cdot(1+(\log|a|)^2) \le \mathrm{c}^{4/\delta}\cdot(1+(\log|a|)^2).$$

Thus,

$$\frac{\|V(a)\|_\delta^{4/\delta}}{(\log|a|)^2} \le \mathrm{c}^{4/\delta}\cdot\frac{1+(\log|a|)^2}{(\log|a|)^2}\,.$$

It turns out that

$$\frac{\|V(a)\|_\delta^{4/\delta}}{(\log|a|)^2} \le \mathrm{c}^{4/\delta} \quad |a| \longrightarrow \infty \quad \text{and} \quad \frac{\|V(a)\|_\delta^{4/\delta}}{(\log|a|)^2} \le \mathrm{c}^{4/\delta} \quad |a| \longrightarrow 0.$$

We can show that

$$\frac{\|V(a)\|_\delta^{4/\delta}}{(\log|a|)^2} = \left(\frac{\|V(a)\|_\delta}{\left|(\log|a|)^{\delta/2}\right|}\right)^{4/\delta}.$$

Therefore,

$$\frac{\|V(a)\|_\delta}{\left|(\log|a|)^{\delta/2}\right|} \le \text{c} \quad |a| \longrightarrow \infty \quad \text{and} \quad \frac{\|V(a)\|_\delta}{\left|(\log|a|)^{\delta/2}\right|} \le \text{c} \quad |a| \longrightarrow 0.$$

□

C_K is abelian, so its irreducible unitary representation is also a character. We use $\tilde{\chi}_0$ to denote a character of C_K since *a character of* $C_K = \chi_0|\cdot|^\rho(\rho \in i\mathbb{R})$ where χ_0 be a character of $C_{K,1}$ which is the maximal compact subgroup: $\{ g \in C_K \,|\, |g| = 1 \}$. C_K is locally compact, so it isn't always that $\hat{C}_K = \{ \tilde{\chi}_0 \}$. We will consider that

$$(V(g)\xi)(x) = c(g)\xi(x) \qquad \forall g \in C_K$$

for $\xi(x) \in L^2{}_\delta(C_K)$. It holds that $c|_{C_{K,1}} = \chi_0$. We have seen that V isn't unitary, so when $c|_{C_{K,1}} = \chi_0$ then $c = \chi_0|\cdot|^\rho$ $(\rho \in \mathbb{C})$. Here

$$|g|^{\mathrm{Re}(\rho)} \le \|V(g)\|_\delta, \quad g \in C_K.$$

Consider

$$\lim_{|g| \to \infty} \frac{|g|^\alpha}{\log|g|} = \infty \quad (\alpha > 0) \text{ and } \lim_{|g| \to 0} \frac{|g|^\alpha}{\log|g|} = \infty \quad (\alpha < 0).$$

Because $|g|^{\mathrm{Re}(\rho)} \le \|V(g)\|_\delta$, $g \in C_K$; if $\mathrm{Re}(\rho) > 0$ or $\mathrm{Re}(\rho) < 0$ then each of them conflicts with the proposition 2.3. (b) and (c). Therefore, it is that $\rho \in i\mathbb{R}$. Namely,

$$c = \chi_0|\cdot|^\rho \qquad \rho \in i\mathbb{R}.$$

There exists $\tilde{\chi}_0 \in \hat{C}_K$ such that $\tilde{\chi}_0(g) = c(g)$.

Let

$$L^2{}_{\delta, \tilde{\chi}_0} = \{\xi(x) \in L^2{}_\delta(C_K) \,|\, \xi(g^{-1}x) = \tilde{\chi}_0(g)\,\xi(x) \ \forall x \in C_K \ \forall g \in C_K\}$$

and let

$$L^2{}_{\delta, \chi_0} = \{\xi(x) \in L^2{}_\delta(C_K) \,|\, \xi(a^{-1}x) = \chi_0(a)\xi(x) \quad \forall x \in C_K \ \forall a \in C_{K,1}\}.$$

We see that

$$L^2{}_{\delta, \chi_0}(C_K) = \bigoplus_{\tilde{\chi}_0} L^2{}_{\delta, \tilde{\chi}_0}$$

since $\hat{C}_K(\chi_0) = \{\pi \in \hat{C}_K \,|\, \pi|_{C_{K,1}} = \chi_0\} = \{ \chi_0|\cdot|^\rho \,|\, \rho \in i\mathbb{R} \}$.

3. Discrete Spectra and Imaginary Parts of Zeros of the L function

We have a following decomposition:

$$C_K \cong C_{K,1} \times N.$$

Here $C_{K,1}$ is the maximal compact subgroup: $\{ g \in C_K \mid |g| = 1 \}$ and $N = \{ |g| \mid g \in C_K \} = \mathbb{R}^*_{>0}$. Since $C_{K,1}$ is abelian and compact, its irreducible unitary representation is a character. Let χ_0 be a character of $C_{K,1}$. We may say that $\hat{C}_{K,1} = \{\chi_0\}$.

We will think of the left regular representation $(V, L^2{}_\delta(C_K))$ of C_K. $C_{K,1}$ acts by the restriction of V to $C_{K,1}$ and it is unitary. Recall $\|V(a)\xi\|_\delta^2 = \|\xi\|_\delta^2 \ \forall a \in C_{K,1}$. For $\xi(x) \in L^2{}_\delta(C_K)$, we will consider that

$$(V(a)\xi)(x) = c(a)\xi(x) \qquad \forall a \in C_{K,1}.$$

Since the left regular representation of $C_{K,1}$ is unitary, there exists $\chi_0 \in \hat{C}_{K,1}$ such that $\chi_0(a) = c(a)$. Thus, fix $\chi_0 \in \hat{C}_{K,1}$ and put

$$L^2{}_{\delta,\chi_0} = \left\{ \xi(x) \in L^2{}_\delta(C_K) \mid \xi(a^{-1}x) = \chi_0(a)\xi(x) \quad \forall x \in C_K \ \forall a \in C_{K,1} \right\}.$$

When $V_{\chi_0} = V|_{L^2{}_{\delta,\chi_0}}$ then $(V_{\chi_0}, L^2{}_{\delta,\chi_0})$ gives a finite dimensional subrepresentation. It turns out that

$$L^2{}_\delta(C_K) = \bigoplus_{\chi_0 \in \hat{C}_{K,1}} L^2{}_{\delta,\chi_0}$$

since $\hat{C}_{K,1} = \{\chi_0\}$.

The dual space $(L^2{}_\delta(C_K))^*$ of $L^2{}_\delta(C_K)$ can be identified with $L^2{}_{-\delta}(C_K)$. It is also decomposed in the direct sum of the subspaces,

$$L^2{}_{-\delta,\chi_0} = \left\{ \eta(x) \in L^2{}_{-\delta}(C_K) \mid \eta(ax) = \chi_0(a)\eta(x) \quad \forall x \in C_K \ \forall a \in C_{K,1} \right\}.$$

Here, we use the transposed of V

$$(V^\tau(a)\eta)(x) = \eta(ax); \quad \eta(x) \in (L^2{}_\delta(C_K))^*.$$

The pairing between $L^2{}_\delta(C_K)$ and its dual $(L^2{}_\delta(C_K))^* = L^2{}_{-\delta}(C_K)$ is given by

$$\langle f, \eta \rangle = \int_{C_K} f(x)\eta(x)d^*x\,.$$

We can obtain the following exact sequences:

$$0 \to L^2_\delta(X)_0 \overset{\mathrm{T}}{\to} L^2_\delta(C_K) \to \mathcal{H} \to 0\,.$$

Let

$$\mathrm{Im}(\mathrm{T})^0 = \{\eta \in (L^2{}_\delta(C_K))^* \mid \langle \mathrm{T}f, \eta\rangle = 0 \quad \forall f \in \mathcal{S}(\mathbb{A}_K)_0\}.$$

It holds that

$$\eta(x) \in \mathrm{Im}(\mathrm{T})^0 \iff \int_{C_K} \mathrm{T}f(a)\eta(a)d^*a = 0, \;\; \forall f \in \mathcal{S}(\mathbb{A}_K)_0.$$

Proposition 3.1. Fix an extension $\tilde{\chi}_0$ of χ_0 as being equal to 1 on N. For any $\eta(g) \in L^2{}_{-\delta,\,\chi_0}$, we can write it as

$$\eta(g) = \tilde{\chi}_0(g)\Psi(|g|) \text{ where } \int_{C_K} |\Psi(|g|)|^2 (1+(\log|g|)^2)^{-\delta/2} d^*g < \infty.$$

$\Psi(|g|)$ is a tempered distribution on $\mathbb{R}^*{}_{>0}$. Let

$$\hat{\Psi}(\mathrm{t}) = \int_{C_K} \Psi(a)|a|^{it}\, d^*a\,.$$

Then $\hat{\Psi}(\mathrm{t})$ has compact support.

Proof. We have seen that an extension $\tilde{\chi}_0$ of χ_0 as a character of C_K has the form $\tilde{\chi}_0 = \chi_0 |\cdot|^\rho\, (\rho \in i\mathbb{R})$. Then

$$\tilde{\chi}_0(g) = \chi_0(a)|g|^\rho \quad g \in C_K,$$

where $a = g/|g| \in C_{K,1}$. Fix an extension $\tilde{\chi}_0$ as being equal to 1 on N, then

$$\tilde{\chi}_0(|g|) = \chi_0(1)||g||^\rho = 1.$$

Thus $\rho = 0$. We can consider that it has the form $\tilde{\chi}_0 = \chi_0|\cdot|^0$.

For $g \in C_K$, put $g = a \cdot a^{-1}g$. Since $a^{-1}g \in N$, it turns out that $|g| = a^{-1}g$. Then,

$$\eta(g) = \chi_0(a) \cdot \chi_0(a^{-1}) \cdot \eta(g) = \chi_0(a) \cdot \eta(a^{-1}g) = \chi_0(a) \cdot \eta(|g|).$$

Now, $\tilde{\chi}_0(g) = \chi_0(a)|g|^0 = \chi_0(a)$. Thus we can write it as

$$\eta(g) = \tilde{\chi}_0(g)\Psi(|g|)$$

Since $\int_{C_K} |\Psi(|g|)|^2 (1+(\log|g|)^2)^{-\delta/2} d^*g < \infty$, we can say that $\Psi(|g|)$ is a tempered distribution.

Denote h's Fourier transform by $\hat{h}$ (or $\mathcal{F}(h)$):

$$\mathcal{F}(h)(\chi, z) = \int_{C_K} h(\mu)\chi(\mu)|\mu|^z\, d^*\mu\,.$$

Let $\tilde{h}(x) = h(x^{-1})$. Then

$$\begin{aligned}(V^\tau(h)\eta)(x) &= \int_{C_K} h(a)\eta(ax)d^*a \\ &= \int_{C_K} h(yx^{-1})\eta(y)d^*y \;=\; \int_{C_K} \tilde{h}(xy^{-1})\eta(y)d^*y \;=(\tilde{h} * \eta)(x).\end{aligned}$$

One has

Lemma 3.1. There exists an approximate unit $f_n \in \mathcal{S}(C_K)$, such that $\hat{f}_n$ has compact support, $\|V(f_n)\|_\delta \leq \mathrm{C}\ \ \forall n$, and

$$V(f_n) \longrightarrow 1 \text{ strongly in } L^2{}_\delta(C_K).$$

From this lemma, we can say that for any $\xi(x) \in L^2{}_\delta(C_K)$

$$(V(f)\xi)(x) = \xi(x)$$

for some f such that $\hat{f}$ has compact support. Consider its dual case, then we can say that for any $\eta(g) \in L^2{}_{-\delta,\,\chi_0}$

$$(V^\tau(h)\eta)(g) = \eta(g)$$

for some h such that $\hat{h}$ has compact support.

We have

$$\mathcal{F}((\tilde{h} * \eta)) = \mathcal{F}(V^\tau(h)\eta) = \mathcal{F}(\eta).$$

Here $\mathcal{F}((\tilde{h} * \eta)) = \mathcal{F}(\tilde{h}) \cdot \mathcal{F}(\eta)$. Since $\hat{h}$ has compact support, $\mathcal{F}(\tilde{h})$ also has compact support. Therefore $\mathcal{F}((\tilde{h} * \eta))$ has compact support. It means that $\mathcal{F}(\eta)$ has compact support from the above equation. We may identify $\eta(g)$ with $\Psi(|g|)$, so we can say that $\hat{\Psi}(\mathrm{t})$ has compact support.

□

Fix an extension $\tilde{\chi}_0$ of χ_0 as being equal to 1 on N. For any $\eta(g) \in L^2{}_{-\delta,\,\chi_0}$, write it as

$$\eta(g) = \tilde{\chi}_0(g)\Psi(|g|) \text{ where } \int_{C_K} |\Psi(|g|)|^2 (1+(\log|g|)^2)^{-\delta/2} d^*g < \infty.$$

Put the "Fourier expansion" of $\Psi(|g|)$;

$$\Psi(|g|) = \int_{-\infty}^{\infty} \hat{\Psi}(\mathrm{t})|g|^{it}\, d\mathrm{t} \text{ where } \hat{\Psi}(\mathrm{t}) = \int_{C_K} \Psi(a)|a|^{it}\, d^*a\,.$$

Thus,

$$\eta(g) = \int_{-\infty}^{\infty} \tilde{\chi}_0(g)|g|^{it}\, \hat{\Psi}(\mathrm{t})d\mathrm{t}\,.$$

Since $\hat{\Psi}(\mathrm{t})$ has compact support, we can compute as follows;

$$\eta(g) \in \mathrm{Im(T)}^0 \Longleftrightarrow \langle \mathrm{T}f, \eta \rangle = \int_{C_K} \mathrm{T}f(a) \int_{-\infty}^{\infty} \tilde{\chi}_0(a)|a|^{it} \hat{\Psi}(\mathrm{t}) d\mathrm{t}\, d^*a$$
$$= \int_{-\infty}^{\infty} \int_{C_K} \mathrm{T}f(a) \tilde{\chi}_0(a)|a|^{it} \hat{\Psi}(\mathrm{t}) d^*a\, d\mathrm{t} = 0, \quad \forall f \in \mathcal{S}(\mathrm{A}_K)_0 .$$

Lemma 3.2. For $\mathrm{Re}(s) > 0$, and any character $\tilde{\chi}_0$ of C_K,

$$\int_{C_K} (\mathrm{T}f)(a) \tilde{\chi}_0(a)|a|^{s-1/2} d^*a = c\, L(\tilde{\chi}_0, s) \Delta'_{\tilde{\chi}_0 s}(f), \quad \forall f \in \mathcal{S}(\mathbb{A}_K)_0.$$

where the non zero constant c depends upon the normalization of the Haar measure d^*a on C_K.

Proof. Consider a fundamental domain D for the action of K^* on $\mathbb{A}_K{}^*$. Then $\mathbb{A}_K{}^* = D \cup r_1 D \cup r_2 D \cup \cdots$. We may identify D with C_K. It holds that

$$\int_{C_K} (\mathrm{T}f)(a) \tilde{\chi}_0(a)|a|^{s-1/2} d^*a = \int_{C_K} \sum_{r\in K^*} f(ra) \tilde{\chi}_0(a)|a|^{s} d^*a$$
$$= \sum_{r\in K^*} \int_{C_K} f(ra) \tilde{\chi}_0(a)|a|^{s} d^*a ,$$

we can consider that $\tilde{\chi}_0(a)|a|^s$ is a quasi-character of C_K then

$$\tilde{\chi}_0(ra)|ra|^s = \tilde{\chi}_0(a)|a|^s \qquad a \in C_K \;\; r \in K^*,$$

so

$$= \sum_{r\in K^*} \int_{rC_K} f(a) \tilde{\chi}_0(a)|a|^{s} d^*a$$
$$= c \int_{\mathbb{A}_K{}^*} f(a) \tilde{\chi}_0(a)|a|^{s} d^*a .$$

From the lemma 1.1, for $\mathrm{Re}(s) > 1$

$$c \int_{\mathbb{A}_K{}^*} f(a) \tilde{\chi}_0(a)|a|^{s} d^*a = c\, L(\tilde{\chi}_0, s) \Delta'_{\tilde{\chi}_0 s}(f).$$

Thus

$$\int_{C_K} (\mathrm{T}f)(a) \tilde{\chi}_0(a)|a|^{s-1/2} d^*a = c\, L(\tilde{\chi}_0, s) \Delta'_{\tilde{\chi}_0 s}(f).$$

It is said that

$$\Pi_v \Delta_{\tilde{\chi}_0, v}{}^s(f_v) = \int_{\mathbb{A}_K{}^*} f(a) \tilde{\chi}_0(a)|a|^{s} d^*a$$

at $\mathrm{Re}(s) > 1$. Since the left term makes sense whenever $\mathrm{Re}(s) > 0$, the equation: $\int_{C_K} (\mathrm{T}f)(a) \tilde{\chi}_0(a)|a|^{s-1/2} d^*a = c\, L(\tilde{\chi}_0, s) \Delta'_{\tilde{\chi}_0 s}(f)$ holds for $\mathrm{Re}(s) > 0$.

□

Theorem 3.1. Suppose $\eta(g) \in L^2_{-\delta,\chi_0}$. Fix an extension $\tilde{\chi}_0$ of χ_0 as being equal to 1 on N.

$$\eta(g) \in \mathrm{Im(T)}^0 \iff L(\tilde{\chi}_0, 1/2+it)\hat{\Psi}(\mathrm{t}) = 0;\ \mathrm{t} \in \mathbb{R}.$$

Proof. To any function $b \in C_c^{\infty}(\mathbb{R}^*{}_+)$, we can assign a test function $f \in \mathcal{S}(\mathbb{A}_K)_0$ such that

$$\Delta'_{\tilde{\chi}_0{}^s}(f) = \int_{\mathbb{R}^*_+} b(x)|x|^s d^*x, \ \mathrm{Re}(s)>0 .$$

From the lemma 3.2,

$$\int_{C_K} \mathrm{T}f(a)\tilde{\chi}_0|a|^{it} d^*a = L(\tilde{\chi}_0, 1/2+it)\,\Delta'_{\tilde{\chi}_0{}^{1/2+it}}(f).$$

Thus it turns out that

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{C_K} \mathrm{T}f(a)\tilde{\chi}_0(a)|a|^{it}\hat{\Psi}(\mathrm{t})d^*a\,d\mathrm{t} &= \int_{-\infty}^{\infty} L(\tilde{\chi}_0, 1/2+it)\Delta'_{\tilde{\chi}_0{}^{1/2+it}}(f)\hat{\Psi}(\mathrm{t})d\mathrm{t} \\ &= \int_{-\infty}^{\infty}\int_{\mathbb{R}^*_+} L(\tilde{\chi}_0, 1/2+it)\hat{\Psi}(\mathrm{t})b(x)|x|^{1/2+it} d^*x\,d\mathrm{t}\end{aligned}$$

for an arbitrary $b \in C_c^{\infty}(\mathbb{R}^*{}_+)$. Then it holds that

$$\eta(g) \in \mathrm{Im(T)}^0 \iff \int_{-\infty}^{\infty}\int_{\mathbb{R}^*_+} L(\tilde{\chi}_0, 1/2+it)\hat{\Psi}(\mathrm{t})b(x)|x|^{1/2+it} d^*x\,d\mathrm{t} = 0$$

for an arbitrary b. Therefore

$$\eta(g) \in \mathrm{Im(T)}^0 \iff L(\tilde{\chi}_0, 1/2+it)\hat{\Psi}(\mathrm{t}) = 0.$$

□

Lemma 3.3. Suppose that $L(\tilde{\chi}_0, 1/2+it)\hat{\Psi}(\mathrm{t}) = 0$. We get that $\hat{\Psi}(\mathrm{t})$, as a distribution, is a finite linear combination of the distributions:

$$\delta_{\mathrm{t}}^{(k)};\ \mathrm{t} \text{ satisfies } L(\tilde{\chi}_0, 1/2+it) = 0,\ k < \text{order of the zero and } k < \frac{\delta-1}{2}.$$

proof. Let

$$\int_{-\infty}^{\infty}\int_{C_K} \mathrm{T}f(a)\tilde{\chi}_0(a)|a|^{it}\hat{\Psi}(\mathrm{t})d^*a\,d\mathrm{t} = \langle \hat{\Psi}(\mathrm{t}),\ \int_{C_K} \mathrm{T}f(a)\tilde{\chi}_0(a)|a|^{i\rho} d^*a \rangle,$$

and we shall consider that $\hat{\Psi}(\mathrm{t})$ is a distribution. Suppose that $L(\tilde{\chi}_0, 1/2+it)\hat{\Psi}(\mathrm{t}) = 0$. If $\hat{\Psi}(\mathrm{t}) \neq 0$ then $L(\tilde{\chi}_0, 1/2+it) = 0$. We may say that $\hat{\Psi}(\mathrm{t})$ is a distribution supported on $\{\mathrm{t} \mid L(\tilde{\chi}_0, 1/2+it) = 0\}$ consisting of a single point. Therefore there are coefficients c_k such that

$$\hat{\Psi}(\mathrm{t}) = \Sigma\, c_k\, \delta_{\mathrm{t}}^{(k)}.$$

□

Corollary 3.1. Suppose that $\eta(g) \in L^2{}_{-\delta,\,\chi_0}$ and $\eta(g) \in \mathrm{Im}(\mathrm{T})^0$. Then $\eta(g)$ is a finite linear combination of functions of the form,

$$\eta_{\mathrm{t},\,k}(g) = \tilde{\chi}_0(g)|g|^{it}(\log|g|)^k$$

where t satisfies $L(\tilde{\chi}_0, 1/2 + i\mathrm{t}) = 0$ and $k <$ order of the zero. Moreover

$$\eta_{\mathrm{t},\,k}(g) \in \mathrm{Im}(\mathrm{T})^0.$$

Proof. From the above lemma

$$\langle \hat{\Psi}(\mathrm{t}), \int_{C_K} \mathrm{T}f(a)\tilde{\chi}_0(a)|a|^{i\rho}\, d^*a \rangle = \langle \Sigma\, c_k\, \delta_{\mathrm{t}}{}^{(k)}, \int_{C_K} \mathrm{T}f(a)\tilde{\chi}_0(a)|a|^{i\rho}\, d^*a \rangle.$$

We can compute as follows;

$$\langle \delta_{\mathrm{t}}{}^{(k)}, \int_{C_K} \mathrm{T}f(a)\tilde{\chi}_0(a)|a|^{i\rho}\, d^*a \rangle = (-1)^k \langle \delta_{\mathrm{t}}, \left(\tfrac{\partial}{\partial\rho}\right)^k \int_{C_K} \mathrm{T}f(a)\tilde{\chi}_0(a)|a|^{i\rho}\, d^*a \rangle$$

and

$$\begin{aligned}\langle \delta_{\mathrm{t}}, \left(\tfrac{\partial}{\partial\rho}\right)^k \int_{C_K} \mathrm{T}f(a)\tilde{\chi}_0(a)|a|^{i\rho}\, d^*a \rangle &= \left(\tfrac{\partial}{\partial\rho}\right)^k \int_{C_K} \mathrm{T}f(a)\tilde{\chi}_0(a)|a|^{i\rho}\, d^*a\,\Big|_{\rho=\mathrm{t}} \\ &= \int_{C_K} \mathrm{T}f(a)\tilde{\chi}_0(a)|a|^{it}(\log|a|^{i})^k d^*a \\ &= (i)^k \int_{C_K} \mathrm{T}f(a)\tilde{\chi}_0(a)|a|^{it}(\log|a|)^k d^*a\,.\end{aligned}$$

Therefore

$$\langle \delta_{\mathrm{t}}{}^{(k)}, \int_{C_K} \mathrm{T}f(a)\tilde{\chi}_0(a)|a|^{i\rho}\, d^*a \rangle = (-\,i)^k \int_{C_K} \mathrm{T}f(a)\tilde{\chi}_0(a)|a|^{it}(\log|a|)^k d^*a\,.$$

It turns out that

$$\begin{aligned}\int_{C_K} \mathrm{T}f(a)\int_{-\infty}^{\infty} \tilde{\chi}_0(a)|a|^{it}\hat{\Psi}(\mathrm{t})d\mathrm{t}\,d^*a &= \int_{-\infty}^{\infty}\int_{C_K} \mathrm{T}f(a)\tilde{\chi}_0(a)|a|^{it}\hat{\Psi}(\mathrm{t})d^*a\,d\mathrm{t} \\ &= \langle \hat{\Psi}(\mathrm{t}), \int_{C_K} \mathrm{T}f(a)\tilde{\chi}_0(a)|a|^{i\rho}\, d^*a \rangle \\ &= \int_{C_K} \mathrm{T}f(a)\sum(-i)^k c_k\tilde{\chi}_0(a)|a|^{it}(\log|a|)^k d^*a\,,\end{aligned}$$

so we see that

$$\int_{-\infty}^{\infty} \tilde{\chi}_0(a)|a|^{it}\hat{\Psi}(\mathrm{t})d\mathrm{t} = \sum(-i)^k c_k\tilde{\chi}_0(a)|a|^{it}(\log|a|)^k\,.$$

Here, $\eta(g) = \int_{-\infty}^{\infty} \tilde{\chi}_0(g)|g|^{it}\hat{\Psi}(\mathrm{t})d\mathrm{t}$.

Let $L(\tilde{\chi}_0, s) = 0$ and let $k <$ order of the zero. From the lemma 3.2,

$$\int_{C_K} (\mathrm{T}f)(a)\tilde{\chi}_0(a)|a|^{s-1/2}\, d^*a = c\,L(\tilde{\chi}_0, s)\,\Delta'_{\tilde{\chi}_0{}^s}(f).$$

Thus we can say that

$$\left(\frac{\partial}{\partial s}\right)^k \int_{C_K} (\mathrm{T}f)(a)\tilde{\chi}_0(a)|a|^{s-1/2}\, d^*a = 0.$$

Here

$$\left(\frac{\partial}{\partial s}\right)^k \int_{C_K} (\mathrm{T}f)(a)\tilde{\chi}_0(a)|a|^{s-1/2}\, d^*a = \int_{C_K} \mathrm{T}f(a)\tilde{\chi}_0(a)|a|^{s-1/2}(\log|a|)^k d^*a\,.$$

Therefore

$$\int_{C_K} \mathrm{T}f(a)\tilde{\chi}_0(a)|a|^{s-1/2}(\log|a|)^k d^*a = 0.$$

It means that $\tilde{\chi}_0(g)|g|^{s-1/2}(\log|g|)^k \in \mathrm{Im}(\mathrm{T})^0$. In our case $s = 1/2 + i\mathrm{t}$.

□

Let $\mathcal{H} \cong L^2{}_\delta(C_K)/\mathrm{Im}(\mathrm{T})$. We shall think of the left regular representation W of C_K on $\mathcal{H}$: $(W, \mathcal{H})$ where one deduces W from V. We will consider its dual $\mathcal{H}^*$: $(W^\tau, \mathcal{H}^*)$ where one deduces W^τ from V^τ. Here

$$\mathcal{H}^* \cong (L^2{}_\delta(C_K)/\mathrm{Im}(\mathrm{T}))^* \cong \mathrm{Im}(\mathrm{T})^0.$$

One decomposes $\mathcal{H}^*$ in the direct sum of the subspaces

$$\mathcal{H}^* = \bigoplus_{\chi_0 \in \hat{C}_{K,1}} \mathcal{H}^*{}_{\chi_0};\ \mathcal{H}^*{}_{\chi_0} = \{\xi \mid \xi(ag) = \chi_0(a)\xi(g) \ \ \forall a \in C_{K,1}\},$$

where $\mathcal{H}^*{}_{\chi_0} \subseteq L^2{}_{-\delta,\chi_0}$. We see that the functions of the form $\eta_{\mathrm{t},k}(g)$ consists of a basis of $\mathcal{H}^*$. Here

$$\begin{aligned}(W^\tau(g)\eta_{\mathrm{t},k})(x) &= \tilde{\chi}_0(gx)|gx|^{i\mathrm{t}}(\log|gx|)^k \\ &= \tilde{\chi}_0(g)|g|^{i\mathrm{t}}\tilde{\chi}_0(x)|x|^{i\mathrm{t}}(\log|g|+\log|x|)^k \qquad \forall g, x \in C_K.\end{aligned}$$

It turns out that

$$(W^\tau(g)\eta_{\mathrm{t},k})(x) = \sum_{n=0}^{k} {}_n\mathrm{C}_k\, \eta_{\mathrm{t},n}(g)\cdot\eta_{\mathrm{t},k-n}(x).$$

Thus W^τ isn't semi simple in general. Now, we see that

$$(W^\tau(g)\eta_{\mathrm{t},k})(x) = \sum_{n=0}^{k} {}_n\mathrm{C}_k\, g^{i\mathrm{t}}(\log g)^n\cdot\eta_{\mathrm{t},k-n}(x) \quad \forall g \in N.$$

Let $W^\tau{}_{\chi_0} = W^\tau|_{\mathcal{H}^*{}_{\chi_0}}$ and $e^{\mathrm{t}} = g$ $(g \in N)$. We will write the action of N on $\mathcal{H}^*{}_{\chi_0}$ as

$$W^\tau{}_{\chi_0}(e^{\mathrm{t}}):\ \mathbb{R} \longrightarrow \mathcal{H}^*{}_{\chi_0}.$$

The following things

(a) $\quad W^\tau{}_{\chi_0}(e^0) = 1,$

(b) $\quad W^\tau{}_{\chi_0}(e^{\mathrm{t}+\mathrm{s}}) = W^\tau{}_{\chi_0}(e^{\mathrm{t}})W^\tau{}_{\chi_0}(e^{\mathrm{s}})$

are satisfied. Thus $W^\tau{}_{\chi_0}(e^{\mathrm{t}})$ is a semi-group. From the theory of semi-group, we can say that

$$W^\tau{}_{\chi_0}(e^{\mathrm{t}}) = e^{\mathrm{t}D^\tau{}_{\chi_0}}$$

where

$$D^{\tau}{}_{\chi_0}\xi = \lim_{t\to 0^+}\frac{W^{\tau}{}_{\chi_0}(e^{t})\xi - W^{\tau}{}_{\chi_0}(e^{0})\xi}{t} = \frac{dW^{\tau}{}_{\chi_0}(e^{t})\xi}{dt}\Bigg|_{t=0}.$$

Here

$$\frac{d(W^{\tau}{}_{\chi_0}(e^{t})\eta_{t,k})(x)}{dt} = \sum_{n=0}^{k} {}_{n}C_{k}\,\{(e^{t\cdot it})'\cdot t^{n} + e^{t\cdot it}\cdot (t^{n})'\}\cdot\eta_{t,k-n}(x)$$
$$= \sum_{n=0}^{k} {}_{n}C_{k}\,\{it\, e^{t\cdot it}\cdot t^{n} + e^{t\cdot it}\cdot n t^{n-1}\}\cdot\eta_{t,k-n}(x).$$

First,

$$\frac{d\,{}_{0}C_{k}\,e^{t\cdot it}\cdot\eta_{t,k}(x)}{dt}\Bigg|_{t=0} = {}_{0}C_{k}\,it\,e^{t\cdot it}\cdot\eta_{t,k}(x)\,|_{\,t=0} = it\cdot\eta_{t,k}(x).$$

Secondly,

$$\frac{d\,{}_{1}C_{k}\,e^{t\cdot it}\,t\cdot\eta_{t,k-1}(x)}{dt}\Bigg|_{t=0} = {}_{1}C_{k}\,\{it\,e^{t\cdot it}\cdot t + e^{t\cdot it}\cdot t^{0}\}\cdot\eta_{t,k-1}(x)\,|_{\,t=0}.$$

We shall think that $t^{0} = \lim_{a\to 0} t^{a}$. Then $\{it\,e^{t\cdot it}\cdot t + e^{t\cdot it}\cdot t^{0}\}\,|_{\,t=0} = 0$. So

$$\frac{d\,{}_{1}C_{k}\,e^{t\cdot it}\,t\cdot\eta_{t,k-1}(x)}{dt}\Bigg|_{t=0} = 0.$$

Finally, when $n > 1$

$$\frac{d\,{}_{n}C_{k}\,e^{t\cdot it}\,t^{n}\cdot\eta_{t,k-n}(x)}{dt}\Bigg|_{t=0} = {}_{n}C_{k}\{it\,e^{t\cdot it}\cdot t^{n} + e^{t\cdot it}\cdot n t^{n-1}\}\cdot\eta_{t,k-n}(x)\,|_{\,t=0} = 0.$$

So we see that

$$\frac{d(W^{\tau}{}_{\chi_0}(e^{t})\eta_{t,k})(x)}{dt}\Bigg|_{t=0} = it\cdot\eta_{t,k}(x).$$

Thus

$$D^{\tau}{}_{\chi_0}\eta_{t,k}(x) = it\cdot\eta_{t,k}(x).$$

The operator $D^{\tau}{}_{\chi_0}$ has discrete spectra. We may say that the discrete spectrum is given by the element $\eta_{t,k}(x)$ of $\mathcal{H}^{*}{}_{\chi_0}$.

Theorem 3.2. $\chi_0 \in \hat{C}_{K,1}$, $\delta > 1$. Then D_{χ_0} has discrete spectra, $\mathrm{sp}D_{\chi_0} \subset i\mathbb{R}$ is the set of imaginary parts of zeros of the L function with Grössencharakter $\tilde{\chi}_0$ which have real part equal to $1/2$;

$$\rho \in \mathrm{sp}D_{\chi_0} \iff L(\tilde{\chi}_0, 1/2+\rho) = 0 \text{ and } \rho \in i\mathbb{R}, \text{ where } \tilde{\chi}_0 \text{ is the unique extension of } \chi_0 \text{ to } C_K \text{ which is equal to } 1 \text{ on } N.$$

Moreover the multiplicity of ρ in $\mathrm{sp}D_{\chi_0}$ is equal to the largest integer of $k < \frac{\delta - 1}{2}$, $k <$ *multiplicity of* $1/2+\rho$ as a zero of L.

Now, let h be a test function on C_K and set

$$W(h) = \int_{C_K} h(g) W(g)\, d^* g\,.$$

Denote h's Fourier transform by $\hat{h}$:

$$\hat{h}(\chi, z) = \int_{C_K} h(\mu)\chi(\mu)|\mu|^z\, d^*\mu\,.$$

We can compute

$$\langle\, (W^\tau(g)\eta_{\mathrm{t},k})(x),\, \eta_{\mathrm{t},k}(x)\,\rangle = \langle\, \sum_{n=0}^{k} {}_n\mathrm{C}_k\, \eta_{\mathrm{t},n}(g)\cdot \eta_{\mathrm{t},k-n}(x),\, \eta_{\mathrm{t},k}(x)\,\rangle$$
$$= \eta_{\mathrm{t},0}(g) = \tilde{\chi}_0(g)|g|^{it}.$$

Therefore,

$$\mathrm{tr}W^\tau(h) = \sum_{L(\tilde{\chi}_0,\, 1/2+it) = 0} \hat{h}(\tilde{\chi}_0, it).$$

Here, $\mathrm{tr}W = \mathrm{tr}W^\tau$.

Corollary 3.2. For any Schwartz function $h \in \mathcal{S}(C_K)$ the operator $\int_{C_K} h(g)W(g)\, d^*g$ in $\mathcal{H}$ is of trace class, and its trace is given by

$$\mathrm{tr}W(h) = \sum_{\substack{L(\tilde{\chi}_0,\, 1/2+\rho) = 0 \\ \rho \in i\mathbb{R}}} \hat{h}(\tilde{\chi}_0, \rho)$$

where the multiplicity is counted as in the theorem 3.2 and where Fourier transform $\hat{h}$ of h is defined by $\hat{h}(\chi, z) = \int_{C_K} h(\mu)\chi(\mu)|\mu|^z\, d^*\mu$.

We can obtain the following exact sequences:

$$0 \to L^2_\delta(X)_0 \to L^2_\delta(X) \to \mathbb{C} \oplus \mathbb{C}(1) \to 0$$

and

$$0 \to L^2_\delta(X)_0 \overset{\mathrm{T}}{\to} L^2_\delta(C_K) \to \mathcal{H} \to 0 \,.$$

We will compute $\mathrm{tr}U(h)$ for $(U, L^2{}_\delta(X))$ from spectral side. From the above first sequence, considering *Lefchetz formula*, we will see that

$$\mathrm{A} = \mathrm{tr}U(h)|_{L^2{}_\delta(X)_0} - \mathrm{tr}U(h)|_{L^2{}_\delta(X)} + \mathrm{tr}U(h)|_{\mathbb{C}\oplus\mathbb{C}(1)}.$$

From the second sequence, we will obtain

$$\mathrm{A}' = \mathrm{tr}U(h)|_{L^2{}_\delta(X)_0} - \mathrm{tr}U(h)|_{L^2{}_\delta(C_K)} + \mathrm{tr}U(h)|_{\mathcal{H}}.$$

Therefore, it is satisfied that

$$\mathrm{tr}U(h)|_{L^2{}_\delta(X)} = \mathrm{tr}U(h)|_{\mathbb{C}\oplus\mathbb{C}(1)} - \mathrm{tr}U(h)|_{\mathcal{H}} + \mathrm{tr}U(h)|_{L^2{}_\delta(C_K)} + \mathrm{A}' - \mathrm{A}.$$

We try to compute $\mathrm{tr}U(h)$ spectrally. Here,

$$U(h) = \int_{C_K} h(g)U(g)\, d^*g \,.$$

The first term $\mathrm{tr}U(h)|_{\mathbb{C}\oplus\mathbb{C}(1)}$ gives

$$\hat{h}(0) + \hat{h}(1)$$

since

$$\mathrm{tr}U(h)|_{\mathbb{C}} = \int_{C_K} h(g)1|g|^0\, d^*g = \hat{h}(1,0) = \hat{h}(0)$$

and

$$\mathrm{tr}U(h)|_{\mathbb{C}(1)} = \int_{C_K} h(g)1|g|^1\, d^*g = \hat{h}(1,1) = \hat{h}(1).$$

Consider that $\mathrm{T}(U(g)\xi)(a) = |g|^{1/2}(V(g)\mathrm{T}\xi)(a)$ then it turns out that $\mathrm{T}U(g)\mathrm{T}^{-1} = |g|^{1/2}V(g)$. We will see that $(U, L^2{}_\delta(C_K))$ coincides with $(|\cdot|^{1/2}V, L^2{}_\delta(C_K))$. Thus

$$U|_{L^2{}_\delta(C_K)} \text{ is } (|\cdot|^{1/2}V, L^2{}_\delta(C_K)) \text{ and } U|_{\mathcal{H}} \text{ is } (|\cdot|^{1/2}V, \mathrm{Im}(\mathrm{T})^0).$$

So, from the corollary 3.2, we will understand that the second term gives

$$\sum_{\substack{L(\tilde{\chi}_0,\rho)=0 \\ \mathrm{Re}\rho = 1/2}} \hat{h}(\tilde{\chi}_0, \rho)\,.$$

Finally, the term $\mathrm{tr}U(h)|_{L^2{}_\delta(C_K)} + \mathrm{A}' - \mathrm{A}$ gives $\infty \cdot h(1)$. Here

$$\begin{aligned} \mathrm{tr}U(h)\,|_{L^2_\delta(C_K)} \;&=\; \int_{C_K} h(g)|g|^{1/2}\,\tilde{\chi}_0(g)d^*g \;+\; \int_{C_K} h(g)|g|^{1/2}\,\tilde{\chi}_0'(g)d^*g \;+\;\cdots \\ &= \int_{C_K} h(g)|g|^{1/2} \sum_{\chi_0 \in \hat{C}_{K,\,1}} \tilde{\chi}_0(g)d^*g\,; \end{aligned}$$

since

$$\sum_{\chi_0 \in \hat{C}_{K,\,1}} \tilde{\chi}_0(g) \;=\; \begin{cases} |C_{K,1}| & g=1 \\ 0 & g \neq 1 \end{cases},$$

$$= \infty \cdot h(1).$$

Therefore

$$\mathrm{tr}U(h) = \hat{h}(0) + \hat{h}(1) \;-\; \sum_{\substack{L(\tilde{\chi}_0,\,\rho)=0 \\ \mathrm{Re}\,\rho = 1/2}} \hat{h}(\tilde{\chi}_0, \rho) + \infty \cdot h(1).$$

4. tr$U$ and Riemann Hypothesis

We try to compute tr$U$ geometrically. Let us start with the computation of the distribution theoretic trace of the operator U: $C^\infty(M) \longrightarrow C^\infty(M)$,

$$(U\xi)(x) = \xi(\varphi(x)).$$

Let $k(x, y)$ be the Schwartz distribution on $M \times M$ such that

$$(U\xi)(x) = \int_M k(x, y)\xi(y)\,dy\,.$$

One gets $k(x, y) = \delta(y - \varphi(x))$ where δ is the Dirac distribution. Then

$$\mathrm{tr}U = \int_M k(x, x)\,dx\,.$$

Here $k(x, x) = \delta(x - \varphi(x))$. Put $g(x) = x - \varphi(x)$. It is known that

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}$$

where sum extends over all roots x_i of $g(x)$, and that

$$\int_M \xi(x)\delta(g(x))\,dx = \sum_i \frac{\xi(x_i)}{|g'(x_i)|}.$$

Therefore, one can compute the trace as a finite sum $\sum_{x,\ \varphi(x)=x}$ and get

$$\mathrm{tr}U = \sum_{x,\ \varphi(x)=x} \frac{1}{|1 - \varphi'(x)|}.$$

Let

$$\pi\colon \mathbb{A}_K \longrightarrow X, \qquad c\colon \mathbb{A}_K^{\,*} \longrightarrow C_K.$$

Put

$$\pi(\tilde{x}) = x,\ x \in X \quad \text{and} \quad c(j) = \lambda,\ j \in \mathbb{A}_K^{\,*}.$$

Consider

$$f: X \times C_K \longrightarrow X, \quad f(x, \lambda) = \lambda x.$$

It corresponds to the above φ. Let $Z = \mathrm{Graph}(f) = \{(x, \lambda, f(x, \lambda))\}$. It corresponds to the above $k(x, y)$. The diagonal map is

$$\theta: X \times C_K \longrightarrow X \times C_K \times X, \quad \theta(x, \lambda) = (x, \lambda, x).$$

We see that $Z \cap (X \times C_K \times X)$ corresponds to $k(x, x)$. Here $\theta^{-1}(Z)$ consists of the pair $(x, \lambda) \in X \times C_K$ such that $x \in X$, $x = \lambda x$. There exist $r, q \in K^*$ such that $x = r\tilde{x}$ and $\lambda = qj$. Thus $r\tilde{x} = qj \cdot r\tilde{x}$. Let $\tilde{j} = qj$. We obtain

$$\tilde{j}\,\tilde{x} = \tilde{x}.$$

Recall

$$\mathbb{A}_K = \prod\nolimits'_{v<\infty} K_v \times \prod\nolimits_{v\,|\,\infty} K_v = \Pi'_v K_v$$

where $\prod'_{v<\infty} K_v = \{(x_v) \in \prod_{v<\infty} K_v \mid x_v \in \mathcal{O}_v \textit{ for almost all } v\}$. The equality $\tilde{j}\,\tilde{x} = \tilde{x}$ means that $\tilde{j}_v\,\tilde{x}_v = \tilde{x}_v$. If $\tilde{x}_v \neq 0$ for all v, it follows that $\tilde{j}_v = 1$ for all v and $\tilde{j} = 1$. So the projection of $\theta^{-1}(Z) \cap (C_K - \{1\})$ on X is the union of the hyperplanes

$$\cup H_v\,;\; H_v = \pi(\tilde{H}_v),\; \tilde{H}_v = \{x \in \mathbb{A}_K \mid x_v = 0\}.$$

Each $\tilde{H}_v$ is closed in $\mathbb{A}_K$ and is invariant under multiplication by elements of K^*. Thus each H_v is a closed subset of X. Namely the fixed points of X under f come from the union on the hyperplanes. Let x be a generic point of H_v;

$$x \in H_v, \text{ where } x_\mu = 0 \text{ iff } \mu = v.$$

Then H_v is the closure of the orbit of x, where the orbit of x is $\{gx \mid g \in C_K\}$. Denote the orbit of such point x by γ_x and its isotropy group $\{g \in C_K \mid gx = x\}$ by I_x. It turns out

$$\mathrm{tr}U(h) = \sum_{\gamma_x}\sum_{\lambda} \#\gamma_x \frac{h(\lambda)}{|1-\lambda|}$$

where $\#\gamma_x$ is the length of the γ_x, λ varies in I_x and h is a test function on C_K which vanishes at 1. We have seen that the fixed points of X come from the union on the hyperplanes. Although $H_v \neq \gamma_x$, we can justify the above computation. It means that not every point of the hyperplane contributes to the computation of $\mathrm{tr}U(h)$.

Here

$$I_x = K_\nu{}^*$$

by the map $\lambda \in K_\nu{}^* \longrightarrow (1, \cdots, 1, \lambda, 1, \cdots)$. Then C_K/I_x is compact. There exists a natural bijection between the orbit of x and C_K/I_x. Thus $\#\gamma_x = \int_{C_K/I_x} d^*\lambda$. We shall normalize the Haar measures to be $\int_{C_K/I_x} d^*\lambda = 1$. This is insured by normalizing the Haar measure of the multiplicative group C_K

$$\int_{|g| \in [1, \Lambda]} d^*g \sim \log \Lambda \qquad \Lambda \to +\infty.$$

We shall identify H_ν with ν. We can write down the above sum with

$$\sum_\nu \int_{K_\nu{}^*} \frac{h(\mu)}{|1-\mu|} d^*\mu.$$

We used

$$(U(\lambda)\xi)(x) = \xi(\lambda^{-1}x).$$

If $x = \lambda x$ then $x = \lambda^{-1}x$. Therefore, this amounts to replace the test function $h(\mu)$ by $h(\mu^{-1})$. It holds that

$$\mathrm{tr}U(h) = \sum_\nu \int_{K_\nu{}^*} \frac{h(\mu^{-1})}{|1-\mu|} d^*\mu.$$

We will compute $\mathrm{tr}U(h)$ using Fourier transformations. For the simpler situation, we shall only consider a finite set S of places of K. Let S be a finite set of places of K containing all infinite places. The S-units of K is given by

$$\mathcal{O}^*_{\mathrm{S}} = \{q \in K^* \mid |q_\nu| = 1 \quad \nu \notin \mathrm{S}\}.$$

$\mathcal{O}^*_{\mathrm{S}} \backslash \mathrm{J}_{\mathrm{S}}{}^1$ is compact where

$$\mathrm{J}_{\mathrm{S}} = \prod_{\nu \in \mathrm{S}} K_\nu{}^*$$

and

$$\mathrm{J}_{\mathrm{S}}{}^1 = \{j \in \mathrm{J}_{\mathrm{S}} \mid |j| = 1\}.$$

Let $C_{\mathrm{S}} = \mathcal{O}^*_{\mathrm{S}} \backslash \mathrm{J}_{\mathrm{S}}$ and $X_{\mathrm{S}} = \mathcal{O}^*_{\mathrm{S}} \backslash \mathrm{A}_{\mathrm{S}}$ where

$$\mathrm{A}_{\mathrm{S}} = \prod_{\nu \in \mathrm{S}} K_\nu.$$

We normalize the Haar measure of the multiplicative group C_{S} by

$$\int_{|g|\in[1,\Lambda]} d^*g \sim \log\Lambda \qquad \Lambda\to+\infty\,.$$

We will think of $L^2(X_S)$ which is obtained by a completion of $\mathcal{S}(\mathrm{A}_S)$ with the norm

$$\|f\|^2 = \int_{C_S}\left|\sum_{q\in\mathcal{O}^*_S} f(qx)\right|^2 |x|\,d^*x\,.$$

We define $U(\lambda)$ $\lambda\in C_S$ by

$$(U(\lambda)\xi)(x) = \xi(\lambda^{-1}x) \qquad \forall x\in\mathrm{A}_S.$$

We will think of the sum $\Sigma_{q\in\mathcal{O}^*_S} f(qx)$, $f\in\mathcal{S}(\mathrm{A}_S)$ and let T be an operator acting on functions on A_S:

$$\begin{aligned} T(\Sigma_{q\in\mathcal{O}^*_S} f(qx)) &= \int_{C_S} k(x,y)\sum_{q\in\mathcal{O}^*_S} f(qy)d^*y \\ &= \int_D k(x,y)\sum_{q\in\mathcal{O}^*_S} f(qy)d^*y \\ &= \sum_{q\in\mathcal{O}^*_S}\int_D k(x,q^{-1}y)f(y)d^*y\,. \end{aligned}$$

Here D is a fundamental domain for the action of $\mathcal{O}^*_S$ on J_S. Then we see that the trace of its action on $L^2(X_S)$ is given by

$$\mathrm{tr}T = \sum_{q\in\mathcal{O}^*_S}\int_D k(x,q^{-1}x)\,dx$$

since $k(qx, qy) = k(x, y)$ $q\in\mathcal{O}^*_S$,

$$= \sum_{q\in\mathcal{O}^*_S}\int_D k(qx,x)\,dx\,.$$

For a given smooth compactly supported function h on C_S, let

$$U(h) = \int_{C_S} h(g)U(g)\,d^*g$$

as an operator acting on $L^2(X_S)$. For any $h\in\mathcal{S}(C_S)$ having compact support, there exists a smooth compactly supported function f on J_S such that

$$\sum_{q\in\mathcal{O}^*_S} f(qg) = h(g) \qquad \forall g\in C_S.$$

$\mathrm{J}_S = D\cup q_1D\cup q_2D\cup\cdots$. Since the integral which is performed on C_S is equivalent to that on the fundamental domain D, so it holds that $U(f) = U(h)$. Let $T = U(f) = U(h)$ as an operator acting on $L^2(X_S)$. The Schwartz kernel of T is

$$k(x,y) = \int_{C_S} h(\lambda^{-1})\delta(y-\lambda x)d^*\lambda\,.$$

On the other hand, let P_Λ be the orthogonal projection onto the subspace,

$$P_\Lambda = \{ f \in L^2(X_S) \mid f(x) = 0,\ \forall x,\ |x| > \Lambda \}.$$

P_Λ is the multiplication operator by the function

$$\rho_\Lambda = \begin{cases} \rho_\Lambda(x) = 1 & \cdots\ |x| \le \Lambda \\ \rho_\Lambda(x) = 0 & \cdots\ |x| > \Lambda \end{cases} .$$

Put $\hat{P}_\Lambda = \mathcal{F}^{-1} P_\Lambda \mathcal{F}$ [*1] where $\mathcal{F}$ is the Fourier transform which depends upon the basic character α. Define the operator $\mathcal{R}$ by $\mathcal{R}f(x) = f(-x)$ then $\mathcal{F}^{-1} = \mathcal{R}\mathcal{F} = \mathcal{F}\mathcal{R}$. Here

$$\begin{aligned} \int_{\mathrm{As}} \rho_\Lambda(\xi) \int_{\mathrm{As}} f(x)\alpha(x\xi)\,dx\ \alpha(\xi\cdot -\eta)d\xi &= \int_{\mathrm{As}} f(x) \int_{\mathrm{As}} \rho_\Lambda(\xi)\alpha(x\xi)\alpha(\xi\cdot -\eta)d\xi dx \\ &= \int_{\mathrm{As}} f(x) \int_{\mathrm{As}} \rho_\Lambda(\xi)\alpha(\xi(x-\eta))d\xi dx \\ &= \int_{\mathrm{As}} f(x)\mathcal{F}(\rho_\Lambda)(x-\eta)\,dx . \end{aligned}$$

We see that $\hat{P}_\Lambda$ is the operator acting on $f \in L^2(X_S)$ like

$$(\hat{P}_\Lambda f)(x) = \int_{\mathrm{As}} \mathcal{F}(\rho_\Lambda)(y-x) f(y)\,dy .$$

Let

$$R_\Lambda = \hat{P}_\Lambda P_\Lambda \qquad \Lambda \in \mathbb{R}_+ .$$

Denote the Schwartz kernel of R_Λ by $r_\Lambda(x, y)$. We see that

$$(R_\Lambda f)(x) = \int_{\mathrm{As}} \mathcal{F}(\rho_\Lambda)(y-x)\rho_\Lambda(y) f(y)\,dy .$$

So

$$r_\Lambda(x, y) = \mathcal{F}(\rho_\Lambda)(y-x)\rho_\Lambda(y) = \rho_\Lambda(y)\mathcal{F}(\rho_\Lambda)(y-x).$$

Moreover, for $R_\Lambda T = R_\Lambda U(h)$,

$$\begin{aligned} ((R_\Lambda U(h))f)(x) &= \int_{\mathrm{As}} r_\Lambda(x, z) \int_{\mathrm{As}} k(z, y) f(y)\,dy\,dz \\ &= \int_{\mathrm{As}}\int_{\mathrm{As}} r_\Lambda(x, z)k(z, y)\,dz\, f(y)\,dy . \end{aligned}$$

Its kernel will be $\int_{\mathrm{As}} r_\Lambda(x, z)k(z, y)\,dz$.

In practice, it is more convenient to define by means of transpose. The Schwartz kernel of the transpose R^τ_Λ is

$$r_\Lambda{}^\tau(x, y) = \rho_\Lambda(x)\mathcal{F}(\rho_\Lambda)(x-y).$$

*1 $\hat{P}_\Lambda = \mathcal{F} P_\Lambda \mathcal{F}^{-1}$ in the original paper. However it must give a coherent explanation to define $\hat{P}_\Lambda = \mathcal{F}^{-1} P_\Lambda \mathcal{F}$.

Here, $\mathcal{F}(\rho_\Lambda)(x-y) = \int_{\mathrm{As}} \rho_\Lambda(z)\alpha(z(x-y))\,dz = \int_{z\in \mathrm{As},\,|z|\le\Lambda} \alpha(z(x-y))\,dz$. Thus

$$\rho_\Lambda(x)\mathcal{F}(\rho_\Lambda)(x-y) = \begin{cases} \int_{z\in \mathrm{As},\,|z|\le\Lambda} \alpha(z(x-y))\,dz & \cdots\ |x|\le\Lambda \\ 0 & \cdots\ |x|>\Lambda \end{cases}.$$

Moreover, $\langle U(h)\varphi, \Psi\rangle = \int_{\mathrm{As}}\int_{\mathrm{As}} k(x, y)\varphi(y)\,dy\,\Psi(x)\,dx = \int_{\mathrm{As}}\varphi(y)\int_{\mathrm{As}} k(x, y)\Psi(x)\,dx\,dy$. So, $\int_{\mathrm{As}} k(x, y)\Psi(x)\,dx = \int_{\mathrm{As}} k^\tau(x, y)\Psi(y)\,dy$ since $\langle U(h)\varphi, \Psi\rangle = \langle \varphi, U^\tau(h)\Psi\rangle$. It means that

$$k^\tau(x, y) = k(y, x).$$

Now,

$$((U(h)R_\Lambda) f)(\varphi) =_{\mathrm{def}} f((R^\tau{}_\Lambda U^\tau(h))\varphi).$$

Here,

$$\begin{aligned} \text{the left term} &= \langle (U(h)R_\Lambda) f, \varphi\rangle \\ &= \int_{\mathrm{As}}\int_{\mathrm{As}}\int_{\mathrm{As}} k(x, z) r_\Lambda(z, y)\,dz\, f(y)\,dy\ \varphi(x)\,dx \end{aligned}$$

and

$$\begin{aligned} \text{the right term} &= \langle f(x), (R^\tau{}_\Lambda U^\tau(h))\varphi\rangle \\ &= \int_{\mathrm{As}} f(x)\int_{\mathrm{As}}\int_{\mathrm{As}} r_\Lambda{}^\tau(x, z) k^\tau(z, y)\,dz\,\varphi(y)\,dy\,dx \\ &= \int_{\mathrm{As}}\int_{\mathrm{As}}\int_{\mathrm{As}} r_\Lambda{}^\tau(x, z) k^\tau(z, y)\,dz\, f(x)\,dx\ \varphi(y)\,dy. \end{aligned}$$

Therefore, we see that the Schwartz kernel of $U(h)R_\Lambda$ will be

$$\begin{aligned} \int_{\mathrm{As}} r_\Lambda{}^\tau(x, z) k^\tau(z, y)\,dz &= \int_{\mathrm{As}} k^\tau(z, y) r_\Lambda{}^\tau(x, z)\,dz \\ &= \int_{\mathrm{As}} k(y, z) r_\Lambda{}^\tau(x, z)\,dz \\ &= \int_{\mathrm{As}} k(y, z)\rho_\Lambda(x)\mathcal{F}(\rho_\Lambda)(x-z)\,dz. \end{aligned}$$

For any $q\in\mathcal{O}^*{}_S$, put

$$I_q = \int_{x\in D}\int_{\mathrm{As}} r_\Lambda{}^\tau(x, z) k^\tau(z, qx)\,dz\,dx$$

then

$$\mathrm{tr}(R_\Lambda U(h)) = \mathrm{tr}(U(h)R_\Lambda) = \sum_{q\in\mathcal{O}^*{}_S} I_q.$$

We shall evaluate the I_q. Let $z = x + a$. Put

$$k^\tau(z, qx) = k(qx, x + a)$$

and

$$r_\Lambda{}^\tau(x, z) = \rho_\Lambda(x)\mathcal{F}(\rho_\Lambda)((x - (x + a)) = \rho_\Lambda(x)\mathcal{F}(\rho_\Lambda)(- a).$$

Then

$$\int_{\mathrm{As}} r_\Lambda{}^\tau(x, z)k_\Lambda{}^\tau(z, qx)\,dz = \int_{\mathrm{As}} k(qx, x+a)\rho_\Lambda(x)\mathcal{F}(\rho_\Lambda)(-a)\,da.$$

Now, let $(k(qx, x + 2a)*\rho_\Lambda(x)\mathcal{F}(\rho_\Lambda)(- a))(t - a)$ be

$$\int_{\mathrm{As}} k(qx, x+(t-a)+2a)\rho_\Lambda(x)\mathcal{F}(\rho_\Lambda)(-a)\,da.$$

The Fourier transform of $(k(qx, x + 2a)*\rho_\Lambda(x)\mathcal{F}(\rho_\Lambda)(- a))(t - a)$ is

$$\begin{aligned}
&\mathcal{F}((k(qx, x + 2a)*\rho_\Lambda(x)\mathcal{F}(\rho_\Lambda)(- a))(t - a))(\xi)\\
&= \int_{\mathrm{As}}\int_{\mathrm{As}} k(qx, x+(t-a)+2a)\rho_\Lambda(x)\mathcal{F}(\rho_\Lambda)(-a)\,da\ \alpha(t\xi)dt\\
&= \int_{\mathrm{As}} k(qx, x+(t-a)+2a)\alpha((t-a)\xi)\,dt \int_{\mathrm{As}} \rho_\Lambda(x)\mathcal{F}(\rho_\Lambda)(-a)\,\alpha(a\xi)\,da\\
&= \int_{\mathrm{As}} k(qx, x+a'+2a)\alpha(a'\xi)\,da' \int_{\mathrm{As}} \rho_\Lambda(x)\mathcal{F}(\rho_\Lambda)(-a)\,\alpha(a\xi)\,da.
\end{aligned}$$

Let $a = 0$. Then

$$\begin{aligned}
&\mathcal{F}((k(qx, x + 2a)*\rho_\Lambda(x)\mathcal{F}(\rho_\Lambda)(- a))(t - a))(\xi)\\
&= \int_{\mathrm{As}} k(qx, x+a')\,\alpha(a'\xi)\,da' \int_{\mathrm{As}} \rho_\Lambda(x)\mathcal{F}(\rho_\Lambda)(-a)\,\alpha(a\xi)\,da.
\end{aligned}$$

Here, $\int_{\mathrm{As}} k(qx, x+a')\,\alpha(a'\xi)\,da'$ is the Fourier transform in a of $k(qx, x+a)$:

$$\sigma(x, \xi) = \int_{\mathrm{Cs}} f(\lambda^{-1})\Big(\int_{\mathrm{As}} \delta(x+a-\lambda qx)\alpha(a\xi)\,da\Big)d^*\lambda.$$

On the other hand, $\int_{\mathrm{As}} \rho_\Lambda(x)\mathcal{F}(\rho_\Lambda)(-a)\,\alpha(a\xi)\,da$ is the Fourier transform in a of $\rho_\Lambda(x)\mathcal{F}(\rho_\Lambda)(- a)$:

$$\sigma_\Lambda(x, \xi) = \rho_\Lambda(x)\rho_\Lambda(\xi)$$

since $\mathcal{F}(\rho_\Lambda)(- a) = \mathcal{F}^{-1}(\rho_\Lambda)(a)$. Therefore

$$\mathcal{F}((k(qx, x + 2a)*\rho_\Lambda(x)\mathcal{F}(\rho_\Lambda)(- a))(t - a))(\xi) = \sigma(x, \xi)\sigma_\Lambda(x, \xi).$$

Think of its Fourier inverse transform:

$$\begin{aligned}
&\mathcal{F}^{-1}(\mathcal{F}((k(qx, x + 2a)*\rho_\Lambda(x)\mathcal{F}(\rho_\Lambda)(- a))(t - a))(\xi))(t)\\
&= \int_{\mathrm{As}} \sigma(x, \xi)\sigma_\Lambda(x, \xi)\,\alpha(-\xi t)\,d\xi.
\end{aligned}$$

Since

$$\mathcal{F}^{-1}(\mathcal{F}((k(qx,\, x+2a)*\rho_\Lambda(x)\mathcal{F}(\rho_\Lambda)(-a))(t-a))(\xi))(t)$$
$$= \int_{\mathrm{A_S}} k(qx,\, x+(t-a)+2a)\rho_\Lambda(x)\mathcal{F}(\rho_\Lambda)(-a)\,da\,,$$

when $t=0$ then

$$\int_{\mathrm{A_S}} k(qx,\, x+a)\rho_\Lambda(x)\mathcal{F}(\rho_\Lambda)(-a)\,da \;=\; \int_{\mathrm{A_S}} \sigma(x,\, \xi)\sigma_\Lambda(x,\, \xi)\,d\xi\,.$$

Thus

$$\int_{\mathrm{A_S}} r_\Lambda{}^\tau(x,\, z)k_\Lambda{}^\tau(z,\, qx)\,dz \;=\; \int_{\mathrm{A_S}} \sigma(x,\, \xi)\sigma_\Lambda(x,\, \xi)\,d\xi\,.$$

Therefore,

$$\int_{x\in D}\int_{\mathrm{A_S}} r_\Lambda{}^\tau(x,\, z)k^\tau(z,\, qx)\,dz\,dx \;=\; \int_{x\in D}\int_{\mathrm{A_S}} \sigma(x,\, \xi)\,\sigma_\Lambda(x,\, \xi)\,d\xi dx$$
$$=\; \int_{x\in D,\, |x|\le\Lambda,\, |\xi|\le\Lambda} \sigma(x,\xi)\,dx\,d\xi\,.$$

Then

$$\sum_{q\in\mathcal{O}^*{}_{\mathrm{S}}} I_q \;=\; \sum_{q\in\mathcal{O}^*{}_{\mathrm{S}}} \int_{x\in D,\, |x|\le\Lambda,\, |\xi|\le\Lambda} \sigma(x,\xi)\,dx\,d\xi\,.$$

From this formula, we can obtain the following theorem.

Theorem 4.1. Let $h\in\mathcal{S}(C_{\mathrm{S}})$ have compact support. Then when $\Lambda\longrightarrow\infty$, one has

$$\mathrm{tr}(R_\Lambda U(h)) = 2\log'(\Lambda)h(1) + \sum_{v\in \mathrm{S}}\int'_{K_v{}^*}\frac{h(u^{-1})}{|1-u|}d^*u \;+ o(1)$$

where $2\log'(\Lambda) = \int_{\lambda\in C_{\mathrm{S}},\, |\lambda|\in[\Lambda^{-1},\, \Lambda]} d^*\lambda$, each $K_v{}^*$ is embedded in C_{S} by the map $u\longrightarrow(1,\, 1,\, \cdots,\, u,\, \cdots,\, 1)$ and $\int'$ means the principal value.

Let Q_Λ be the orthogonal projection on the subspace of $L^2(X_{\mathrm{S}})$ spanned by the $f(x)\in\mathcal{S}(\mathrm{A_S})$ such that $f(x)$ and $(\mathcal{F}f)(x)$ vanish for $|x|>\Lambda$. Here,

$$\mathrm{Im}(P_\Lambda) = \{\, f\in L^2(X_{\mathrm{S}}) \mid f(x)=0,\ \forall x,\ |x|>\Lambda \,\}.$$

On the other hand,

$$\mathrm{Im}(\hat{P}_\Lambda) = \{ \mathcal{F}^{-1}(\mathcal{F}f) \in L^2(X_S) \mid (\mathcal{F}f)(\xi) = 0,\ \forall\xi,\ |\xi| > \Lambda \}.$$

Put

$$B_\Lambda = \mathrm{Im}(P_\Lambda) \cap \mathrm{Im}(\hat{P}_\Lambda).$$

Here $\hat{P}_\Lambda$ and P_Λ are commutative on B_Λ. We will say that $\hat{P}_\Lambda$ and P_Λ are commutative if we can obtain B_Λ. We see that Q_Λ becomes the orthogonal projection on the subspace B_Λ of $L^2(X_S)$. Let $f \in L^2(X_S)$. It yields that $\mathcal{F}^{-1}(\rho_\Lambda(\xi)\mathcal{F}(\rho_\Lambda(x)f(x))(\xi))(x) \in \mathrm{Im}(\hat{P}_\Lambda P_\Lambda)$. Its Fourier transform vanishes for $|\xi| > \Lambda$, but it itself doesn't always vanish for $|x| > \Lambda$. So $B_\Lambda \subseteq \mathrm{Im}(\hat{P}_\Lambda P_\Lambda)$. It yields that $\rho_\Lambda(x)\mathcal{F}^{-1}(\rho_\Lambda(\xi)\mathcal{F}(f(x))(\xi))(x) \in \mathrm{Im}(P_\Lambda \hat{P}_\Lambda)$. It itself vanishes for $|x| > \Lambda$, but its Fourier transform doesn't always vanish for $|\xi| > \Lambda$. So $B_\Lambda \subseteq \mathrm{Im}(P_\Lambda \hat{P}_\Lambda)$. Since $B_\Lambda \subseteq \mathrm{Im}(\hat{P}_\Lambda P_\Lambda)$, we can replace R_Λ of the Theorem 4.1 by Q_Λ. Suppose that $\hat{P}_\Lambda$ and P_Λ are commutative, then we can show the following.

Corollary. Let Q_Λ be the orthogonal projection on the subspace of $L^2(X_S)$ spanned by the $f \in \mathcal{S}(\mathbb{A}_S)$, which vanish as well as Fourier transform for $|x| > \Lambda$. Let $h \in \mathcal{S}(C_S)$ have compact support. Then when $\Lambda \longrightarrow \infty$, one has

$$\mathrm{tr}(Q_\Lambda U(h)) = 2\log'(\Lambda)h(1) + \sum_{v \in S} \int'_{K_v^*} \frac{h(u^{-1})}{|1-u|} d^*u + o(1).$$

We can get from this corollary an S-independent global formulation:

$$\mathrm{tr}(Q_\Lambda U(h)) = 2\log'(\Lambda)h(1) + \sum_{v} \int'_{K_v^*} \frac{h(u^{-1})}{|1-u|} d^*u + o(1) \quad \Lambda \longrightarrow \infty$$

where $h \in \mathcal{S}(C_K)$ has compact support.

Let Q_Λ be the orthogonal projection on the subspace of $L^2(X)$ spanned by the $f \in \mathcal{S}(\mathbb{A}_K)$, which vanish as well as Fourier transform for $|x| > \Lambda$. Let $h \in \mathcal{S}(C_K)$ have compact support. Let S_Λ be the orthogonal projection on the subspace of $L^2(C_K)$:

$$S_\Lambda = \{ \xi \in L^2(C_K) \mid \xi(x) = 0,\ \forall x,\ |x| \notin [\Lambda^{-1}, \Lambda] \}.$$

Let $B_{\Lambda,0}$ be the subspace of $L^2(X)_0$ spanned by the $f \in \mathcal{S}(\mathbb{A}_K)_0$, which vanish as well as Fourier transform for $|x| > \Lambda$ and let $Q_{\Lambda,0}$ be the orthogonal projection on $B_{\Lambda,0}$. Let $f \in \mathcal{S}(\mathbb{A}_K)_0$ be such that $f(x)$ and $(\mathcal{F}f)(x)$ vanish for $|x| > \Lambda$. Then $\mathrm{T}f(x)$ vanishes for $|x| > \Lambda$. From Lemma 2.1, $\mathrm{T}f(x) = \mathrm{T}\mathcal{F}f(x^{-1})$. So $\mathrm{T}f(x)$ vanishes for $|x| < \Lambda^{-1}$. This shows that $\mathrm{T}(B_{\Lambda,0}) \subseteq S_\Lambda$. Analogously let $R_{\Lambda,0}$ be the orthogonal projection, and we will think of $f \in \mathrm{Im}(R_{\Lambda,0})$. It doesn't always hold that $f(x)$ vanishes for $|x| > \Lambda$, so we can't always say that $\mathrm{T}f(x)$ vanishes for $|x| > \Lambda$. Thus we can't always state that $\mathrm{T}(\mathrm{Im}(R_{\Lambda,0})) \subseteq S_\Lambda$. It must be instructive to understand the difference between Q_Λ and R_Λ.

Put $Q'_{\Lambda,0} = \mathrm{T}\, Q_{\Lambda,0}\, \mathrm{T}^{-1}$. It holds that $Q'_{\Lambda,0} \leq S_\Lambda$. Then the following distribution on C_K of positive type is given

$$\Delta_\Lambda(f) = \mathrm{tr}((S_\Lambda - Q'_{\Lambda,0})V(f)).$$

Here

$$\Delta_\Lambda(f * f^*) \geq 0 \qquad f^*(x) = \bar{f}(x^{-1}),\ x \in C_K.$$

Let $h \in \mathcal{S}(C_K)$ have compact support. Set $f(x) = |x|^{-1/2}h(x^{-1})$. Since $\mathrm{T}U(a) = |a|^{1/2}V(a)\mathrm{T}$,

$$\begin{aligned}\mathrm{T}U(h) &= \int_{C_K} h(g)\mathrm{T}U(g)d^*g = \int_{C_K} h(g)|g|^{1/2}V(g)\mathrm{T}\, d^*g \\ &= \int_{C_K} f(g^{-1})V(g)\mathrm{T}\, d^*g = V(\tilde{f})\mathrm{T}\end{aligned}$$

where $\tilde{f}(g) = f(g^{-1})$. Then

$$U(h)\mathrm{T}^{-1} = \mathrm{T}^{-1}V(\tilde{f}).$$

It holds that

$$Q'_{\Lambda,0}V(\tilde{f}) = \mathrm{T}\, Q_{\Lambda,0}\, \mathrm{T}^{-1}V(\tilde{f}) = \mathrm{T}\, Q_{\Lambda,0}\, U(h)\mathrm{T}^{-1}.$$

We see that

$$\mathrm{tr}(Q'_{\Lambda,0}V(\tilde{f})) = \mathrm{tr}(\mathrm{T}\, Q_{\Lambda,0}\, U(h)\mathrm{T}^{-1}) = \mathrm{tr}(Q_\Lambda U(h)).$$

Let

$$s_\Lambda = \begin{cases} s_\Lambda(x) = 1 \ \cdots\ |x| \in [\Lambda^{-1},\ \Lambda] \\ s_\Lambda(x) = 0 \ \cdots\ |x| \notin [\Lambda^{-1},\ \Lambda] \end{cases}.$$

We can write the Schwartz kernel of $S_\Lambda V(\tilde{f})$ as

$$s_\Lambda(x)\, k(x,\, y) = s_\Lambda(x) \int_{C_K} \tilde{f}(\lambda^{-1}) \delta(y - \lambda x)\, d^*\lambda\,.$$

Then

$$\begin{aligned} \operatorname{tr}(S_\Lambda V(\tilde{f})) &= \int_{C_K} s_\Lambda(x) k(x,\, x)\, dx \;=\; \int_{C_K} s_\Lambda(x) \int_{C_K} \tilde{f}(\lambda^{-1}) \delta(x - \lambda x) d^*\lambda\, dx \\ &= \int_{|x| \in [\Lambda^{-1},\, \Lambda]} \frac{f(1)}{|x|}\, dx \;=\; \int_{|x| \in [\Lambda^{-1},\, \Lambda]} f(1)\, d^*x \;=\; 2h(1) \cdot \log'(\Lambda). \end{aligned}$$

Therefore, if we can show that

$$\operatorname{tr}(\mathrm{Q}_\Lambda U(h)) = 2\log'(\Lambda) h(1) + \sum_v \int'_{K_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u \;+ o(1) \quad \Lambda \longrightarrow \infty$$

then Δ_∞ (Δ_Λ $\Lambda \longrightarrow \infty$) becomes the Weil distribution. Since Δ_Λ is a positive type, Δ_∞ is also a positive type. It means that the Weil distribution is positive, so the Riemann Hypothesis is valid.

We will think of the space

$$\mathcal{H}_\Lambda \;\cong\; \operatorname{Im}(S_\Lambda)/\mathrm{T}(B_{\Lambda,0}).$$

Then

$$\operatorname{tr} U(h)|_{\mathcal{H}_\Lambda} = \operatorname{tr}((S_\Lambda - \mathrm{Q}'_{\Lambda,0}) |\cdot|^{1/2} V(h)) = \operatorname{tr}((S_\Lambda - \mathrm{Q}'_{\Lambda,0}) V(\tilde{f}))$$

and $\operatorname{tr} U(h)|_{\mathcal{H}_\Lambda}$ gives $\sum_{\substack{L(\tilde{\chi}_0,\, \rho) = 0 \\ \operatorname{Re}\rho = 1/2}} \hat{h}(\tilde{\chi}_0, \rho)$. When $\Lambda \longrightarrow \infty$, $\operatorname{Im}(S_\Lambda)/\mathrm{T}(B_{\Lambda,0})$ is identified with $L^2(C_K)/\mathrm{T}(L^2(X)_0)$. Therefore

$$\operatorname{tr}((S_\Lambda - \mathrm{Q}'_{\Lambda,0}) V(\tilde{f})) = \sum_{\substack{L(\tilde{\chi}_0,\, \rho) = 0 \\ \operatorname{Re}\rho = 1/2}} \hat{h}(\tilde{\chi}_0, \rho) \qquad \Lambda \longrightarrow \infty.$$

Suppose that the Riemann Hypothesis is valid. Then, from the Weil explicit formula, it holds that

$$\sum_{\substack{L(\tilde{\chi}_0,\, 1/2+\rho) = 0 \\ \rho \in i\mathbb{R}}} \hat{h}(\tilde{\chi}_0, \rho) - \hat{h}(0) - \hat{h}(1) \;=\; -\sum_v \int'_{K_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u\,.$$

We may say that

$$\operatorname{tr}((S_\Lambda - \mathrm{Q}'_{\Lambda,0}) V(\tilde{f})) = -\sum_v \int'_{K_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u \qquad \Lambda \longrightarrow \infty.$$

It yields that

$$\mathrm{tr}(Q_\Lambda U(h)) = 2\log'(\Lambda)h(1) + \sum_v \int'_{K_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u + o(1) \quad \Lambda \longrightarrow \infty.$$

Therefore, if $\hat{P}_\Lambda$ and P_Λ are commutative then the Riemann Hypothesis is valid.

5. Riemann Hypothesis and Prolate Spherodial Wave Functions

We have been supposing that B_Λ sufficiently well behaves. However we have some difficulties which one has to overcome.

Let ν be a finite place and let's think of K_ν. We will restrict ourselves to $\mathbb{Q}_p$. The Fourier transform of a function $f \in L^1(\mathbb{Q}_p)$ is

$$\hat{f}(\omega) = \int_{\mathbb{Q}_p} f(x) e^{-2\pi i \{x\omega\}_p} dx$$

where $\{\cdot\}_p$ is the fractional part of a p-adic number

$$\{\sum_{i=-n}^{\infty} a_i p^i\}_p = \sum_{i=-n}^{-1} a_i p^i .$$

We will think of the function space $C_c^\infty(\mathbb{Q}_p)$ of compactly supported, locally constant functions. Let $\mathrm{B}_{\leq p^n}(a) = \{ x \in \mathbb{Q}_p \mid |x-a|_p \leq p^n \}$ and let $\mathrm{B}_{\leq p^n}(0) = \mathrm{B}_{\leq p^n}$.

Lemma 5.1. If $f \in L^1(\mathbb{Q}_p)$, $x \neq 0$ then

$$\int_{\mathbb{Q}_p} f(x^{-1}y)dy = |x|_p \int_{\mathbb{Q}_p} f(y)dy .$$

Proposition 5.1. Denote the characteristic function of $\mathrm{B}_{\leq p^n}$ by ξ_{p^n}. Then

$$\hat{\xi}_{p^n}(\omega) = p^n \xi_{p^{-n}}(\omega).$$

Proof.

$$\hat{\xi}_{p^n}(\omega) = \int_{\mathbb{Q}_p} e^{-2\pi i \{x\omega\}_p} \xi_{p^n}(x) dx$$

from the lemma 5.1

$$= |\omega|_p^{-1} \int_{\mathbb{Q}_p} e^{-2\pi i \{x\}_p} \xi_{p^n}(\omega^{-1}x) dx ;$$

when $|\omega^{-1}x|_p \leq p^n$ then $|x|_p \leq p^n |\omega|_p$,

$$= |\omega|_p^{-1} \int_{\mathbb{Q}_p} e^{-2\pi i \{x\}_p} \xi_{p^n|\omega|_p}(x) dx = |\omega|_p^{-1} \int_{|x|_p \leq p^n|\omega|_p} e^{-2\pi i \{x\}_p} dx .$$

Let m be the integer such that $p^m = p^n|\omega|_p$. If $|\omega|_p \leq p^{-n}$ then $m \leq 0$, so

$$\int_{|x|_p \leq p^m} e^{-2\pi i\{x\}_p}\, dx \;= p^m.$$

If $m > 0$ then there exists an y with $|y|_p \leq p^m$ such that $e^{2\pi i\{y\}_p} \neq 1$. $\mathrm{B}_{\leq p^m} = \mathrm{B}_{\leq p^m}(y)$ since $y \in \mathrm{B}_{\leq p^m}$. Thus

$$\int_{|x|_p \leq p^m} e^{-2\pi i\{x\}_p}\, dx \;= \int_{|x|_p \leq p^m} e^{-2\pi i\{y+x\}_p}\, dx \;=\; e^{-2\pi i\{y\}_p} \int_{|x|_p \leq p^m} e^{-2\pi i\{x\}_p}\, dx\,.$$

So

$$\int_{|x|_p \leq p^m} e^{-2\pi i\{x\}_p}\, dx \;= 0.$$

We see that

$$\text{if } |\omega|_p \leq p^{-n} \text{ then } \hat{\xi}_{p^n}(\omega) = |\omega|_p{}^{-1} p^m = p^n$$

and that

$$\text{if } |\omega|_p > p^{-n} \text{ then } \hat{\xi}_{p^n}(\omega) = 0.$$

□

Suppose that f is supported on $\mathrm{B}_{\leq p^m}$ and constant on the cosets of $\mathrm{B}_{\leq p^{-n}}$. We can choose a finite set of $\{a_k\} \subseteq \mathrm{B}_{\leq p^m}$ such that

$$\mathrm{B}_{\leq p^m} = \coprod_{k=0}^{l} (a_k + \mathrm{B}_{\leq p^{-n}})$$

where f is equal to zero outside $\mathrm{B}_{\leq p^m}$ and f is constant on each set $\mathrm{B}_{\leq p^{-n}}(a_k)$. Then f has the form

$$\sum_{k=0}^{l} c_k \xi_{p^{-n}}(x - a_k)\,.$$

Since the Fourier transform of the characteristic function ξ_{p^n} is $\hat{\xi}_{p^n}(\omega) = p^n \xi_{p^{-n}}(\omega)$ and it yields that $\widehat{f(x-a)}(\omega) = e^{-2\pi i\{a\omega\}_p} \hat{f}(\omega)$,

$$\hat{f}(x) = \begin{cases} \sum_{k=0}^{l} c_k e^{-2\pi i\{a_k x\}_p} p^{-n} & \cdots \quad |x|_p \leq p^n \\ 0 & \cdots \quad |x|_p > p^n \end{cases}.$$

Let $m \geq 0$. There exists a non-zero function f supported on $\mathrm{B}_{\leq p^m}$ and constant on the cosets of $\mathrm{B}_{\leq p^{-m}}$. Then it turns out that f is a function on $\mathbb{Q}_p$ which vanishes as well as its Fourier transform for $|x|_p > p^m$. On the other hand, if $m < 0$ then such a function is identically zero because $m < -m$. We see that B_Λ for $\mathbb{Q}_p$ makes sense for large Λ. Especially, we will think of a function

$$\eta\chi(x) = \sum_{k=0}^{l} \chi(a_k)\xi_{p^{-m}}(x-a_k),$$

where χ is a character of $\mathbb{Q}_p^*$. If m (≥ 0) is sufficiently large, we may consider $\eta\chi(x)$ as the function which vanishes as well as its Fourier transform for $|x|_p > p^m$ and agrees with χ on $\mathrm{B}_{\leq p^m}$.

When ν is an Archimedean place there exists no non-zero function on K_ν, e.g. $\mathbb{R}$, which vanishes as well as its Fourier transform for $|x| > \Lambda$. Namely B_Λ for $\mathbb{R}$ makes no sense. The work of Landau, Pollak and Slepian allows to overcome this difficulty. The results are as follows.

Given any $T > 0$ and any $\Omega > 0$, we can find a countably infinite set of real functions $\psi_0(t)$, $\psi_1(t)$, $\psi_2(t)$, $\cdots$ and a set of real positive numbers

$$\lambda_0 > \lambda_1 > \lambda_2 > \cdots$$

with the following properties:

i. The $\psi_i(t)$ are bandlimited, i.e. its Fourier transform $\mathcal{F}(\psi_i)(\omega)$ vanishes for $|\omega| > \Omega$, orthogonal on the real line and complete in $B = \{ f(t) \in L^2(\mathbb{R}) \mid (\mathcal{F}f)(\omega) = 0, \forall\omega, |\omega| > \Omega \}$:

$$\int_{-\infty}^{\infty} \psi_i(t)\psi_j(t)\,dt = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \qquad i, j = 0, 1, 2, \cdots.$$

ii. In the interval $-T/2 \leq t \leq T/2$, the ψ_i are orthogonal and complete in $L^2{}_{T/2}$:

$$\int_{-T/2}^{T/2} \psi_i(t)\psi_j(t)\,dt = \begin{cases} 0 & i \neq j \\ \lambda_i & i = j \end{cases} \qquad i, j = 0, 1, 2, \cdots.$$

Here $L^2{}_{T/2}$ is the class of all complex valued function $f(t)$ defined for $-T/2 \leq t \leq T/2$ and integrable in absolute square in the interval $(-T/2, T/2)$.

iii. For all values of t, real or complex,

$$\lambda_i \psi_i(t) = \int_{-T/2}^{T/2} \frac{\sin(\Omega(t-s))}{\pi(t-s)} \psi_i(s)\,ds \quad i = 0, 1, 2, \cdots.$$

Both the ψ's and the λ's are functions of $c = \Omega T/2$. In order to make this dependence explicit, we write

$$\lambda_i = \lambda_i(c),\ \psi_i(t) = \psi_i(c, t), \quad i = 0, 1, 2, \cdots.$$

For any $f(t) \in B$, we can write, from i.,

$$f(t) = \sum_{n=0}^{\infty} a_n \psi_n(t)$$

where

$$a_n = \int_{-\infty}^{\infty} f(t)\psi_n(t)\,dt\,.$$

Since $\int_{-T/2}^{T/2} f(t)\psi_i(t)\,dt = \int_{-T/2}^{T/2} a_i\psi_i(t)\,\psi_i(t)\,dt = \lambda_i a_i$,

$$a_n = 1/\lambda_n \int_{-T/2}^{T/2} f(t)\psi_n(t)\,dt\,.$$

This means that we can write $f(t)$ from values of $f(t)$ in the interval $(-T/2, T/2)$.

Fix $\Omega = \Lambda$ for a given Λ. For any $f(t) \in L^2(\mathbb{R})$, we can obtain

$$\rho_\Lambda f(t) = \begin{cases} f(t) & \cdots\ |t| \le \Lambda \\ 0 & \cdots\ |t| > \Lambda \end{cases}.$$

Denote the function $\rho_\Lambda f(t)$ in the interval $(-T/2, T/2)$ by $\rho_\Lambda f(t)_{T/2}$. We may say that $\rho_\Lambda f(t)_{T/2} \in L^2{}_{T/2}$, thus we can write

$$\rho_\Lambda f(t)_{T/2} = \sum_{n=0}^{\infty} a_n \psi_n(c, t) \qquad c = \Omega T/2, \quad t \in \mathbb{R}.$$

This description $\rho_\Lambda f(t)_{T/2} = \sum_{n=0}^{\infty} a_n \psi_n(c, t)$ is valid only for $|t| \le T/2$. The right term, if it converges, gives a function over the whole real line, so $\rho_\Lambda f(t)_{T/2}$ is extended to a function over the whole real line. Namely, $\sum_{n=0}^{\infty} a_n \psi_n(c, t)$ describes the function over a whole real line which fits $\rho_\Lambda f(t)_{T/2}$ in the interval $(-T/2, T/2)$. We also denote it by $\rho_\Lambda f(t)_{T/2}$. Since the $\psi(c, t)$'s are bandlimited, if $\rho_\Lambda f(t)_{T/2}$ is a bandlimited function then $\sum_{n=0}^{\infty} a_n \psi_n(t)\ t \in \mathbb{R}$ converges and give a bandlimited function $\rho_\Lambda f(t)_{T/2}$. So we will see that the series $\sum_{n=0}^{\infty} a_n \psi_n(t)\ t \in \mathbb{R}$ does not converge in general. However it must give a formal description of $\rho_\Lambda f(t)_{T/2}$. Even if it is formal, we might say that $\rho_\Lambda f(t)_{T/2}$ is bandlimited actually.

Now, we will see that it does not always hold that $\rho_\Lambda f(t)_{T/2} = \rho_\Lambda f(t)$. However, when $T \longrightarrow \infty$ then $\rho_\Lambda f(t)_{T/2} = \rho_\Lambda f(t)$. Thus we could say that $\rho_\Lambda f(t)_{T/2} = \rho_\Lambda f(t)$ for sufficient large T. Thus we can admit B_Λ formally and it will sufficiently well behave.

Even if $\hat{P}_\Lambda$ and P_Λ do not commute exactly, we may be allowed to consider that $\hat{P}_\Lambda$ and P_Λ are commutative actually.

We will think of the case $K = \mathbb{Q}$. Let $\mathrm{S} = \{\infty, p_1, \cdots, p_d\}$ be a finite set of places of K containing all infinite places.

We will think of the left regular representation $(U, L^2(X_\mathrm{S}))$ of $C_{\mathrm{S},1}$ which is the subgroup: $\{g \in C_\mathrm{S} \mid |g| = 1\}$. We see that U isn't always unitary since $L^2(X_\mathrm{S})$ is based on the additive measure $dx = |x| d^*x$. However, if U is restricted to $C_{\mathrm{S},1}$ then

$$dg^{-1}x = |g^{-1}x| d^* g^{-1}x = |x| d^*x = dx,$$

so the restriction of U to $C_{\mathrm{S},1}$ is unitary. For $\xi(x) \in L^2(X_\mathrm{S})$, we will consider that

$$(U(a)\xi)(x) = c(a)\xi(x) \qquad \forall a \in C_{\mathrm{S},1}.$$

Since the left regular representation of $C_{\mathrm{S},1}$ is unitary, there exists $\chi_0 \in \hat{C}_{\mathrm{S},1}$ such that $\chi_0(a) = c(a)$. Fix $\chi_0 \in \hat{C}_{\mathrm{S},1}$ and put

$$L^2{}_{\chi_0} = \left\{ \xi \in L^2(X_\mathrm{S}) \middle| \xi(a^{-1}x) = \chi_0(a)\xi(x) \quad \forall x \in X_\mathrm{S},\, a \in C_{\mathrm{S},1} \right\}.$$

One decomposes $L^2(X_\mathrm{S})$ into the direct sum of subspaces:

$$L^2(X_\mathrm{S}) = \bigoplus_{\chi_0 \in \hat{C}_{\mathrm{S},1}} L^2{}_{\chi_0}.$$

For Λ large enough, we can find

$$p_i{}^{m_i} \le \Lambda, \ \ m_i \ge 0; \quad 1 \le i \le d.$$

We can choose a finite set of $\{ a_{k,p_i} \} \subseteq \mathrm{B}_{\le p_i{}^{m_i}}$ such that

$$\mathrm{B}_{\le p_i{}^{m_i}} = \coprod_{k=0}^{l} (a_{k,p_i} + \mathrm{B}_{\le p_i{}^{-m_i}}).$$

We will find

$$\eta_{\chi_{p_i}}(x) = \sum_{k=0}^{l} \chi_{p_i}(a_{k,p_i}) \xi_{p_i^{-m_i}}(x - a_{k,p_i})$$

where $\chi = \prod \chi_{p_i}$, and we will find a vector

$$\eta_{\chi_0}(x) = \prod \eta_{\chi_{0\,p_i}}(x) \in L^2{}_{\chi_0}.$$

Put

$$\eta_{\chi_0}(x - a_k);\;\; a_k = \prod a_{k,p_i}\,.$$

On the other hand, let $\Omega = \Lambda$ for a given Λ. We can obtain a countably infinite set of real functions $\psi_0(c,\,t),\,\psi_1(c,\,t),\,\cdots\;;\, c = \Omega T/2$. We will see that the linear span of

$$\{\psi_0(c,\,t),\,\psi_1(c,\,t),\,\psi_2(c,\,t),\,\cdots\,;\,\eta_{\chi_0}(x-a_0),\,\cdots,\,\eta_{\chi_0}(x-a_l)\}$$

makes a subspace B_Λ of $L^2{}_{\chi_0}$. Denote it by $B_\Lambda{}^{\chi_0}$. As we have seen, $B_\Lambda{}^{\chi_0}$ is given formally. Thus $\hat{P}_\Lambda$ and P_Λ don't commute on $L^2{}_{\chi_0}$ exactly, but $B_\Lambda{}^{\chi_0}$ behaves well. One decomposes $L^2(X_{\mathrm{S}})$ into the direct sum of subspaces: $L^2(X_{\mathrm{S}}) = \bigoplus_{\chi_0 \in \hat{C}_{\mathrm{S},1}} L^2{}_{\chi_0}$. Thus we can say that $\hat{P}_\Lambda$ and P_Λ commute on $L^2(X_{\mathrm{S}})$ actually.

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