The Universal Constructions of Heterotic Vacua in Complex and Hermitian Moduli Superspaces

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Abstract

In the present article we construct universal representations of a heterotic vacua in the conditions of the complex structure and hermitian moduli spaces. We show that it agrees with the results that have recently been obtained from a ten-dimensional perspective where supersymmetric Minkowski solutions including the Bianchi identity correspond to an integrable holomorphic structure, with infinitesimal moduli calculated by its first cohomology. As has recently been noted, interplay of complex structure and bundle deformations through holomorphic and anomaly constraints can lead to fewer moduli than may have been expected. We derive a relation between the number of complex structure and bundle moduli removed from the low energy theory in this way, and give conditions for there to be no complex structure moduli or bundle moduli remaining in the low energy theory. The link between Yukawa couplings and obstruction theory is also briefly discussed.

1 Introduction

Heterotic geometry is the geometry associated with the moduli space of a heterotic vacua of superstring theory. The geometrical background, associated with the vacua, is understood, at large volume, as $\mathbb{R}^{1,3}\times X$, where X is a complex 3-dimensional manifold with vanishing first Chern class. This geometry is endowed with a holomorphic vector bundle $\mathcal{E} \to X$, admitting a connection A that satisfies the Hermitian Yang-Mills equations. The metric on the moduli space of heterotic supergravity metric was computed, correct to $\mathcal{O}(\alpha^{`})$ by a dimensional reduction of heterotic supergravity. This metric has to be Kähler as a consequence of supersymmetry. It should not be surprising, therefore, that verifying that the moduli superspace is in fact Kähler requires taking into account the relations between H, the connection on the bundle $\mathcal{E}$, and the hermitian form ω on X, since these relations follow from both the anomaly cancellations condition and the requirement of supersymmetry. The purpose of this article is to show that considering structure is worthwhile. Before entering into technical matters it may be helpful to indicate why this might be expected to be the case. To start, consider for example the deformation of a manifold, which is part of our data. In general relativity one often thinks of a three-geometry that evolves in time. We think of time as a parameter which governs the evolution. Our aim is to describe the metric on the space of heterotic vacua. This parameter space geometry, which we term heterotic geometry, is the generalisation of the special geometry of type II string theory. The heterotic vacua of concern here derive from compactifying heterotic string theory, at large radius, on $\mathbb{R}^{3,1}\times\mathcal{X}$, where $\mathcal{X}$ is a smooth complex threefold with vanishing first Chern-class that is endowed with a holomorphic vector bundle $\mathcal{E}$, that has a connection satisfying the Hermitian–Yang–Mills (HYM) equation and a gauge invariant three-form H. These quantities satisfy an anomaly condition, that will be discussed shortly. These vacua are of physical interest since, at low-energies, they realise quasi-realistic four-dimensional theories of relevance to observable particle physics. These equations already imply that the moduli space has a recondite character, since the deformations of F, ω and H are intricately related. By contrast to the case of type II vacua, where the roles of the complex structure parameters and the Kähler class parameters are strictly separated, there seems to be no useful distinction, in the heterotic context, between what are conventionally labelled the complex structure moduli, hermitian moduli and bundle moduli. The deformations of a heterotic structure, within a given topological class, correspond to the points of the moduli space M, which is itself a complex manifold. The geometry of the moduli superspaces of these vacua are also of mathematical interest. The metric on the local moduli space of holomorphic hermitian Yang-Mills (HYM) bundles $\mathcal{E}$ constructed over an arbitrary but fixed complex manifold goes back to Kobayashi and collaborators. By constructing local coordinates in the spirit of Kodaira-Spencer, this metric was shown to be Kähler by Itoh. However, the restriction to a fixed CY manifold is artificial from the point of view of string theory: the moduli space includes deformations of $\mathcal{X}$, the gauge-invariant three-form H and the vector bundle $\mathcal{E}$ simultaneously. We call the triple $(\mathcal{X}, \mathcal{E}, H)$ a heterotic structure. The present work is complementary to a series of papers, which describe this heterotic structure and identify the moduli of the vacua with certain cohomology groups. Within the fibration $\mathcal{U}$ lies the fibration of the manifold X over M. This is the natural context in which to discuss the Ehresmann connection — equivalently, the projection ∂ — the metric $\mathbb{g}$ and complex structure $\mathbb{J}$ for the extended space. The connection ${c_a}^m$ allows us to restrict

$\mathbb{g}$ and $\mathbb{J}$ to fibres covariantly and, when this is done, they are identified with the metric g and complex structure J on X. Furthermore, using ∂ we can also project $\mathbb{g}$ and $\mathbb{J}$ to the moduli space metric $g^\sharp$ and complex structure $J^\sharp$. We describe the differential calculus of $\mathbb{X}$ and its relation to deformations. For example, we show that the covariant derivatives are identified as Lie derivatives acting tensors on $\mathbb{X}$. This leads to an interpretation of deformations with flows on $\mathbb{X}$. We start to see the profits of our labour. We introduce on $\mathbb{X}$ extensions of the connections A and Θ, denoted $\mathbb{A}$ and $\mathbb{\Theta}$ respectively, which allows to discuss the extended symmetry groups mentioned above. The fields $\mathbb{A}$ and $\mathbb{\Theta}$ are holomorphic connections for the vector bundle $\mathcal{U} \to \mathbb{X}$. Moreover, we define the extensions of ω and H, denoted $\mathbb{\omega}$ and $\mathbb{H}$ respectively, and suppose a relation $\mathbb{H} = \mathbb{d}^c\mathbb{\omega}$, as the extension of the supersymmetry relation. Surprisingly, this relation together with its Bianchi identity, encapsulate in a simple pair of tensor equations, a set of long and otherwise complicated equations relating covariant derivatives which were crucial to the derivation of the Kähler moduli space metric. This is similar to how the laws of electrodynamics when viewed relativistically are unified into a simple tensor equation.

2 The Universal Geometry of Heterotic Moduli

The central results of this paper were originally derived with the goal of finding the natural Kähler metric on the moduli of heterotic structures. In the broader context of the heterotic superstring we show how this is extended to the holotypical derivative and how it is interpreted as a connection. Remarkably, however, we have found they have a natural geometric interpretation when viewed in the context of what is known as the universal bundle. The salient point is that geometrising our algebraic structures is a powerful way of viewing the moduli superspace. Finally, we remark that the same results for the moduli problem of heterotic structures was obtained in the present literature from first and second order deformations of a heterotic superpotential.

2.1 The extension of A

The covariant derivative for A defined in [1] transforms covariantly under gauge transformations. It needs to be generalised to transform, additionally, under bundle diffeomorphisms. To do this we define an extended connection $\mathbb{A}$ for the extended vector bundle $\mathcal{U} \to \mathbb{X}$

$$\mathbb{A} \;=\; A_m e^m + A^\sharp_a \mathrm{d}y^a \ , \qquad A^\sharp_a \;=\; \Lambda_a - A_m c_a{}^m \ ,$$

where the components of the corpus A_m are identified with the connection along X. In the following, we will denote the corpus of $\mathbb{A}$ by $A = A_m e^m$ in the e-basis, the animus by $A^\sharp = A^\sharp_a \mathrm{d}y^a$. We can divide the form into holomorphic type

$$\mathbb{A} \;=\; \boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{A}}^\dagger \ , \quad \boldsymbol{\mathcal{A}} \;=\; \mathbb{A}^{(0,1)} \ .$$

We will not be specific about the structure group of the universal bundle $\mathcal{U}$ beyond requiring it contain $\mathfrak{G}$ as a subgroup when restricted to X appropriately. This restriction is important in later sections when we discuss deformations of $\mathcal{T}_X$.

The form $\boldsymbol{\mathcal{A}}$ can be decomposed into its animus and corpus

$$\boldsymbol{\mathcal{A}} \;=\; \mathcal{A}^\sharp_{\overline{\alpha}}\,\mathrm{d}y^{\overline{\alpha}} + \mathcal{A}_{\overline{\mu}}\, e^{\overline{\mu}} \ .$$

The field strength of $\mathbb{A}$ is defined as usual

$$\mathbb{F} \;=\; \mathbb{d}\mathbb{A} + \mathbb{A}^2 \,. \tag{2.0}$$

This can be decomposed according to tangibility and in terms of the covariant derivatives $\text{Đ}\,, \text{Đ}^\sharp$, respectively

$$\mathbb{F} \;=\; (\text{Đ} + \text{Đ}^\sharp - S)(A + A^\sharp) + (A + A^\sharp)^2 \;=\; \frac{1}{2} F_{mn} e^m e^n + \text{d}y^a\, \mathbb{F}^\sharp_{\;a} + \frac{1}{2}\, \mathbb{F}^\sharp_{\;ab}\, \text{d}y^a \text{d}y^b \,. \tag{2.1}$$

Let us unpackage each of the three components of $\mathbb{F}$. The corpus is the field strength of A on X,

$$F_{mn} \;=\; \partial_m A_n - \partial_n A_m + A_m A_n - A_n A_m \,.$$

The second term defines a covariant derivative that transforms homogeneously under gauge transformations and is invariant under bundle diffeomorphisms:

$$\mathbb{F}^\sharp_{\;a} \;=\; \mathfrak{D}_a A \,, \quad \text{where} \quad \mathfrak{D}_a A \;=\; e_a(A) - (\text{Đ}\, c_a{}^m) A_m - \text{Đ}_{\,A} A^\sharp_{\;a} \,, \tag{2.2}$$

here

$$\text{Đ}_{\,A} A^\sharp_{\;a} \;=\; \text{Đ}\, A^\sharp_{\;a} + [A, A^\sharp_{\;a}] \,,$$

and

$$e_a(A) \;=\; \partial_a A - c_a{}^m \partial_m A \,.$$

On a gauge neutral object, $\mathfrak{D}_a$ reduces to $\text{Đ}^\sharp_{\;a}$.

In holomorphic coordinates, using the identification of Δ_α, we find it is the appropriate generalisation of the holotypical derivative introduced in [1]:

$$\mathfrak{D}_\alpha \mathcal{A} \;=\; e_\alpha(\mathcal{A}) - \Delta_\alpha{}^\nu\, \mathcal{A}^\dagger_\nu - \overline{\eth}_{\mathcal{A}} A^\sharp_{\;\alpha} \,.$$

The third equation of (2.1) is

$$\mathbb{F}^\sharp_{\;ab} \;=\; 2\, \text{Đ}^\sharp_{\;[a} A^\sharp_{\;b]} + [A^\sharp_{\;a}, A^\sharp_{\;b}] - S_{ab}{}^m A_m \,, \quad \text{where} \quad \text{Đ}^\sharp_{\;a} A^\sharp_{\;b} = e_a(A^\sharp_{\;b}) \,.$$

We take $\mathcal{U}$ to be holomorphic meaning

$$\mathbb{F}^{(0,2)} \;=\; 0 \,.$$

The corpus of $\mathbb{F}$ automatically satisfies this requirement in virtue of $F^{(0,2)} = 0$. The tangibility $[1,1]$ component is the condition that $\mathcal{A}$ depend holomorphically on parameters

$$\mathfrak{D}_{\overline{\beta}} \mathcal{A} \;=\; 0 \,.$$

The tangibility $[2,0]$ component implies $\mathbb{F}^\sharp_{\;\overline{\alpha}\overline{\beta}} = 0$. That is, that the bundle $\mathcal{U}$ restricted to M is holomorphic. In deducing this we have used $S_{\overline{\alpha}\overline{\beta}} = 0$.

Consider now the Bianchi identity for $\mathbb{F}$

$$\mathbb{d}_{\mathbb{A}} \mathbb{F} \;=\; 0 \,. \tag{2.3}$$

The corpus realises the Bianchi identity on X. The animus gives two further identities

$$
Ð_A(\mathfrak{D}_a A) \;=\; \mathfrak{D}_a F \quad \text{and} \quad [\mathfrak{D}_a, \mathfrak{D}_b] A \;=\; -Ð_A(\mathbb{F}^\sharp_{ab}) + S_{ab}{}^m\, F_m \;=\; 0 \;,
$$

where

$$
\begin{aligned}
Ð_A(\mathfrak{D}_a A) \;&=\; Ð(\mathfrak{D}_a A) + [A, \mathfrak{D}_a A] \;, \qquad & \mathfrak{D}_a F \;&=\; Ð^\sharp_a F + [A^\sharp_a, F] \;,\\
\mathfrak{D}_a(\mathfrak{D}_b A) \;&=\; Ð^\sharp_a(\mathfrak{D}_b A) + [A^\sharp_a, \mathfrak{D}_b A] \;, \qquad & Ð_A \mathbb{F}^\sharp_{ab} \;&=\; Ð\mathbb{F}^\sharp_{ab} + [A, \mathbb{F}^\sharp_{ab}] \;.
\end{aligned}
$$

The relations (2.4) can be derived directly from the definition of the covariant derivative as in [1] with some labour. What we see here is an alternative derivation through the Bianchi identity. This also has the advantage of unification, reducing a pair of identities to a single identity.

The Atiyah constraint comes from taking $a = \alpha$ in the first equation of (2.4), and considering the $(0,2)$-component together with the identification of Δ_α:

$$
\bar{\eth}_{\mathcal{A}}(\mathfrak{D}_\alpha \mathcal{A}) \;=\; \Delta_\alpha{}^\mu\, F_\mu \;.
$$

2.2 The extension of $\boldsymbol{B}$ and $\boldsymbol{H}$

The field $\mathbb{H}$ is the extension of H, and defined as

$$
\mathbb{H} \;=\; \mathbb{d}\mathbb{B} - \frac{\alpha'}{4}\Big(\mathrm{CS}[\mathbb{A}] - \mathrm{CS}[\mathbb{\Theta}]\Big) \;, \quad \text{where} \quad \mathrm{CS}[\mathbb{A}] \;=\; \mathrm{Tr}\left(\mathbb{A}\,\mathbb{d}\mathbb{A} + \frac{2}{3}\mathbb{A}^3\right) \;, \tag{2.4}
$$

where $\mathbb{B}$ is the extension of the Kalb–Ramond field

$$
\mathbb{B} \;=\; \frac{1}{2}B_{mn}\, e^m e^n + \mathbb{B}^\sharp_{am}\, \mathrm{d}y^a e^m + \frac{1}{2}\mathbb{B}^\sharp_{ab}\, \mathrm{d}y^a \mathrm{d}y^b \;=\; B + \mathbb{B}^\sharp_a\, \mathrm{d}y^a + B^\sharp \;.
$$

$\mathbb{H}$ decomposes as

$$
\mathbb{H} \;=\; \frac{1}{3!}\,\mathrm{d}y^{abc}\, \mathbb{H}^\sharp_{abc} + \frac{1}{2}\,\mathrm{d}y^{ab}\, \mathbb{H}^\sharp_{ab} + \mathrm{d}y^a\, \mathbb{H}^\sharp_a + H \;,
$$

where the $[1,2]$ term will be relevant in what follows. It is given by

$$
\mathbb{H}^\sharp_a = Ð^\sharp_a B - Ð\mathbb{B}^\sharp_a - \frac{\alpha'}{4}\Big(\mathrm{Tr}\,(A^\sharp_a\, Ð A) - \mathrm{Tr}\,(\Theta^\sharp_a\, Ð\Theta)\Big) + \frac{\alpha'}{4}\Big(\mathrm{Tr}\,(A\,\mathfrak{D}_a A) - \mathrm{Tr}\,(\Theta\,\mathfrak{D}_a\Theta)\Big) \;. \tag{2.5}
$$

We can now rewrite this in terms of covariant derivatives

$$
\mathbb{H}^\sharp_a \;=\; \mathfrak{D}_a B + \frac{\alpha'}{4}\Big(\mathrm{Tr}\,(A\,\mathfrak{D}_a A) - \mathrm{Tr}\,(\Theta\,\mathfrak{D}_a\Theta)\Big) - Ð\mathbb{B}^\sharp_a \;, \tag{2.6}
$$

with the covariant derivative $\mathfrak{D}_a B$ is defined as

$$
\mathfrak{D}_a B \;=\; Ð^\sharp_a B - \frac{\alpha'}{4}\Big(\mathrm{Tr}\,(A^\sharp_a\, Ð A) - \mathrm{Tr}\,(\Theta^\sharp_a\, Ð\Theta)\Big) \;,
$$

which sharpens the relation derived in [1]. We will see why this is a covariant derivative shortly.

By demanding $\mathbb{H}$ be gauge invariant, we see that the field $\mathbb{B}$ transforms under gauge transformations:

$$
\mathbb{B} \to {}^{\Phi,\Psi}\mathbb{B} \;=\; \mathbb{B} + \frac{\alpha'}{4}\Big\{\mathrm{Tr}\,(\mathbb{Y}\mathbb{A} - \mathbb{Z}\,\mathbb{\Theta}) + \mathbb{U} - \mathbb{W}\Big\} \;. \tag{2.7}
$$

Given the above relations the field strength $\mathbb{H}$ is invariant. As the animus of $\mathbb{B}$ transforms inhomogeneously, it is inconsistent to try to set it to zero. Here $\mathbb{Y}$, $\mathbb{U}$ are the extensions of Y and U:

$$\mathbb{Y} \;=\; \Phi^{-1}\, \mathbb{d}\Phi \;, \quad \mathbb{d}\mathbb{U} \;=\; \frac{1}{3}\,\mathrm{Tr}\,\mathbb{Y}^3 \;,$$

with $\mathbb{Z}, \mathbb{W}$ being the spin connection counterpart.

The right hand side of (2.6) is the combination of terms identified in [1] as being gauge invariant. This we now understand since $\mathcal{B}_a = \mathbb{H}^\sharp_a$ and $\mathbb{H}$ is gauge invariant.

The covariant derivative is defined such that it transforms in a manner parallel to the B-field itself:

$$^{(\Phi,\Psi)}\mathfrak{D}_a B \;=\; \mathfrak{D}_a B + \frac{\alpha^\backprime}{4}\Big(\mathrm{Tr}\,(Y\,\mathfrak{D}_a A) + \mathfrak{Y}_a - \mathrm{Tr}\,(Z\,\mathfrak{D}_a\Theta) - \mathfrak{Z}_a\Big).$$

We have also defined

$$\begin{aligned}
\mathfrak{Y}_a \;&=\; \text{Đ}^\sharp_a U - \mathrm{Tr}\,(Y^\sharp_a Y^2) + \text{Đ}\,\big(\mathrm{Tr}\,(Y^\sharp_a A - A Y^\sharp_a)\big) \;, \\
\mathfrak{Z}_a \;&=\; \text{Đ}^\sharp_a Z - \mathrm{Tr}\,(Z^\sharp_a Z^2) + \text{Đ}\,\big(\mathrm{Tr}\,(Z^\sharp_a \Theta - \Theta^\sharp_a Z)\big) \;,
\end{aligned}$$

Using that the form $\mathbb{Y}$ satisfies $\mathbb{d}\mathbb{Y} = -\mathbb{Y}^2$, we find that this quantity is Đ-closed

$$\text{Đ}\mathfrak{Y}_a \;=\; 0\;.$$

In addition to the gauge transformations above the field strength $\mathbb{H}$ is invariant under an additional symmetry, in which $\mathbb{B}$ shifts by a $\mathbb{d}$-exact amount,

$$\mathbb{B} \to \mathbb{B} + \mathbb{d}\beta \;, \quad \beta = \beta_m\, e^m + \beta^\sharp_a\, \mathrm{d}y^a \;,$$

where the one-form β is gauge-invariant. Decomposing this into tangibilities we have

$$\begin{aligned}
&B \to B + \text{Đ}\beta \;, \\
&\mathbb{B}^\sharp_a \to \mathbb{B}^\sharp_a + \text{Đ}^\sharp_a\beta - \text{Đ}\,\beta^\sharp_a \;, \\
&\mathbb{B}^\sharp_{ab} \to \mathbb{B}^\sharp_{ab} + \text{Đ}^\sharp_a\beta^\sharp_b - \text{Đ}^\sharp_b\beta^\sharp_a - S_{ab}{}^m\,\beta_m \;.
\end{aligned} \tag{2.8}$$

The first line corresponds to shifting B by a Đ-exact term. The second line corresponds to shifts of $\mathbb{B}^\sharp_a$. The way to think of $\mathbb{B}^\sharp_a$ is that it is another connection; its purpose to is define an invariant quantity $\mathcal{B}_a$ as in (2.6). This invariance can be checked directly, but an easier way to see this is to note that $\mathbb{H}$ is invariant and so $\mathbb{H}^\sharp_a = \mathcal{B}_a$ is invariant. The quantity $\mathcal{B}_a$, mentioned in the introduction, plays an important role as $\mathcal{B}_a + \mathrm{i}\mathfrak{D}_a\omega$ plays the role in heterotic geometry analogous to the role of complexified Kähler class in special geometry. All this goes to show that the animi of $\mathbb{A}$ and $\mathbb{B}$ are connections which are needed to define covariant derivatives on the moduli space.

Although we have not fully explored this aspect, we believe the quantity $\mathbb{B}^\sharp_{ab}$ with the transformation rules as in the third line above, provide connections that enable one to define second and higher order derivatives. For example, see [1] where a second order covariant derivative was defined.

2.3 The extension of $\mathrm{d}^c\omega$

We will shortly have need for the quantity

$$\mathbb{d}^c \mathbb{\omega} \;=\; \frac{1}{3!}\mathbb{J}^P\,\mathbb{J}^Q\,\mathbb{J}^R(\mathbb{d}\mathbb{\omega})_{PQR}~.$$

In a holomorphic basis $\mathbb{\omega}$ is $(1,1)$ and so

$$\mathbb{d}^c\mathbb{\omega} \;=\; \mathrm{i}(\mathbb{d}\mathbb{\omega})^{(2,1)} - \mathrm{i}(\mathbb{d}\mathbb{\omega})^{(1,2)}~.$$

The term $\mathbb{d}^c\mathbb{\omega}$ has vanishing $[3,0]$ term due to the fact that $g^\sharp_{\alpha\overline{\beta}}$ is Kähler, while the remaining components are given by

$$\begin{aligned}
(\mathbb{d}^c\mathbb{\omega})_\alpha &= \mathrm{i}\,\mathfrak{D}_\alpha\omega^{(1,1)} - \mathrm{i}\mathfrak{D}_\alpha\omega^{(0,2)}~,\\
(\mathbb{d}^c\mathbb{\omega})_{\alpha\beta} &= -\,\mathrm{i}\,S_{\alpha\beta}{}^\mu\,\omega_\mu~, \qquad\qquad (\mathbb{d}^c\mathbb{\omega})_{\overline{\alpha}\overline{\beta}} \;=\; \mathrm{i}\,S_{\overline{\alpha}\overline{\beta}}{}^{\overline{\mu}}\,\omega_{\overline{\mu}}~,\\
(\mathbb{d}^c\mathbb{\omega})_{\alpha\overline{\beta}} &= -\,\mathrm{i}\,S_{\alpha\overline{\beta}}{}^{\overline{\mu}}\,\omega_{\overline{\mu}} + \mathrm{i}\,S_{\alpha\overline{\beta}}{}^{\mu}\,\omega_{\mu}~.
\end{aligned} \tag{2.9}$$

Note that the action of the covariant derivative $\mathfrak{D}_\alpha$ on a gauge neutral object is the same as $\text{Đ}^\sharp$ so that $\mathfrak{D}_\alpha\omega^{(p,q)} = \text{Đ}^\sharp_a\omega^{(p,q)}$. In the sections to follow, where no ambiguity will arise we will use $\mathfrak{D}_\alpha$ to prevent an unnecessary proliferation of symbols.

On setting $S=0$ the expression simplifies significantly

$$\mathbb{d}^c\mathbb{\omega} \;=\; \mathrm{i}(\eth-\overline{\eth})\omega + \mathrm{i}\mathrm{d}y^\alpha(\mathfrak{D}_\alpha\omega^{(1,1)} - \mathfrak{D}_\alpha\omega^{(0,2)}) + \mathrm{i}\mathrm{d}y^{\overline{\beta}}(\mathfrak{D}_{\overline{\beta}}\omega^{(2,0)} - \mathfrak{D}_{\overline{\beta}}\omega^{(1,1)})~.$$

While ω is type $(1,1)$, its derivative $\mathfrak{D}_\alpha$ is type $(2,1)\oplus(1,2)$: $\mathfrak{D}_\alpha\omega = \mathfrak{D}_\alpha\omega^{(1,1)} + \mathfrak{D}_\alpha\omega^{(0,2)}$, and this expresses the type changing property of variations with respect to complex structure.

2.4 The relation $\mathbb{H} = \mathbb{d}^c\mathbb{\omega}$, Bianchi identity and second order relations

We suppose that the extended supersymmetry relation (3.7) holds on $\mathbb{X}$ This imposes some constraints on the variations of a heterotic structure. The tangibility $[1,2]$ part of this relation gives

$$\begin{aligned}
\mathcal{B}_\alpha{}^{(2,0)} &= 0~,\\
\mathcal{B}_\alpha{}^{(1,1)} - \mathrm{i}\,\mathfrak{D}_\alpha\omega^{(1,1)} &= 0~,\\
\mathcal{B}_\alpha{}^{(0,2)} + \mathrm{i}\,\mathfrak{D}_\alpha\omega^{(0,2)} &= 0~.
\end{aligned} \tag{2.10}$$

We define

$$\mathcal{Z}_\alpha \;=\; \mathcal{B}_\alpha + \mathrm{i}\mathfrak{D}_\alpha\omega~, \quad \text{and} \quad \overline{\mathcal{Z}}_\alpha \;=\; \mathcal{B}_\alpha - \mathrm{i}\mathfrak{D}_\alpha\omega~,$$

which are the generalisation to heterotic geometry of the variation of the complexified Kähler class familiar in special geometry $\delta B + \mathrm{i}\delta\omega$. In terms of $\mathcal{Z}, \overline{\mathcal{Z}}$, (2.10) can be written as

$$\begin{aligned}
\mathcal{Z}_\alpha{}^{(2,0)} &= \overline{\mathcal{Z}}_\alpha{}^{(2,0)} \;=\; 0~,\\
\overline{\mathcal{Z}}_\alpha{}^{(1,1)} &= 0~,\\
\mathcal{Z}_\alpha{}^{(0,2)} &= 0~.
\end{aligned} \tag{2.11}$$

These equations described first order conditions on the heterotic moduli which were derived in [13–16] and in this notation in [1] by taking partial derivatives of the supersymmetry relation $H = \mathrm{d}^c\omega$. We identify $\mathbb{B}_a$ with b_a and note that $\mathbb{H} = \mathbb{d}^c\mathbb{\omega}$ captures all of the moduli equations except one. For the remaining one we turn to the Bianchi identity for $\mathbb{d}\mathbb{H}$ on $\mathbb{X}$:

$$\mathbb{d}\mathbb{H} \;=-\frac{\alpha^{\backprime}}{4}\Big(\mathrm{Tr}\,\mathbb{F}^2 - \mathrm{Tr}\,\mathbb{R}^2\Big) \;=\; \mathbb{d}(\mathbb{d}^c\mathbb{\omega})\ .$$

The curvatures $\mathbb{F}$ and $\mathbb{R}$ are of type $(1,1)$ and so only the type $(2,2)$ part of this relation is non-vanishing.

We start with tangibility $[1,3]$, focusing on holomorphic variation with index α. The first equality of the previous equation is

$$(\mathbb{d}\mathbb{H})_\alpha \;=-\frac{\alpha^{\backprime}}{2}\Big(\mathrm{Tr}\,(\mathfrak{D}_\alpha\mathcal{A}\;F) - \mathrm{Tr}\,(\mathfrak{D}_\alpha\vartheta\;R)\Big)\ .$$

Meanwhile $(\mathbb{d}\mathbb{d}^c\mathbb{\omega})_\alpha$ is simplified using

$$\begin{aligned}
\text{Đ}\,(\mathbb{d}^c\mathbb{\omega})_\alpha \;&=\; \mathrm{i}\text{Đ}\,(\mathfrak{D}_\alpha\omega^{1,1} - \mathfrak{D}_\alpha\omega^{0,2})\ ,\\
\mathfrak{D}_\alpha(\text{Đ}^c\omega) \;&=\; 2\mathrm{i}\,\Delta_\alpha{}^\mu\,(\eth\omega)_\mu - 2\mathrm{i}\,\eth(\Delta_\alpha{}^\mu\,\omega_\mu) + \mathrm{i}(\eth - \overline{\eth})\mathfrak{D}_\alpha\omega\ ,
\end{aligned}$$

and by using (2.10) we get

$$\overline{\eth}(\mathcal{Z}_\alpha^{(1,1)}) \;=\; 2\mathrm{i}\,\Delta_\alpha{}^\mu\,(\eth\omega)_\mu + \frac{\alpha^{\backprime}}{2}\Big(\mathrm{Tr}\,(\mathfrak{D}_\alpha\mathcal{A}\;F) - \mathrm{Tr}\,(\mathfrak{D}_\alpha\vartheta\;R)\Big)\ . \tag{2.12}$$

Let us now turn our attention to tangibility $[2,2]$. Assuming that $S=0$, this consists of two relations

$$\begin{aligned}
\mathfrak{D}_\alpha(\mathbb{d}^c\mathbb{\omega})_\beta - \mathfrak{D}_\beta(\mathbb{d}^c\mathbb{\omega})_\alpha \;&=\; -\,\frac{\alpha^{\backprime}}{2}\,\Big(\mathrm{Tr}\,(\mathfrak{D}_\alpha\mathcal{A}\,\mathfrak{D}_\beta\mathcal{A}) - \mathrm{Tr}\,(\mathfrak{D}_\alpha\vartheta\,\mathfrak{D}_\beta\theta)\Big)\ ,\\
\mathfrak{D}_\alpha(\mathbb{d}^c\mathbb{\omega})_{\overline{\beta}} - \mathfrak{D}_{\overline{\beta}}(\mathbb{d}^c\mathbb{\omega})_\alpha \;&=\; -\,\frac{\alpha^{\backprime}}{2}\,\Big(\mathrm{Tr}\,(\mathfrak{D}_\alpha\mathcal{A}\,\mathfrak{D}_{\overline{\beta}}\mathcal{A}^\dagger) - \mathrm{Tr}\,(\mathfrak{D}_\alpha\theta\,\mathfrak{D}_{\overline{\beta}}\theta^\dagger)\Big)\\
&\qquad -\frac{\alpha^{\backprime}}{4}\Big(\mathrm{Tr}\,(\mathbb{F}^\sharp_{\alpha\overline{\beta}}\,F) - \mathrm{Tr}\,(\mathbb{R}^\sharp_{\alpha\overline{\beta}}\,R)\Big)\ .
\end{aligned} \tag{2.13}$$

The second relation forms part of a critical algebraic relation needed to derive the moduli space metric in [1] and so we focus on this one. It becomes

$$\begin{aligned}
\mathfrak{D}_\alpha(\mathbb{d}^c\mathbb{\omega})_{\overline{\beta}} - \mathfrak{D}_{\overline{\beta}}(\mathbb{d}^c\mathbb{\omega})_\alpha \;&=\; -2\mathrm{i}\,\big(\mathfrak{D}_\alpha\mathfrak{D}_{\overline{\beta}}\omega\big)^{(1,1)} + 2\mathrm{i}\,\Delta_\alpha{}^\mu\,(\mathfrak{D}_{\overline{\beta}}\omega^{(2,0)})_\mu + 2\mathrm{i}\,\Delta_{\overline{\beta}}{}^{\overline{\nu}}\,(\mathfrak{D}_\alpha\omega^{(0,2)})_{\overline{\nu}}\\
&\quad + \mathrm{i}\,\mathfrak{D}_\alpha(\Delta_{\overline{\beta}}{}^{\overline{\nu}}\,\omega_{\overline{\nu}}) - \mathrm{i}\,\Delta_{\overline{\beta}}{}^{\overline{\nu}}(\mathfrak{D}_\alpha\omega^{(1,1)})_{\overline{\nu}} + \mathrm{i}\,\mathfrak{D}_{\overline{\beta}}(\Delta_\alpha{}^\mu\,\omega_\mu) - \mathrm{i}\,\Delta_\alpha{}^\mu(\mathfrak{D}_{\overline{\beta}}\omega^{(1,1)})_\mu\ .
\end{aligned}$$

The last equation can be simplified by noticing a further relation

$$\mathfrak{D}_\alpha(\Delta_{\overline{\beta}}{}^{\overline{\nu}}\,\omega_{\overline{\nu}}) -\; \Delta_{\overline{\beta}}{}^{\overline{\nu}}(\mathfrak{D}_\alpha\omega^{(1,1)})_{\overline{\nu}} \;=\; (\eth S_{\alpha\overline{\beta}}{}^{\overline{\nu}})\,\omega_{\overline{\nu}} \;=\; 0\ ,$$

which sets the last line to zero. Putting everything together, we can rearrange (2.13) to obtain

$$\begin{aligned}
\big(\mathfrak{D}_\alpha\mathfrak{D}_{\overline{\beta}}\omega\big)^{(1,1)} \;&=\; -\,\frac{\mathrm{i}\alpha^{\backprime}}{4}\,\Big(\mathrm{Tr}\,(\mathfrak{D}_\alpha\mathcal{A}\,\mathfrak{D}_{\overline{\beta}}\mathcal{A}^\dagger) - \mathrm{Tr}\,(\mathfrak{D}_\alpha\theta\,\mathfrak{D}_{\overline{\beta}}\theta^\dagger)\Big)\\
&\quad -\frac{\mathrm{i}\alpha^{\backprime}}{8}\Big(\mathrm{Tr}\,(\mathbb{F}^\sharp_{\alpha\overline{\beta}}\,F) - \mathrm{Tr}\,(\mathbb{R}^\sharp_{\alpha\overline{\beta}}\,R)\Big) + \Delta_\alpha{}^\mu\,(\mathfrak{D}_{\overline{\beta}}\omega^{(2,0)})_\mu + \Delta_{\overline{\beta}}{}^{\overline{\nu}}\,(\mathfrak{D}_\alpha\omega^{(0,2)})_{\overline{\nu}}\ .
\end{aligned}$$

This shows that the Bianchi identity for $\mathbb{H}$ incorporates the second order algebraic relation for the variation of the hermitian form that is crucial in deriving the $\alpha^{\backprime}$-corrected moduli metric.

2.5 The covariant derivative of $\Theta^{(\mathrm{e'}\rho)}$

We now compute $\mathfrak{D}_\alpha\Theta^{(\mathrm{e'}\rho)}$ to zeroth order in α'. We will find that only when $\mathrm{e}^{-}\rho = 1$ is the connection holomorphic, that is $\mathbb{R}^{(0,2)} = 0$. For the remainder of the paper we set $S_{ab} = 0$, and work in harmonic gauge, the conventional choice in supergravity: $\nabla^m \delta g_{mn} = 0$ where $\nabla_m = \partial_m + \Theta_m$ is computed with respect to the affine spin connection on X that is discussed in Appendix **??**. This gauge fixing decomposes into

$$\Delta_\alpha{}^\mu\,\omega_\mu \;=\; 0 \;, \quad \nabla_\mu\,\Delta_\alpha{}^\mu \;=\; 0 \;, \quad \partial_m\big(\omega^{\mu\overline{\nu}}\mathfrak{D}_\alpha\omega_{\mu\overline{\nu}}\big) \;=\; 0 \;,$$

provided X has $h^{(0,2)} = 0$. Interestingly, without vanishing curvature $S = 0$ and gauge fixing, the connection is not holomorphic for any choice of e'ρ.

First, we demand that the connection is holomorphic $\mathcal{D}_\alpha\Theta_\mu = 0$. We find the following components are not immediately zero:

$$\mathfrak{D}_\alpha\Theta^{(\mathrm{e'}\rho)}{}_\mu{}^{\overline{\nu}}{}_\sigma \;=\; \frac{(1-\mathrm{e}^{+}\rho)}{2\mathrm{i}}\, g^{\nu\overline{\lambda}}\,\nabla_\mu\,\mathfrak{D}_\alpha\omega_{\sigma\overline{\lambda}} \;, \quad \mathfrak{D}_\alpha\Theta^{(\mathrm{e'}\rho)}{}_\mu{}^{\overline{\nu}}{}_{\overline{\sigma}} \;=\; -\frac{(1-\mathrm{e}^{+}\rho)}{2\mathrm{i}}\, g^{\overline{\nu}\lambda}\,\nabla_\mu\,\mathfrak{D}_\alpha\omega_{\lambda\overline{\sigma}} \;.$$

We see that the covariant derivatives of the variations appear

$$\nabla_\sigma\,\Delta_{\alpha\overline{\mu}}{}^\nu \;=\; \partial_\sigma\,\Delta_{\alpha\overline{\mu}}{}^\nu + \Theta_\sigma{}^\nu{}_\lambda\,\Delta_{\alpha\overline{\mu}}{}^\lambda \;, \quad \nabla_{\overline{\mu}}\,\mathfrak{D}_\alpha\omega_{\sigma\overline{\nu}} \;=\; \partial_{\overline{\mu}}\,\mathfrak{D}_\alpha\omega_{\sigma\overline{\nu}} - \Theta_{\overline{\mu}}{}^{\overline{\lambda}}{}_{\overline{\nu}}\,\mathfrak{D}_\alpha\omega_{\sigma\overline{\lambda}} \;.$$

For the connection to be holomorphic we need to set $\mathrm{e}^{-}\rho = 1$. It can be checked that this relation is sufficient to ensure that $\mathbb{R}^{(0,2)} = 0$. So we have found a 1-parameter family of holomorphic connections on $\mathbb{X}$.

Computing, we find the following non-zero components for the physical deformations $\mathfrak{D}_\alpha\Theta_{\overline{\mu}}$:

$$\begin{aligned}
\mathfrak{D}_\alpha\Theta^{(\mathrm{e'}\mathrm{e}^{-}1)}{}_{\overline{\mu}}{}^\nu{}_\sigma \;&=\; \nabla_\sigma\,\Delta_{\alpha\overline{\mu}}{}^\nu + i\,\nabla^\nu\,\mathfrak{D}_\alpha\omega_{\sigma\overline{\mu}} \;, \\
\mathfrak{D}_\alpha\Theta^{(\mathrm{e'}\mathrm{e}^{-}1)}{}_{\overline{\mu}}{}^{\overline{\nu}}{}_{\overline{\sigma}} \;&=\; -g^{\overline{\nu}\lambda}\big(\nabla_\lambda\,\Delta_{\alpha\overline{\mu}}{}^\rho + i\,\nabla^\rho\mathfrak{D}_\alpha\omega_{\lambda\overline{\mu}}\big)\,g_{\rho\overline{\sigma}} \;.
\end{aligned} \tag{2.14}$$

Before we continue, let us pause to make some comments. Firstly, we have not computed terms which have vertical indices, such as $\mathfrak{D}_\alpha\Theta_\mu{}^\alpha$, as they do not appear in (3.6).

Second, it is straightforward to show that $\mathfrak{D}_\alpha\Theta$ satisfies the Atiyah condition:

$$\nabla^{(0,1)}\mathfrak{D}_\alpha\Theta^{(0,1)} \;=\; \Delta_\alpha{}^\mu R_\mu \;. \tag{2.15}$$

Third, for the Hull connection (e'ρ)=$(1,0)$ if we compute the covariant derivative of the fibre metric, we find it vanishes since we have set S to zero:

$$\nabla_\alpha(\mathrm{d}s^2_X) \;=\; \nabla_\alpha\big(2\,g_{\mu\overline{\nu}}\,e^{(\mu}\otimes e^{\overline{\nu})}\big) \;=\; -2g_{\mu\overline{\nu}}\big(S_{\alpha\overline{\beta}}{}^\mu\,\mathrm{d}y^{\overline{\beta}}\otimes e^{\overline{\nu}} + S_{\alpha\overline{\beta}}{}^{\overline{\nu}}\,e^\mu\otimes\mathrm{d}y^{\overline{\beta}}\big) \;=\; 0 \;.$$

These covariant derivatives do not mix components of the fibre metric with components of the base metric under parallel transport along the moduli space.

Fourth, the extended connection $\mathbb{\Theta}$ defines a covariant derivative of tensors, and it might be tempting to interpret this parallel transport as the appropriate deformation theory of tensors. However, this does not reduce to known expressions derived in [1] for the appropriate deformations of tensors on X. Note also, if one were to impose that $\mathbb{\nabla}$ and π commute, then this would imply $\mathbb{\Theta}^m{}_a$ and $\mathbb{\Theta}^a{}_m$ vanish. This would mean that $\mathfrak{D}_\alpha g_{\mu\overline{\nu}}$ and $\Delta_{\alpha\overline{\mu}}{}^\nu$ both vanish, which is a condition we do not want.

3 The Metric for Heterotic Moduli

We come to computing the parameter space metric. We compute it in two ways: the first is by computing the metric deriving from a Kähler potential which we propose with some prescience. The second is to dimensionally reduce α'-corrected heterotic supergravity. The dependence of the bundle parameters arises through the mixing of fields implied by supersymmetry as dictated by (6.1). The metric can be written in the form

$$\mathrm{d}s^2 = 2G_{\xi\overline{\eta}}\,\mathrm{d}y^\xi \mathrm{d}y^{\overline{\eta}} + 2G^0_{\alpha\overline{\beta}}\,\mathrm{d}z^\alpha \mathrm{d}z^{\overline{\beta}} ~,$$

where

$$\begin{aligned} G^0_{\alpha\overline{\beta}} &= -\frac{\displaystyle\int \chi_\alpha \overline{\chi}_{\overline{\beta}}}{\displaystyle\int \Omega\,\overline{\Omega}} ~, \\ G_{\xi\overline{\eta}} &= \frac{1}{V}\int \mathcal{D}_\xi\omega \star \mathcal{D}_{\overline{\eta}}\omega + \frac{\mathrm{i}\alpha'}{8V}\int \omega^2 \mathrm{Tr}\left(\mathcal{D}_\xi\mathcal{A}\,\mathcal{D}_{\overline{\eta}}\mathcal{A}^\dagger\right) - \frac{\mathrm{i}\alpha'}{8V}\int \omega^2\mathrm{Tr}\left(\mathcal{D}_\xi\vartheta\,\mathcal{D}_{\overline{\eta}}\vartheta^\dagger\right) ~. \end{aligned} \tag{3.1}$$

Here the χ_α form a basis of closed $(2,1)$-forms, and the second term in the last line is the Kobayashi metric, extended to the entire parameter space. The metric is the natural inner product of $\mathcal{D}_\xi\omega$ and $\mathcal{D}_\xi\mathcal{A}$ together with the inner product of representatives of deformations of complex structure. As expected, the B-field does not make an explicit appearance, being determined by the other fields in the heterotic structure through the anomaly and supersymmetry constraints. The construction of this metric did not assume any underlying special geometry, and its simplicity leads us to conjecture it holds for a general heterotic structure satisfying the equations of motion and so for the Strominger system. The result (3.1) builds on [9] who studied the moduli space metric to $\mathcal{O}(\alpha'^{\,2})$ restricted to a locus of the parameter space in which only the hermitian part of the metric varies $\delta g_{\mu\overline{\nu}} \neq 0$ with the remaining fields remain fixed, $\delta A = \delta B = 0$. On this sub-locus, the leading correction to the moduli space metric is $\mathcal{O}(\alpha'^{\,2})$ and not $\mathcal{O}(\alpha')$. As can be seen from (3.1), this result is a manifestation of demanding the gauge field remain fixed — which is in our language $\mathcal{D}_\xi B = 0$ and $\mathcal{D}_\xi A = 0$. In general, we need to allow all the fields to vary, even when considering Kähler parameter variations. As shown in (3.1), this means the metric is corrected at $\mathcal{O}(\alpha')$, with the property, for example, that the special geometry metric is corrected through a mixing the complex structure and hermitian parameter sectors.

We propose a Kähler that describes the α'-corrected moduli space metric. It is remarkably similar to the special geometry Kähler potential, in which the Kähler form is replaced by the α'-corrected hermitian form:

$$K = K_1 + K_2 = -\log\left(\mathrm{i}\int \Omega\,\overline{\Omega}\right) - \log\left(\frac{4}{3}\int \omega^3\right) ~. \tag{3.2}$$

Although it is remarkably similar to the special geometry Kähler potential, in the derivation of the moduli space metric and Kähler potential, no assumptions are made about special geometry. The fact we arrived at such a similar Kähler potential is a surprising conclusion from our calculation.

In this section we compute the metric using the results constructed in the previous two sections. The answer agrees with known mathematics literature in the situation with the CY is fixed. It also agrees with the answer we get from dimensionally reducing α'-corrected supergravity in the next section. We conclude this is the Kähler parameter space metric and Kähler potential as dictated to us by α'-corrected supergravity.

The first term, K_1, gives the complex structure metric:

$$G^0_{\alpha\overline{\beta}} \;=\; \partial_\alpha\partial_{\overline{\beta}}K_1 \;=\; \frac{1}{4V}\int \partial_\alpha g_{\overline{\mu\nu}}\partial_{\overline{\beta}}g^{\overline{\mu\nu}}\star 1 \;=\; -\frac{\mathrm{i}}{V\,\|\Omega\|^2}\int \chi_\alpha\star\chi_{\overline{\beta}}. \tag{3.3}$$

The second term K_2 contains all the α'-corrections. Differentiating twice

$$\partial_\xi\partial_{\overline{\eta}}K_2 \;=\; \frac{1}{V}\int \partial_\xi\omega\star\partial_{\overline{\eta}}\omega - \frac{1}{2V}\int \omega^2\partial_\xi\partial_{\overline{\eta}}\omega\ . \tag{3.4}$$

We need to turn these terms into appropriate holotypical derivatives in order to express the metric in gauge invariant quantities that reflect the physical moduli fields that arise in the dimensional reduction. The first term uses

$$\partial_\xi\omega \;=\; \mathcal{D}_\xi\omega^{1,1}+\mathcal{D}_\xi\omega^{0,2}\ .$$

For the second, we use ω is a $(1,1)$-form and so

$$\omega^2\partial_\xi\partial_{\overline{\eta}}\omega \;=\; \omega^2\mathcal{D}_\xi\mathcal{D}_{\overline{\eta}}\omega^{1,1} \;=\; \frac{1}{2}\omega^2\{\mathcal{D}_\xi,\mathcal{D}_{\overline{\eta}}\}\omega^{1,1}.$$

The second equality follows from $[\partial_\xi,\partial_{\overline{\eta}}]\omega=0$.

Returning to the Kähler potential,

$$\begin{aligned}\partial_\xi\partial_{\overline{\eta}}K_2 \;&=\; \frac{1}{V}\int(\mathcal{D}_\xi\omega^{1,1}+\mathcal{D}_\xi\omega^{0,2})\star(\mathcal{D}_{\overline{\eta}}\omega^{1,1}+\mathcal{D}_{\overline{\eta}}\omega^{2,0}) - \frac{1}{4V}\int\omega^2\{\mathcal{D}_\xi,\mathcal{D}_{\overline{\eta}}\}\omega^{1,1}\\ &=\; \frac{1}{V}\int\Big(\mathcal{D}_\xi\omega^{1,1}\star\mathcal{D}_{\overline{\eta}}\omega^{1,1}+\mathcal{D}_\xi\omega^{0,2}\star\mathcal{D}_{\overline{\eta}}\omega^{2,0}\Big)+\frac{\mathrm{i}}{4V}\int\omega^2\mathrm{i}\{\mathcal{D}_\xi,\mathcal{D}_{\overline{\eta}}\}\omega^{1,1}\ .\end{aligned}$$

In the second term $\mathcal{D}_\xi\omega^{0,2}\star\mathcal{D}_{\overline{\eta}}\omega^{2,0}$ is $\mathcal{O}(\alpha'^{\,2})$. For the third term we use (??):

$$\frac{\mathrm{i}}{4V}\int\omega^2\mathrm{i}\{\mathcal{D}_\xi,\mathcal{D}_{\overline{\eta}}\}\omega = \frac{\mathrm{i}}{4V}\int\omega^2\Big[\frac{\alpha'}{2}\mathrm{Tr}\,\mathcal{D}_\xi\mathcal{A}\mathcal{D}_{\overline{\eta}}\mathcal{A}^\dagger+\Delta_\xi{}^{\mu}\Upsilon^{1,0}_{\overline{\eta}\,\mu}-\Delta_{\overline{\eta}}{}^{\overline{\nu}}\Upsilon^{0,1}_{\xi\,\overline{\nu}}\Big]\ ,$$

where Υ_ξ is defined, and also

$$\frac{1}{2}\omega^2\Delta_{\overline{\eta}}{}^{\overline{\nu}}\Upsilon^{0,1}_{\xi\,\overline{\nu}} \;=\; \mathcal{D}_{\overline{\eta}}\omega^{2,0}\star\mathcal{B}^{0,2}_\xi-\mathrm{i}\mathcal{D}_{\overline{\eta}}\omega^{2,0}\star\mathcal{D}_\xi\omega^{0,2} \;=\; \mathcal{O}(\alpha'^{\,2})\ .$$

In this way we see that

$$\frac{\mathrm{i}}{4V}\int\omega^2\mathrm{i}\{\mathcal{D}_\xi,\mathcal{D}_{\overline{\eta}}\}\omega \;=\; \frac{\mathrm{i}\alpha'}{8V}\int\omega^2\mathrm{Tr}\,\mathcal{D}_\xi\mathcal{A}\mathcal{D}_{\overline{\eta}}\mathcal{A}^\dagger\ .$$

Hence,

$$\partial_\xi\partial_{\overline{\eta}}K_2 \;=\; \frac{1}{V}\int\mathcal{D}_\xi\omega^{1,1}\star\mathcal{D}_{\overline{\eta}}\omega^{1,1}+\frac{\mathrm{i}\alpha'}{8V}\int\omega^2\mathrm{Tr}\,\mathcal{D}_\xi\mathcal{A}\mathcal{D}_{\overline{\eta}}\mathcal{A}^\dagger$$

Including the complex structure special geometry metric (3.3) we get

$$\mathrm{d}s^2 = 2G^K_{\xi\overline{\eta}}\,\mathrm{d}y^\xi\mathrm{d}y^{\overline{\eta}}+2G^0_{\alpha\overline{\beta}}\,\mathrm{d}z^\alpha\mathrm{d}z^{\overline{\beta}}\ , \tag{3.5}$$

where $\mathrm{d}y^\xi = \{\mathrm{d}z^\alpha, \mathrm{d}t^\rho, \mathrm{d}w^i\}$. So when we choose complex coordinates on $\mathcal{M}$ so that holds, a choice naturally handed to us by string theory as shown in the next section, we find the Kähler potential exactly gives the metric arising from the dimensional reduction.

The upshot is that the complex structure metric $G^0_{\alpha\overline{\beta}}$ is unchanged in $\mathcal{O}(\alpha^\backprime)$, while the complexified Kähler metric $G^K_{\xi\overline{\eta}}$ is corrected, and, as written above, implicitly includes $\alpha^\backprime$-corrections. The complex structure metric can still, as is the case of special geometry, be written as a metric on the cohomology classes. This is not obviously the case for the metric $G^K_{\xi\overline{\eta}}$, we intend to return to this point in future work.

In section 4, we illustrate a utility of $\mathbb{X}$ by showing how the curvature $\mathbb{R}$ in (6.1) can be used to compute the covariant derivative $D_\alpha\Theta$ in terms of the complex structure moduli $\Delta_\alpha{}^\mu$ and hermitian moduli $\mathfrak{D}_\alpha\omega^{1,1}$ to zeroth order in $\alpha^\backprime$. We then use this to compute the last term in the moduli space metric $g^\sharp$ derived in [1] to be

$$\mathrm{d}s^{\sharp\,2} = 2g^\sharp_{\alpha\overline{\beta}}\,\mathrm{d}y^\alpha\otimes\mathrm{d}y^{\overline{\beta}}\ ;$$

$$g^\sharp_{\alpha\overline{\beta}} = \frac{1}{V}\int_X\Big\{\Delta_\alpha{}^\mu\star\Delta_{\overline{\beta}}{}^{\overline{\nu}}\,g_{\mu\overline{\nu}} + \frac{1}{4}\mathcal{Z}_\alpha\star\overline{\mathcal{Z}}_{\overline{\beta}} + \frac{\alpha^\backprime}{4}\mathrm{Tr}\left(\mathfrak{D}_\alpha A\star\mathfrak{D}_{\overline{\beta}}A\right) \\ + \frac{\alpha^\backprime}{2}\left(\Delta_{\alpha\,\overline{\mu}\overline{\nu}}\Delta_{\overline{\beta}\rho\sigma} + \mathfrak{D}_\alpha\omega_{\rho\overline{\mu}}\,\mathfrak{D}_{\overline{\beta}}\omega_{\sigma\overline{\nu}}\right)R^{\overline{\mu}\rho\overline{\nu}\sigma}\Big\}\ , \tag{3.6}$$

which generalises an expression in [9] to include all the moduli.

In section 5, we put all of this together to show how to derive the moduli space metric $g^\sharp_{\alpha\overline{\beta}}$ from its Kähler potential in a concise way, which simplifies much of the analysis of [1].

We have need of derivatives that are covariant under the coordinate transformations, this requires a refinement of the derivatives defined in [1] for which covariance was required only under the simpler transformations $(y,\,x)\to(\tilde{y}(y),\,\tilde{x}(x))$. We are led to construct outer derivatives that descend from $\mathbb{d}$ and covariant derivatives $\text{Đ}^\sharp_a$ and $\mathfrak{D}_a$. For complex manifolds X and M the operators Đ and $\text{Đ}^\sharp$ split further into $\eth + \overline{\eth}$ and $\eth^\sharp + \overline{\eth}^\sharp$, which are the analogues of the familiar split $\mathrm{d} = \partial + \overline{\partial}$.

Furthermore, we overload the derivative symbol so that $\text{Đ}^\sharp_a$, say, should also be covariant with respect to gauge transformations. When we take into account the complex structure of X and M, the $\text{Đ}^\sharp_a$ decomposes further into $\mathfrak{D}_\alpha$ and $\mathfrak{D}_{\overline{\beta}}$, which are suitable generalisations of the holotypical derivatives of [1]. From §2.3 we write $\mathfrak{D}$ in place of $\text{Đ}^\sharp$ even when acting on 'gauge neutral' objects since no ambiguity arises, and this gives cleaner expressions. For example, we understand that $\mathfrak{D}_\alpha\omega = \text{Đ}^\sharp_\alpha\omega$.

In the previous sections we have described in detail how to extend the geometry of X to the larger structure of the fibration $\mathbb{X}$. This also allowed us to describe geometrically the variations of the metric and complex structures on X in terms of Lie derivatives and flows on the moduli space M. We now study the geometry $\mathcal{U}$, the universal bundle, whose base manifold is $\mathbb{X}$. This is a holomorphic bundle with connection $\mathbb{A}$, with $\mathbb{A}$ the natural extension of A. The field strength $\mathbb{F}$ for $\mathbb{A}$ has a tangibility $[1,1]$ part which exactly describes the variation of A. The Bianchi identity for $\mathbb{F}$ efficiently encapsulates otherwise subtle identities derived in [1].

The universal geometry also includes the three-form $H = \mathrm{d}^c\omega$ and its Bianchi identity (6.2).

The extension of H to $\mathbb{X}$ is defined in a natural way

$$\mathbb{H} = \mathbb{d}^c \omega \ . \tag{3.7}$$

We demand that $\mathbb{H}$ obeys an extended Bianchi identity

$$\mathbb{d}\mathbb{H} \ = -\frac{\alpha^\backprime}{4}\Big(\mathrm{Tr}\,(\mathbb{F}^2) - \mathrm{Tr}\,(\mathbb{R}^2)\Big) \ .$$

Remarkably, this equation elegantly captures otherwise complicated algebraic relations derived with much effort in [1]. These identities are important as they are central to the construction of the metric on M and showing that it is Kähler. Using the extended quantities on $\mathbb{X}$ we re-derive the metric on M in a concise fashion.

At this point it is useful to pause, and recall what happens in special geometry when the gauge connection is identified with the spin connection, $\delta A = \delta\Theta$. We do not rely on being connected to this example, but it serves the purpose of illustration for the more general case below. The only independent variations are contained within δg_{mn} and δB_{mn}. Denote the $\alpha^\backprime$-expansion of fields as $B = B^0 + \alpha^\backprime B^1 + \dots$, $\omega = \omega^0 + \alpha^\backprime \omega^1 + \dots$.

A variation of the metric is

$$\partial_\xi(\mathrm{d}s^2) \ = \ 2\Delta_{\xi\overline{\mu}}{}^{\rho} g_{\rho\overline{\nu}}\, \mathrm{d}x^{\overline{\mu}} \otimes \mathrm{d}x^{\overline{\nu}} + 2(\partial_\xi g)_{\mu\overline{\nu}}\, \mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\overline{\nu}} \ .$$

Since $\delta g_{\overline{\mu\nu}}$ and $\delta g_{\mu\overline{\nu}}$ separately solve the Lichnerowicz equation, they can be varied independently of each other, so we can assign independent parameters to these variations. The mixed component, $\delta g_{\mu\overline{\nu}}$, is a zero-mode of the Lichnerowicz operator if and only if it is a harmonic $(1,1)$-form. Similarly, the B-field satisfies $\mathrm{d}\delta B^0 = 0$ and is gauge-fixed $\mathrm{d}^\dagger \delta B^0 = 0$, so δB^0 can be expanded in harmonic $(1,1)$-forms. In sum, we associate parameters to field variations as follows:

$$\begin{aligned} \delta g_{\overline{\mu\nu}} \ &= \ \delta z^\alpha \Delta_{\alpha(\overline{\mu\nu})} \ , \quad \delta z^\alpha \in \mathbb{C}, \ \text{for } \alpha = 1, \dots h^{2,1} \ , \\ \delta\omega \ &= \ \delta v^r e_r, \quad \delta B = \delta u^r e_r, \quad \delta u^r, \delta v^r \in \mathbb{R}, \quad e_r \in H^{1,1}(X,\mathbb{R}) \ , \text{for } r = 1, \dots, h^{1,1}. \end{aligned} \tag{3.8}$$

The conventional choice of gauge fixing, $\nabla^m \delta g_{mn} = 0$, implies $\nabla^{\overline{\mu}} \Delta_{\alpha(\overline{\mu\nu})} = 0$. When this is so, each tensor $\Delta_{\alpha(\overline{\mu\nu})}$ is in one-to-one correspondence with a harmonic representative $\Delta_\alpha{}^\rho \in H^1(X,T)$. To see this vary the Kähler condition $\overline{\partial}\omega = 0$ with respect to complex structure to give

$$\overline{\partial}\, \mathcal{D}_\alpha \omega^{0,2} = 0 \ .$$

As $h^{0,2} = 0$,

$$\mathcal{D}_\alpha \omega^{0,2} = \mathrm{i}\Delta_{\alpha[\overline{\mu\nu}]} \mathrm{d}x^{\overline{\mu}} \mathrm{d}x^{\overline{\nu}} = \overline{\partial} k_\alpha$$

for some $(0,1)$-form k_α. Co-closure of $\Delta_\alpha{}^\rho$ gives $\partial^{\overline{\mu}} \Delta_{\alpha\,\overline{\mu\nu}} = \partial^{\overline{\mu}} \Delta_{\alpha\,[\overline{\mu\nu}]} = 0$ and, as $\mathcal{X}$ is compact, this forces $\Delta_{\alpha[\overline{\mu\nu}]} = 0$. Hence, $\Delta_\alpha{}^\rho$ is in one-to-one correspondence both with the metric variations $\delta g_{\overline{\mu\nu}}$ via

$$g^{\rho\overline{\nu}} \delta g_{\overline{\mu\nu}} \ = \ \delta z^\alpha \Delta_{\alpha\overline{\mu}}{}^{\rho} \ , \tag{3.9}$$

and with harmonic $(2,1)$-forms χ_α via

$$\chi_\alpha \ = \ \frac{1}{2} \Omega_{\rho\sigma}{}^{\overline{\nu}} \Delta_{\alpha\,\overline{\mu\nu}} \mathrm{d}x^\rho \mathrm{d}x^\sigma \mathrm{d}x^{\overline{\mu}} \ . \tag{3.10}$$

The inverse of this last relation is

$$\Delta_\alpha{}^\mu \;=\; \frac{1}{2\,\|\Omega\|^2}\overline{\Omega}^{\mu\tau\rho}\chi_{\alpha\,\tau\rho\overline{\sigma}}\,\mathrm{d}x^{\overline{\sigma}}\ . \tag{3.11}$$

We have seen these relations before, though now we have specialised to the case $\alpha' = 0$, for which $\Delta_{\alpha\,[\overline{\mu\nu}]} = 0$, and this has allowed us to write (3.9) in the given form. It is easy to see χ_α and $\Delta_\alpha{}^\mu$ are also $\overline{\partial}$-closed and co-closed. This establishes an isomorphism $H^1(X,T) \cong H^{2,1}(X,\mathbb{C})$.

Promoting the parameters to dynamical fields, denoted by corresponding capital letters, for example $u^r \to U^r(y)$, $\mathcal{L}_g$ is

$$\begin{aligned}
\mathcal{L}_g \;&= -\frac{1}{2V}\int \mathrm{d}^6x\sqrt{g}g^{\mu\overline{\nu}}g^{\rho\overline{\tau}}\Big(\partial_e(\delta g_{\overline{\nu}\overline{\tau}})\partial^e(\delta g_{\mu\rho}) + \partial_e(\delta g_{\mu\overline{\tau}})\partial^e(\delta g_{\overline{\nu}\rho})\Big)\\
&= -\frac{1}{2V}\int \mathrm{d}^6x\sqrt{g}\left(\partial_e Z^\alpha\partial^e Z^{\overline{\beta}}\Delta_{\alpha(\overline{\mu\nu})}\,\Delta_{\overline{\beta}}{}^{(\mu\nu)} + \partial_e\omega_{\mu\overline{\nu}}\partial^e\omega^{\mu\overline{\nu}}\right)\\
&= -2G^0_{\alpha\overline{\beta}}\,\partial_e Z^\alpha\partial^e Z^{\overline{\beta}} + G^0_{rs}\,\partial_e V^r\partial^e V^s\ .
\end{aligned} \tag{3.12}$$

where we identify the special geometry metrics

$$G^0_{\alpha\overline{\beta}} \;=\; -\frac{\mathrm{i}}{V\,\|\Omega\|^2}\int\chi_\alpha\star\chi_{\overline{\beta}}\ , \qquad G^0_{rs} = \frac{1}{2V}\int e_r\star e_s\ .$$

We have used the Kaluza–Klein ansatz (3.8) in writing $\partial_e\omega = \partial_e V^r e_r$ and $\partial_e(\delta g_{\overline{\mu\nu}}) = \partial_e Z^\alpha\Delta_{\alpha(\overline{\mu\nu})}$ together with (3.11). The H-field gives

$$\mathcal{L}_H \;=\; G^0_{rs}\,\partial U^r\partial U^s\ .$$

The complex structure moduli space automatically gives a Kähler moduli space metric $G^0_{\alpha\overline{\beta}}$. The Kähler moduli space $\mathcal{M}_K$ is also complex but the choice of complex coordinates in terms of u^r, v^r is ambiguous. The canonical choice is to associate a point $p \in \mathcal{M}_K$ with a complexified form $B + \mathrm{i}\omega$. As $\dim_{\mathbb{C}}\mathcal{M}_K = h^{1,1}$ there are local coordinates $t^\rho, t^{\overline{\sigma}}$ for $\rho, \overline{\sigma} = 1,\ldots,h^{1,1}$ to be identified. The tangent space $T_p\mathcal{M}_K^{\mathbb{C}}$ is a complex vector space, and the complex structure facilitates a splitting: $T_p\mathcal{M}_K^{\mathbb{C}} = T_p\mathcal{M}_K^{1,0}\oplus T_p\mathcal{M}_K^{0,1}$. The e_r are a basis for the complexification $H^{1,1}(X,\mathbb{C})$ and so the conventional choice is

$$\delta B + \mathrm{i}\delta\omega \;=\; (\delta u^\rho + \mathrm{i}\delta v^\rho)e_\rho \in T_p^{1,0}\mathcal{M}_K \cong H^{1,1}(X,\mathbb{C})\ .$$

Similarly, deformations of $B - \mathrm{i}\omega$ are identified as

$$\delta B - \mathrm{i}\delta\omega \;=\; (\delta u^{\overline{\sigma}} - \mathrm{i}\delta v^{\overline{\sigma}})e_{\overline{\sigma}} \in T_p^{0,1}\mathcal{M}_K \cong H^{1,1}(X,\mathbb{C})\ .$$

The special geometry metric is then given by identifying the metric of the kinetic terms in the Lagrangian (3.12):

$$\mathrm{d}s^2 \;=\; 2G^0_{\alpha\overline{\beta}}\,\mathrm{d}z^\alpha\mathrm{d}z^{\overline{\beta}} + 2G^0_{\rho\overline{\sigma}}\,\mathrm{d}t_0^\rho\mathrm{d}t_0^{\overline{\sigma}}\ ,$$

where for harmonic forms e_r, χ_α we can write the metrics in a form that depends only on the cohomology classes:

$$G^0_{\alpha\overline{\beta}} \;=\; -\frac{\int\chi_\alpha\,\chi_{\overline{\beta}}}{\int\Omega\,\overline{\Omega}}\ , \qquad G_{\rho\overline{\sigma}} \;=\; \frac{1}{2}\left(\frac{1}{2V}\int e_\rho\,\omega^2\right)\left(\frac{1}{2V}\int e_{\overline{\sigma}}\,\omega^2\right) - \frac{1}{4V}\int\omega\,e_\rho\,e_{\overline{\sigma}}\ .$$

Now we proceed to the general case, including the α'-correction and assuming a general choice of holomorphic semi-stable vector bundle. The Kaluza–Klein ansatz includes a correction that allows for a dependence on all parameters:

$$\delta\omega \;=\; \delta v^r e_r + \alpha'\, \delta y^M \mathcal{D}_M\omega, \qquad \delta H \;=\; \mathrm{d}\left(\delta u^r e_r + \alpha'\, \delta y^M \mathcal{B}_M\right) \;.$$

When substituted into the Ricci-scalar and H-field kinetic term, we identify the metric through the kinetic terms arising from $\mathrm{d}s_g^2$ and $\mathrm{d}s_H^2$ respectively:

$$\mathrm{d}s_g^2 = G^0_{rs}\mathrm{d}v^r\mathrm{d}v^s + \alpha'\left(\frac{1}{V}\int \mathcal{D}_\xi\omega^1 \star e_s\right)\mathrm{d}y^\xi\mathrm{d}v^s + \alpha'\left(\frac{1}{V}\int e_r \star \mathcal{D}_{\overline{\eta}}\omega^1\right)\mathrm{d}v^r\mathrm{d}y^{\overline{\eta}} + 2G^0_{\alpha\overline{\beta}}\mathrm{d}z^\alpha\mathrm{d}z^{\overline{\beta}},$$

$$\begin{aligned}
\mathrm{d}s_H^2 &= G^0_{rs}\mathrm{d}u^r\mathrm{d}u^s + \alpha'\left(\frac{1}{V}\int \mathcal{B}^1_\xi \star e_s\right)\mathrm{d}y^\xi\mathrm{d}u^s + \alpha'\left(\frac{1}{V}\int e_r \star \mathcal{B}^1_{\overline{\eta}}\right)\mathrm{d}u^r\mathrm{d}y^{\overline{\eta}} \\
&= G^0_{rs}\mathrm{d}u^r\mathrm{d}u^s + \mathrm{i}\alpha'\left(\frac{1}{V}\int \mathcal{D}_\xi\omega^1 \star e_s\right)\mathrm{d}y^\xi\mathrm{d}u^s - \mathrm{i}\alpha'\left(\frac{1}{V}\int e_r \star \mathcal{D}_{\overline{\eta}}\omega^1\right)\mathrm{d}u^r\mathrm{d}y^{\overline{\eta}} \\
&\qquad + \alpha'\left(\frac{1}{V}\int e_r \star \gamma^1_\xi\right)\mathrm{d}u^r\mathrm{d}y^\xi + \alpha'\left(\frac{1}{V}\int e_r \star \gamma^1_{\overline{\eta}}\right)\mathrm{d}u^r\mathrm{d}y^{\overline{\eta}} \;,
\end{aligned}$$

where we have substituted spacetime fields kinetic energy terms for metric coordinates on $\mathcal{M}$ e.g. $\partial_e U^r \to \mathrm{d}u^r$. In the last equality, we have used the supersymmetry relation $\mathcal{B}^{1,1}_\xi = \mathrm{i}\mathcal{D}_\xi\omega^{1,1} + \gamma^{1,1}_\xi + [\mathrm{d}(\ldots)]^{1,1}$. We have written the special geometry metric G^0_{rs}, and we identify the α'-correction to it:

$$G^0_{rs} \;=\; \frac{1}{2V}\int e_r \star e_s \;, \qquad G^1_{\xi s} \;=\; \frac{1}{2V}\int \mathcal{D}_\xi\omega^1 \star e_s \;. \tag{3.13}$$

The freedom to shift by d-exact terms means we can expand γ^1_ξ in harmonic $(1,1)$ forms, $\gamma^1_\xi = \gamma^{1\;s}_\xi e_s$ giving

$$\left(\frac{\alpha'}{2V}\int e_r \star \gamma^1_\xi\right)\mathrm{d}u^r\mathrm{d}y^\xi \;=\; \left(\frac{\alpha'}{2V}\int e_r \star e_s\right)\gamma^{1\,s}_\xi\,\mathrm{d}u^r\mathrm{d}y^\xi \;=\; \alpha'\, G^0_{rs}\gamma^{1\,s}_\xi\,\mathrm{d}u^r\mathrm{d}y^\xi \;.$$

Adding $\mathrm{d}s_g^2$ and $\mathrm{d}s_H^2$ together

$$\begin{aligned}
\mathrm{d}s_g^2 + \mathrm{d}s_H^2 \;&=\; 2G^0_{\alpha\overline{\beta}}\,\mathrm{d}z^\alpha\mathrm{d}z^{\overline{\beta}} + G^0_{rs}\Big(\mathrm{d}v^r\mathrm{d}v^s + \mathrm{d}u^r\mathrm{d}u^s + \alpha'\,\gamma^{1\,r}_\xi\mathrm{d}y^\xi\mathrm{d}u^s + \alpha'\,\gamma^{1\,r}_{\overline{\eta}}\mathrm{d}y^{\overline{\eta}}\mathrm{d}u^s\Big) \\
&\qquad + 2\mathrm{i}\alpha'\,G^1_{\xi s}\,\mathrm{d}y^\xi(\mathrm{d}u^s - \mathrm{i}\mathrm{d}v^s) - 2\mathrm{i}\alpha'\,G^1_{r\overline{\eta}}\,(\mathrm{d}u^r + \mathrm{i}\mathrm{d}v^r)\,\mathrm{d}y^{\overline{\eta}} \\
&=\; 2G^0_{\alpha\overline{\beta}}\,\mathrm{d}z^\alpha\mathrm{d}z^{\overline{\beta}} + G^0_{rs}\Big(\mathrm{d}u^r + \mathrm{i}\mathrm{d}v^r + \alpha'\,\gamma^{1\,r}_{\overline{\eta}}\,\mathrm{d}y^{\overline{\eta}}\Big)\Big(\mathrm{d}u^s - \mathrm{i}\mathrm{d}v^s + \alpha'\,\gamma^{1\,s}_\xi\,\mathrm{d}y^\xi\Big) \\
&\qquad + 2\alpha'\,G^1_{\xi s}\mathrm{d}y^\xi(\mathrm{d}v^s + \mathrm{i}\mathrm{d}u^s) + 2\alpha'\,G^1_{r\overline{\eta}}(\mathrm{d}v^r - \mathrm{i}\mathrm{d}u^r)\,\mathrm{d}y^{\overline{\eta}} \;.
\end{aligned}$$

The penultimate line indicates the complex coordinates on the parameter space $\mathcal{M}$ are modified at first order in α'. We can view this as a change in special geometry complex structure.

4 The contribution of $\mathfrak{D}_\alpha\Theta$ to the moduli space metric

We are now in a position to compute the last term of (3.6). The connection in that metric is the Hull connection (e'ρ) $= (1,0)$, though in fact e drops out of the following calculation and so it is valid for a 1-parameter family. The integration is evaluated for a fixed point $y \in M$ giving a simplifying rule $e^m \to \mathrm{d}x^m$.

We use the result (2.14) to find

$$\mathrm{Tr}\left(\mathfrak{D}_\alpha\Theta \star \mathfrak{D}_{\overline{\beta}}\Theta\right) = 2\left(\nabla^{(1,0)}\Delta_\alpha{}^\mu + i\,\nabla^\mu\,\mathfrak{D}_\alpha\omega^{(1,1)}\right) \star \left(\nabla^{(0,1)}\Delta_{\overline{\beta}}{}^{\overline{\nu}} - i\,\nabla^{\overline{\nu}}\,\mathfrak{D}_{\overline{\beta}}\omega^{(1,1)}\right) g_{\mu\overline{\nu}}\ ,$$

where $\nabla^{(1,0)}\Delta_\alpha{}^\mu = \mathrm{d}x^\nu\,\nabla_\nu\,\Delta_\alpha{}^\mu$.

Using $\mathfrak{D}_\alpha\omega^{(0,2)} = \mathcal{O}(\alpha^{\backprime})$ we find

$$\begin{aligned}
&-\frac{\alpha^{\backprime}}{4V}\int_X \mathrm{Tr}\left(\mathfrak{D}_\alpha\Theta \star \mathfrak{D}_{\overline{\beta}}\Theta\right) = \\
&\quad -\frac{\alpha^{\backprime}}{2V}\int_X \nabla^{(1,0)}\Delta_\alpha{}^\mu \star \nabla^{(0,1)}\Delta_{\overline{\beta}}{}^{\overline{\nu}}\, g_{\mu\overline{\nu}} - \frac{\alpha^{\backprime}}{2V}\int_X \nabla^\mu\,\mathfrak{D}_\alpha\omega \star \nabla_\mu\,\mathfrak{D}_{\overline{\beta}}\omega \\
&\quad +\frac{\mathrm{i}\alpha^{\backprime}}{2V}\int_X \nabla^{(1,0)}\Delta_\alpha{}^\mu \star \nabla_\mu\,\mathfrak{D}_{\overline{\beta}}\omega - \frac{\mathrm{i}\alpha^{\backprime}}{2V}\int_X \nabla_{\overline{\nu}}\,\mathfrak{D}_\alpha\omega \star \nabla^{(0,1)}\Delta_{\overline{\beta}}{}^{\overline{\nu}} + \mathcal{O}(\alpha^{\backprime\,2})\ .
\end{aligned}$$

At this point, we notice a series of useful identities. The variation of the complex structure satisfies

$$\nabla^\sigma\,\nabla_\sigma\,\Delta_\alpha{}^\mu = \Delta_\alpha{}^{\sigma\nu}R_{\sigma\overline{\lambda}\nu}{}^\mu\mathrm{d}x^{\overline{\lambda}} \quad \text{and} \quad \nabla_\mu\nabla^{(1,0)}\Delta_\alpha{}^\mu = 0\ .$$

where we used the vanishing of the pure part of the curvature tensor for Θ: $R_{\nu\rho\lambda}{}^\mu = 0$.

For the Kähler form variation we find

$$\nabla_\sigma\nabla^\sigma\mathfrak{D}_\alpha\omega = R_{\mu\overline{\nu}}{}^{\sigma\overline{\tau}}\,\mathfrak{D}_\alpha\omega_{\sigma\overline{\tau}}\,\mathrm{d}x^\mu\mathrm{d}x^{\overline{\nu}} \quad \text{and} \quad g^{\mu\overline{\nu}}\,\nabla^\tau\nabla_{\overline{\nu}}\mathfrak{D}_\alpha\omega_{\tau\overline{\sigma}}\,\mathrm{d}x^{\overline{\sigma}} = 0\ .$$

After integrating by parts, using the terms above and metric compatibility of ∇ we find to first order in $\alpha^{\backprime}$:

$$-\frac{\alpha^{\backprime}}{4V}\int_X \mathrm{Tr}\left(\mathfrak{D}_\alpha\Theta \star \mathfrak{D}_{\overline{\beta}}\Theta\right) = \frac{\alpha^{\backprime}}{2V}\int_X \left(\Delta_{\alpha\,\overline{\mu}\overline{\nu}}\Delta_{\overline{\beta}\rho\sigma} + \mathfrak{D}_\alpha\omega_{\rho\overline{\mu}}\,\mathfrak{D}_{\overline{\beta}}\omega_{\sigma\overline{\nu}}\right) R^{\overline{\mu}\rho\overline{\nu}\sigma}\ .$$

This expression agrees in form with that derived in [9].

We derive the moduli space metric from the Kähler potential in a concise manner using extended forms on $\mathbb{X}$. As in the previous section, we set $S_{ab} = 0$ and within integrals over X we have the rule $e^m \to \mathrm{d}x^m, e_a \to \partial_a$.

The moduli space metric $g^\sharp_{\alpha\overline{\beta}}$ has the associated Kähler form

$$\omega^\sharp = \mathrm{i}g^\sharp_{\alpha\overline{\beta}}\,\mathrm{d}y^\alpha\mathrm{d}y^{\overline{\beta}}\ .$$

We show using $\mathbb{X}$ that

$$\omega^\sharp = \mathrm{i}\,\mathfrak{D}\overline{\mathfrak{D}}\mathcal{K}\ , \quad \text{where} \quad \mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2 = -\log\left(\frac{4}{3}\int\omega^3\right) - \log\left(\mathrm{i}\int\Omega\,\overline{\Omega}\right)\ .$$

That is, we are showing that $\mathcal{K}$ is the Kähler potential for the moduli space metric.

We adopt the convention that when a universal form appears within an integral over X, the only surviving part is that which makes the integrand a top form on X. Some useful statements illustrating this are

$$
\begin{aligned}
\int_X \omega^2 \mathbb{F} \;&=\; \int_X \omega^2 F \;=\; 0\;, \\
\frac{1}{2}\int_X \omega^2 \mathrm{Tr}\,\mathbb{F}^2 \;&=\; \int_X \omega^2 \mathrm{Tr}\left(F\,\mathbb{F}_{\alpha\overline{\beta}} - \mathbb{F}_\alpha\,\mathbb{F}_{\overline{\beta}}\right) \mathrm{d}y^\alpha \mathrm{d}y^{\overline{\beta}} \;=\; \int_X \omega^2 \mathrm{Tr}\,(\mathfrak{D}_\alpha \mathcal{A}\,\mathfrak{D}_{\overline{\beta}}\mathcal{A}^\dagger)\,\mathrm{d}y^\alpha \mathrm{d}y^{\overline{\beta}}, \\
\int_X \omega^2\,\mathfrak{D}\,\overline{\mathfrak{D}}\,\omega \;&=\; \int_X \omega^2 \partial\overline{\partial}\omega \;=\; \int_X \omega^2 \partial\overline{\partial}\boldsymbol{\omega}\;.
\end{aligned}
\tag{4.1}
$$

where we use the relations $\omega^2 F = 0$, $\mathfrak{D}_{\overline{\beta}}\mathcal{A} = 0$ and $\overline{\partial}(\omega^2) = \partial(\omega^2) = 0$. We will also use, within the integrand,

$$
\mathfrak{D}\left(\frac{\omega^2}{2V}\right) \;=-\frac{1}{V}\star\mathfrak{D}\omega\;.
\tag{4.2}
$$

Recall that $\mathbb{d}\Omega \;=-k^\sharp\Omega + \chi$ with $\chi \;=\; \frac{1}{2}\chi_{\alpha\mu\nu\overline{\rho}}\,\mathrm{d}y^\alpha\mathrm{d}x^\mu\mathrm{d}x^\nu\mathrm{d}x^{\overline{\rho}}$ and $k^\sharp = \mathfrak{D}\,\mathcal{K}_2 = \mathrm{d}y^\alpha\partial_\alpha\mathcal{K}_2$ and we use $\mathrm{d}\Omega = \overline{\mathfrak{D}}\Omega = 0$.

Consider first the derivatives of $\mathcal{K}_1$,

$$
\begin{aligned}
\mathfrak{D}\overline{\mathfrak{D}}\mathcal{K}_1 \;&=-\mathfrak{D}\left(\frac{1}{2V}\int_X \omega^2\overline{\mathfrak{D}}\omega\right) \\
&=\uparrow\frac{1}{V}\int_X \mathfrak{D}\omega\star\overline{\mathfrak{D}}\omega - \frac{\mathrm{i}}{2V}\int_X \omega^2\partial\overline{\partial}\boldsymbol{\omega} \\
&=\uparrow\frac{1}{V}\int_X \mathfrak{D}\omega\star\overline{\mathfrak{D}}\omega - \frac{\alpha'}{16V}\int_X \omega^2\left(\mathrm{Tr}\,\mathbb{F}^2 - \mathrm{Tr}\,\mathbb{R}^2\right) \\
&=\;\left(\frac{1}{V}\int_X \mathfrak{D}_\alpha\omega\star\mathfrak{D}_{\overline{\beta}}\omega + \frac{\mathrm{i}\alpha'}{8V}\int_X \omega^2\mathrm{Tr}\left(\mathfrak{D}_\alpha\mathcal{A}\mathfrak{D}_{\overline{\beta}}\mathcal{A}^\dagger - \mathfrak{D}_\alpha\theta\mathfrak{D}_{\overline{\beta}}\theta^\dagger\right)\right)\mathrm{d}y^\alpha\mathrm{d}y^{\overline{\beta}}\;,
\end{aligned}
\tag{4.3}
$$

where we have used (4.1) and (4.2).

While for the derivatives of $\mathcal{K}_2$ we have

$$
\begin{aligned}
\mathrm{i}\mathfrak{D}\overline{\mathfrak{D}}\mathcal{K}_2 \;&=\; -\mathrm{i}\mathfrak{D}\left(\frac{\int_X \Omega\,\overline{\mathfrak{D}\Omega}}{\int_X \Omega\,\overline{\Omega}}\right) \\
&=\uparrow\mathrm{i}\left(\frac{\int_X \mathfrak{D}\Omega\,\overline{\Omega}\;\int_X \Omega\,\overline{\mathfrak{D}\Omega}}{(\int_X \Omega\,\overline{\Omega})^2} - \frac{\int_X \mathfrak{D}\Omega\,\overline{\mathfrak{D}\Omega}}{\int_X \Omega\,\overline{\Omega}}\right) \\
&=\; -\mathrm{i}\,\frac{\int_X \chi_\alpha\,\overline{\chi}_{\overline{\beta}}}{\int_X \Omega\,\overline{\Omega}}\;\mathrm{d}y^\alpha\mathrm{d}y^{\overline{\beta}}\;,
\end{aligned}
\tag{4.4}
$$

where we use $\overline{\mathbb{\partial}}\Omega = \overline{\partial}\Omega + \overline{\mathfrak{D}}\Omega = 0$ and, in the second line, several terms vanish owing to considerations of holomorphic type.

Finally, combining (4.3) and (4.4), we obtain the desired result

$$
\mathrm{i}\,\mathfrak{D}\overline{\mathfrak{D}}K \;=\; \omega^\sharp\;.
$$

4.1 The $\mathbb{\Gamma}$ Symbols for the Levi–Civita connection

These are the $\mathbb{\Gamma}$ symbols for the Levi–Civita connection. We first invert the relation and decompose the indices, giving

$$
\begin{aligned}
\mathbb{\Gamma}^m{}_n &= -c_a{}^m\, \mathbb{\Theta}^b{}_n + \mathbb{\Theta}^m{}_n \ , \\
\mathbb{\Gamma}^a{}_n &= \mathbb{\Theta}^a{}_n \ , \\
\mathbb{\Gamma}^m{}_b &= -c_a{}^m\, \mathbb{\Theta}^a{}_b - c_a{}^m\, \mathbb{\Theta}^a{}_n\, c_b{}^n + \mathbb{\Theta}^m{}_b + \mathbb{\Theta}^m{}_n\, c_b{}^n + \mathbb{d}\, c_b{}^m \ , \\
\mathbb{\Gamma}^a{}_b &= \mathbb{\Theta}^a{}_b + \mathbb{\Theta}^a{}_n\, c_b{}^n \ .
\end{aligned}
\tag{4.5}
$$

Using the symbols for $\mathbb{\Theta}^{\rm LC}$, we have

$$
\begin{aligned}
\mathbb{\Gamma}^n{}_k &= \mathrm{d}x^m\Big(\Gamma^{\rm LC}{}_m{}^n{}_k + \frac12\, c_b{}^n\, g^{\sharp bd}\, \mathfrak{D}_d g_{mk}\Big) + \\
&\qquad + \mathrm{d}y^a\Big(c_a{}^m\, \Gamma^{\rm LC}{}_m{}^n{}_k + c_a{}^m\, c_b{}^n\, g^{\sharp bd}\, \mathfrak{D}_d g_{mk} + \partial_k c_a{}^n + \frac12\, g^{nl}\, \mathfrak{D}_a g_{lk} + c_b{}^n\, g^{\sharp bd}\, S_{ad}{}^l\, g_{lk}\Big) \ , \\
\mathbb{\Gamma}^b{}_k &= -\frac12\, \mathrm{d}x^m\, g^{\sharp bd}\, \mathfrak{D}_d g_{mk} - \frac12 \mathrm{d}y^a\Big(c_a{}^m\, g^{\sharp bd}\, \mathfrak{D}_d g_{mk} + g^{\sharp bd}\, S_{ad}{}^l\, g_{lk}\Big) \ , \\
\mathbb{\Gamma}^n{}_c &= \mathrm{d}x^m\Big(\Gamma^{\rm LC}{}_m{}^n{}_k\, c_c{}^k + \partial_m c_c{}^k + \frac12\, c_b{}^n\, g^{\sharp bd}\, \mathfrak{D}_d g_{mk}\, c_c{}^k + \frac12\, g^{nl}\, \mathfrak{D}_c g_{lm} + \frac12\, c_b{}^n\, g^{\sharp bd}\, S_{cd}{}^l\, g_{lm}\Big) \\
&\qquad + \mathrm{d}y^a\Big(\partial_a c_c{}^n + (\partial_k c_a{}^n) c_c{}^k + c_a{}^m\, c_b{}^n\, c_c{}^k \Gamma^{\rm LC}{}_m{}^n{}_k - c_b{}^n\, \Gamma^{\sharp{\rm LC}}{}_a{}^b{}_c \\
&\qquad\qquad + \frac12\, c_a{}^m\, c_b{}^n\, g^{\sharp bd}\, \mathfrak{D}_d g_{mk}\, c_c{}^k + \frac12\, c_a{}^m\, g^{nl}\, \mathfrak{D}_c g_{lm} + \frac12\, c_c{}^k\, g^{nl}\, \mathfrak{D}_a g_{lk} \\
&\qquad\qquad + \frac12\, S_{ac}{}^n + \frac12\, S_{cd}{}^l\, c_a{}^m\, c_b{}^n\, g^{\sharp bd}\, g_{lm} + \frac12\, S_{ad}{}^l\, c_b{}^n\, c_c{}^k\, g^{\sharp bd}\, g_{lk}\Big) \ , \\
\mathbb{\Gamma}^b{}_c &= -\frac12\, \mathrm{d}x^m\Big(c_c{}^k\, g^{\sharp bd}\, \mathfrak{D}_d\, g_{mk} - S_{dc}{}^l\, g^{\sharp bd}\, g_{lm}\Big) \\
&\qquad + \mathrm{d}y^a\Big(\Gamma^{\sharp{\rm LC}}{}_a{}^b{}_c - \frac12\, c_a{}^m\, c_c{}^k\, g^{\sharp bd}\, \mathfrak{D}_d g_{mk} + \frac12\, S_{dc}{}^l\, c_a{}^m\, g^{\sharp bd}\, g_{lm} - \frac12\, S_{ad}{}^l\, g^{\sharp bd}\, g_{lk}\, c_c{}^k\Big) \ .
\end{aligned}
\tag{4.6}
$$

4.2 The Nijenhuis tensor for $\mathbb{X}$

The Nijenhuis tensor for $\mathbb{J}$ is

$$
N_{\mathbb{J}} = (\mathbb{J}^P \partial_P \mathbb{J}^Q - \mathbb{J}_P{}^Q \mathbb{d} \mathbb{J}^P)\partial_Q \ , \tag{4.7}
$$

where $u^P = (y^a, x^m)$ denotes a point in $\mathbb{X}$ and we write $\mathbb{J}^P = \mathbb{J}_S{}^P \mathrm{d}u^S$. The complex structure is triangular in the coordinate basis:

$$
\mathbb{J} = J_m{}^n e^m \otimes e_n + J^\sharp{}_a{}^b e^a \otimes e_b = J_m{}^n\, \mathrm{d}x^m \otimes \partial_n + (c_a{}^m J_m{}^n - J^\sharp{}_a{}^b c_b{}^n)\, \mathrm{d}y^a \otimes \partial_n + J^\sharp{}_a{}^b\, \mathrm{d}y^a \otimes \partial_b \ .
$$

Thus,

$$
\mathbb{J}_m{}^a = 0, \qquad \mathbb{J}_a{}^m = J_n{}^m c_a{}^m - J^\sharp{}_a{}^b c_b{}^m \ .
$$

The terms in (4.7) decompose according to tangibility. In the following, we suppress the $\otimes$ in writing out the tensor structure of $N_{\mathbb{J}}$ to simplify notation, so for example $N_{\mathbb{J}} = \frac12 N_{\mathbb{J}\,PQ}{}^R \mathrm{d}u^P \mathrm{d}u^Q \partial_R$.

1. The first term, proportional to $\mathrm{d}x^m\mathrm{d}x^n$, reduces to that on X

$$\frac{1}{2}N_{\mathbb{J}\,mn}{}^{Q}\mathrm{d}x^m\mathrm{d}x^n\partial_Q \;=\; N_J \;. \tag{4.8}$$

2. The next term has mixed tangibility $\mathrm{d}y^a\mathrm{d}x^m$

$$\begin{aligned} N_{\mathbb{J}\,am}{}^{Q}\mathrm{d}y^a\mathrm{d}x^m\partial_Q \;&=\; N_{J\,mn}{}^{q}c_a{}^{m}\,\mathrm{d}y^a\mathrm{d}x^n\,\partial_q + (J^{\sharp}{}_a{}^{b}\delta_p{}^{q} - \delta_a{}^{b}J_p{}^{q})\,e_b(J_m{}^{p})\,\partial_q + \\ &+ \Big(J^{\sharp}{}_a{}^{b}J_p{}^{q}\delta_m{}^{n} - J^{\sharp}{}_b{}^{a}J_m{}^{n}\delta_p{}^{q}\Big)[e_n,e_b]^p\,\mathrm{d}y^a\mathrm{d}x^m\partial_q \;. \end{aligned}$$

where

$$e_b(J_m{}^{p}) \;=\; \partial_b J_m{}^{p} - c_b{}^{n}\partial_n J_m{}^{p} \quad \text{and} \quad [e_n,e_b] \;=\; -(\partial_n c_b{}^{q})\,\partial_q \;.$$

We use the projectors to rewrite the $N_{\mathbb{J}\,am}{}^{Q}$ components

$$\begin{aligned} N_{\mathbb{J}\,am}{}^{Q}\mathrm{d}y^a\mathrm{d}x^m\partial_Q \;&=\; N_{J\,mn}{}^{q}c_a{}^{m}\,\mathrm{d}y^a\mathrm{d}x^n\,\partial_q + 2\mathrm{i}\,(P_a{}^{c}Q_p{}^{q} - Q_a{}^{c}P_p{}^{q})e_c(J_m{}^{p})\,\partial_q + \\ &+ 4\Big(P_a{}^{c}P_m{}^{n}Q_p{}^{q} + Q_a{}^{c}Q_m{}^{n}P_p{}^{q}\Big)[e_n,e_c]^p\,\mathrm{d}y^a\mathrm{d}x^m\partial_q \;. \end{aligned} \tag{4.9}$$

3. The final term of (4.7) has tangibility $\mathrm{d}y^a\mathrm{d}y^b$

$$\begin{aligned} \frac{1}{2}N_{\mathbb{J}\,ab}{}^{Q}\mathrm{d}y^a\mathrm{d}y^b\partial_Q \;&=\; \frac{1}{2}N_{J^\sharp\,ab}{}^{d}\mathrm{d}y^a\mathrm{d}y^b\,e_d + \frac{1}{2}N_{J\,mn}{}^{q}c_a{}^{m}c_b{}^{n}\,\mathrm{d}y^a\mathrm{d}y^b\,e_q + \\ &\Big(J^{\sharp}{}_a{}^{c}\,e_c(J_p{}^{q})c_b{}^{p} - J_m{}^{q}\,e_a(J_p{}^{m})c_b{}^{p}\Big)\mathrm{d}y^a\mathrm{d}y^b\partial_q + \\ &\Big(\delta_a{}^{c}\delta_b{}^{d}\delta_p{}^{q} + J^{\sharp}{}_a{}^{c}J_p{}^{q}\delta_b{}^{d} - J^{\sharp}{}_a{}^{c}J^{\sharp}{}_b{}^{d}\delta_p{}^{q} + J^{\sharp}{}_b{}^{d}J_p{}^{q}\delta_a{}^{c}\Big)(\partial_c c_d{}^{p})\mathrm{d}y^a\mathrm{d}y^b\partial_q + \\ &\Big(\delta_a{}^{c}\delta_b{}^{d}J_n{}^{m}J_p{}^{q} - \delta_a{}^{c}J^{\sharp}{}_b{}^{d}J_n{}^{m}\delta_p{}^{q} - J^{\sharp}{}_a{}^{c}\delta_b{}^{d}\delta_n{}^{m}J_p{}^{q} + J^{\sharp}{}_a{}^{c}J^{\sharp}{}_b{}^{d}\delta_n{}^{m}\delta_p{}^{q}\Big)c_c{}^{n}(\partial_m c_d{}^{p})\,\mathrm{d}y^a\mathrm{d}y^b\,\partial_q \;. \end{aligned}$$

In terms of projectors

$$\begin{aligned} \frac{1}{2}N_{\mathbb{J}\,ab}{}^{Q}\mathrm{d}y^a\mathrm{d}y^b\partial_Q \;&=\; \frac{1}{2}N_{J^\sharp\,ab}{}^{d}\mathrm{d}y^a\mathrm{d}y^b\,e_d + \frac{1}{2}N_{J\,mn}{}^{q}c_a{}^{m}c_b{}^{n}\,\mathrm{d}y^a\mathrm{d}y^b\,e_q + \\ &2\mathrm{i}\Big(P_a{}^{c}Q_p{}^{q} - Q_a{}^{c}P_p{}^{q}\Big)\,e_c(J_m{}^{p})c_b{}^{m}\mathrm{d}y^a\mathrm{d}y^b\partial_q \;-2\Big(P^cP^dQ_p + Q^cQ^dP_p\Big)[e_c,e_d]^p - \\ &4\Big(P^cP^dQ_pP_n{}^{m} + Q^cQ^dP_pQ_n{}^{m} + P^cQ^dQ_n{}^{m}P_p + Q^cP^dP_n{}^{m}Q_p\Big)c_c{}^{n}[e_m,e_d]^p \;. \end{aligned} \tag{4.10}$$

Gathering (4.8), (4.9) and (4.10), and simplifying we find

$$\begin{aligned} N_{\mathbb{J}} \;&=\; \frac{1}{2}N_{J\,mn}{}^{q}e^me^ne_q + 2\mathrm{i}\,e_c\,(J_m{}^{p})\Big(P^{\sharp c}e^mQ_p - Q^{\sharp c}e^mP_p\Big) - \\ &4[e_c,e_n]^p\Big(P^{\sharp c}P^nQ_p + Q^{\sharp c}Q^nP_p\Big) - 2[e_c,e_d]^p\Big(P^{\sharp c}P^{\sharp d}Q_p + Q^{\sharp c}Q^{\sharp d}P_p\Big) + \frac{1}{2}N_{J^\sharp\,ab}{}^{c}e^ae^be_c \;. \end{aligned} \tag{4.11}$$

The second and third terms combine in virtue of the relation

$$[P^{\sharp}_a, P_m]^p\mathrm{d}y^a e^m \;=\; -\frac{\mathrm{i}}{2}P^{\sharp c}\,e_c(J_m{}^{p})e^m + P^{\sharp c}P^m[e_c,e_m]^p \;.$$

Note also that

$$[P^\sharp_c, P^\sharp_d]^q \;=\; P^\sharp{}_c{}^a P^\sharp{}_d{}^b [e_a, e_b]^q \;.$$

These relations, together with (4.11), give the final expression

$$\begin{aligned} N_{\mathbb{J}} \;=\; & \frac{1}{2} N_{J\;mn}{}^q\, e^m\, e^n\, e_q - 4[P^\sharp_a, P_m]^q\, e^a\, e^m\, Q_q - 4[Q^\sharp_a, Q_m]^q\, e^a\, e^m\, P_q \\ & - 2[P^\sharp_c, P^\sharp_d]^q\, e^c\, e^d\, Q_q - 2[Q^\sharp_c, Q^\sharp_d]^q\, e^c\, e^d\, P_q + \frac{1}{2} N_{J^\sharp\;cd}{}^e\, e^c\, e^d\, e_e \;. \end{aligned}$$

The first and last term are N_J and $N_{J^\sharp}$.

4.3 The holomorphic form Ω

We define the holomorphic three form on X to have an extension which is purely vertical

$$\Omega \;=\; \frac{1}{3!}\,\Omega_{\mu\nu\rho}\, e^\mu e^\nu e^\rho \;, \tag{4.12}$$

where $\mathrm{e}^{\mu\nu\rho}$ is the constant antisymmetric symbol and the function f depends holomorphically on the coordinates. As $\star\Omega = -\mathrm{i}\Omega$, it follows Ω that is d-harmonic. Supersymmetry implies that it is covariantly constant with respect to the Bismut connection $\nabla^{\mathrm{B}}\Omega = 0$. Decomposing according to holomorphic type yields two relations

$$\begin{aligned} \nabla^{\mathrm{B}}_\mu \Omega \;&=\; \Big(\partial_\mu \log \|\Omega\|^2 - H_{\mu\nu}{}^\nu\Big)\,\Omega \;=\; 0 \;, \\ \nabla^{\mathrm{B}}_{\overline{\mu}} \Omega \;&=\; -g^{\nu\overline{\lambda}}(\partial_{\overline{\mu}} g_{\overline{\lambda}\nu} - \partial_{\overline{\lambda}} g_{\overline{\mu}\nu})\,\Omega \;=\; H_{\overline{\mu}\overline{\lambda}}{}^{\overline{\lambda}}\;\Omega \;=\; 0 \;, \end{aligned}$$

which are solved by

$$H_{\mu\nu}{}^\nu \;=\; 0 \;, \qquad \partial_\mu \log \|\Omega\|^2 \;=\; 0 \;.$$

The three–form Ω is a section of a line bundle over the moduli space M with a $\mathbb{C}^*$–gauge symmetry

$$\Omega \to \lambda(y)\,\Omega \;, \qquad \lambda \in \mathbb{C}^* \;. \tag{4.13}$$

Now consider variations of Ω, which need to be covariant under (4.13). The derivative $\eth^\sharp_\alpha \Omega$ is decomposed into holomorphic type on X:

$$\eth^\sharp_\alpha \Omega = (\text{Đ}^\sharp_\alpha \Omega)^{(3,0)} + (\text{Đ}^\sharp_\alpha \Omega)^{(2,1)} \;,$$

where the superscripts refer to holomorphic type with respect to J. Using $\{\text{Đ}, \text{Đ}^\sharp\} = 0$, applying $\text{Đ}^\sharp$ to $\text{Đ}\Omega = 0$ and decomposing according to

$$\overline{\eth}(\text{Đ}^\sharp_\alpha \Omega)^{(2,1)} \;=\; 0 \quad \text{and} \quad \overline{\eth}(\text{Đ}^\sharp_\alpha \Omega)^{(3,0)} + \eth(\text{Đ}^\sharp_\alpha \Omega)^{(2,1)} \;=\; 0 \;. \tag{4.14}$$

The first equation defines a $\overline{\eth}$-closed form $\chi_\alpha \;=\; \Delta_\alpha{}^\mu \Omega_\mu$. For the second equation the Hodge decomposition of $(\text{Đ}^\sharp_\alpha \Omega)^{(3,0)}$ with respect to the $\eth$-operator gives the sum of a harmonic form and a $\eth$-exact term. As $h^{3,0} = 1$, the harmonic term is Ω multiplied by a parameter dependant coefficient $\mathcal{K}_{2\,\alpha}$

$$(\text{Đ}^\sharp_\alpha \Omega)^{(3,0)} \;=\; -\mathcal{K}_{2\,\alpha}\,\Omega + \eth\zeta_\alpha \;. \tag{4.15}$$

Multiplying this equation by $\overline{\Omega}$ and integrating over X, we see the coefficient $\mathcal{K}_{2\,\alpha}$ can be written as a derivative

$$\mathcal{K}_{2\,\alpha} = \partial_\alpha \mathcal{K}_2 \ ; \qquad \mathcal{K}_2 \ = -\log\left(\int_X \mathrm{i}\,\Omega\,\overline{\Omega}\right) \ .$$

Under small diffeomorphisms, there is a transformation law for $\delta\Omega = \delta y^\alpha \text{Đ}^\sharp_\alpha \Omega$

$$\delta\Omega^{(3,0)} \to \delta\Omega^{(3,0)} - \partial(\mathrm{e}^\mu\,\Omega_\mu) \quad \text{and} \quad \delta\Omega^{(2,1)} \to \delta\Omega^{(2,1)} - \overline{\partial}(\mathrm{e}^\mu\,\Omega_\mu) \ .$$

Comparing this equation with (4.15) we see that we can solve $\mathrm{e}^\mu\,\Omega_\mu = \delta y^\alpha \zeta_\alpha$ with explicit solution given by

$$\varepsilon^\nu \ = \ \frac{1}{2\,\|\Omega\|^2}\,\overline{\Omega}^{\nu\rho\sigma}(\delta y^\alpha \zeta_{\rho\sigma} + (\partial\xi^{(1,0)})_{\rho\sigma}) \ ,$$

where $\xi^{(1,0)}$ is an arbitrary one form. With this choice $\delta\Omega^{(3,0)}$ is harmonic. We see that $\xi^{(1,0)}$ is a residual gauge freedom that does not affect $\delta\Omega$. This choice coincides with $\delta\omega = \delta y^\alpha \text{Đ}^\sharp_\alpha \omega^{(1,1)}$ being also harmonic. Firstly, because ω is closed it follows that $\delta\omega^{(1,1)}$ is $\eth$-closed. Secondly, it is co-closed for the following reason. We write $\text{Đ}^\sharp_\alpha \Omega^{(3,0)}$ in two different ways. The first is in (4.15) with $\zeta_\alpha = 0$. The second is to use that $\Omega_{\mu\nu\rho} = f\varepsilon_{\mu\nu\rho}$ with f a holomorphic function of parameters to give

$$\text{Đ}^\sharp_\alpha \Omega^{(3,0)} \ = \ \Big(\partial_\alpha \log\|\Omega\|^2 + \omega^{\mu\overline{\nu}}\,\text{Đ}^\sharp_\alpha \omega_{\mu\overline{\nu}}\Big)\,\Omega \ . \tag{4.16}$$

We also used $|f|^2 = \sqrt{g}\,\|\Omega\|^2$ and $\text{Đ}^\sharp_\alpha \sqrt{g} = \omega^{\mu\overline{\nu}} \text{Đ}^\sharp_\alpha \omega_{\mu\overline{\nu}}$. Furthermore, $\overline{\eth}\delta\omega = 0$, so the $\text{Đ}^\sharp_\alpha \omega^{1,1}$ form a basis for $H^2(X,\mathbb{R})$. Comparing with (4.15) and using that $\mathcal{K}_{2\alpha}$ and $\|\Omega\|^2$ depend only on parameters, it follows that $\omega^{\mu\overline{\nu}} \text{Đ}^\sharp_\alpha \omega_{\mu\overline{\nu}}$ can also only depend on parameters. The Hodge dual relation is

$$\star\,\text{Đ}^\sharp_\alpha \omega^{(1,1)} \ = \ \frac{1}{2}\left(\omega^{\mu\overline{\nu}} \text{Đ}^\sharp_\alpha \omega_{\mu\overline{\nu}}\right)\,\omega^2 - (\text{Đ}^\sharp_\alpha \omega)\,\omega \ .$$

From this, it follows that $\text{Đ}^\sharp_\alpha \omega^{(1,1)}$ is $\overline{\eth}$-coclosed and so it is harmonic.

Returning to the second equation in (4.14), we see that it implies χ_α is $\eth$-closed.

The derivative of Ω that is covariant both with respect the symmetry (4.13) and diffeomorphisms is

$$\mathfrak{D}_\alpha \Omega \ = \ \left(\text{Đ}^\sharp_\alpha + \mathcal{K}_{2\,\alpha}\right)\Omega \ = \ \chi_\alpha = \Delta_\alpha{}^\mu \Omega_\mu \ .$$

5 Integrability of the Supersymmetry Equations

In this section we provide a general analysis of the relation between supersymmetry and equations of motion in terms of the integrability conditions of the supersymmetry variations. Strictly speaking, gaugino condensates only make sense in the lower dimensional effective actions obtained from compactification. However, we perform the analysis of integrability in the full ten-dimensional theory, treating the condensate as a formal object. The aim is to derive the most general set of constraints, which can then be applied to specific compactifications.

In absence of a condensate, it is possible to build combinations of squares of the supersymmetry variations that reproduce the equations of motion

$$\Gamma^M D^0_{[N} D^0_{M]}\epsilon - \frac{1}{2} D^0_N(\mathcal{P}^0\epsilon) + \frac{1}{2}\mathcal{P}^0 D^0_N\epsilon = -\frac{1}{4}\mathcal{E}^0_{NP}\Gamma^P\epsilon + \frac{1}{8}\mathcal{B}^0_{NP}\Gamma^P\epsilon + \frac{1}{2}\iota_N\delta H^0\epsilon\,,$$
$$\not{D}^0\not{D}^0\epsilon - (D^{0\,M} - 2\partial^M\phi)D^0_M\epsilon = -\frac{1}{8}\mathcal{D}^0\epsilon + \frac{1}{4}\delta H^0\epsilon\,, \tag{5.1}$$

where $\mathcal{E}^0_{NP}$, $\mathcal{B}^0_{NP}$ and $\mathcal{D}^0$ denote the Einstein, B-field and dilaton equations of motion. δH is the Bianchi identity. Since the left-hand side of both equations in (5.1) vanishes on supersymmetric solutions, the equations of motion are also satisfied, provided the Bianchi identity holds.

When the condensate is included, the first equation in (5.1) becomes

$$\begin{aligned}\Gamma^M D_{[N}D_{M]}\epsilon - \frac{1}{2}D_N(\mathcal{P}\epsilon) + \frac{1}{2}\mathcal{P}D_N\epsilon = &-\frac{1}{4}\mathcal{E}_{NP}\Gamma^P\epsilon + \frac{1}{8}\mathcal{B}_{NP}\Gamma^P\epsilon \\ &+ \frac{1}{2}\iota_N\delta H\epsilon - \frac{\alpha}{32}A_N(\Sigma)\epsilon\,,\end{aligned} \tag{5.2}$$

where now $\mathcal{E}_{NM}$, $\mathcal{B}_{NM}$, $\mathcal{D}$ are the Einstein, B-field and dilaton equations of motion with non-zero condensate, and the extra term $A_N(\Sigma)$ is given by

$$\begin{aligned}A_N(\Sigma)\epsilon &= A_{NP}\Gamma^P\epsilon + A_{NPQR}\Gamma^{PQR}\epsilon + A_{NPQRST}\Gamma^{PQRST}\epsilon \\ &= \big[e^{2\phi}\nabla^M(e^{-2\phi}\Sigma_{MNP}) + \frac{1}{2}\Sigma_{NRS}H^{RS}{}_P + H\Sigma\delta_{NP}\big]\Gamma^P\epsilon \\ &\quad + \frac{1}{2}\big[e^{2\phi}\nabla^M(e^{-2\phi}\Sigma_{MPQ})\delta_{NR} + \frac{1}{3}e^{-2\phi}\nabla_N(e^{2\phi}\Sigma_{PQR}) \\ &\qquad + \nabla_R\Sigma_{NPQ} - H_{PST}\Sigma_Q{}^{ST}\delta_{NR}\big]\Gamma^{PQR}\epsilon \\ &\quad - \frac{1}{6}\big[\nabla_S\Sigma_{PQR}\delta_{NT} + \frac{1}{2}H_{NPQ}\Sigma_{RST}\big]\Gamma^{PQRST}\epsilon\,.\end{aligned} \tag{5.3}$$

Note that the tensors $\{A_{NM}, A_{NPQR}, A_{NPQRST}\}$ do not have any symmetry property. The analogue of the second equation in (5.1) contains extra terms in Σ that cancel non-tensorial terms

$$\not{D}\not{D}\epsilon - (D^{0\,M} - 2\partial^M\phi - \frac{\alpha}{16}\not{\Sigma}\Gamma^M - \frac{\alpha}{4}\Sigma^M)D_M\epsilon = -\frac{1}{8}\mathcal{D}\epsilon + \frac{1}{4}\delta H\epsilon - \frac{\alpha}{32}B(\Sigma)\epsilon\,, \tag{5.4}$$

where the extra contribution from Σ is

$$\begin{aligned}B(\Sigma)\epsilon &= B\epsilon + B_{NP}\Gamma^{NP}\epsilon + B_{MNPQ}\Gamma^{MNPQ}\epsilon \\ &= 6H\lrcorner\Sigma\epsilon + 3\big[e^{2\phi}\nabla^M(e^{-2\phi}\Sigma_{MNP}) + H_{NQR}\Sigma_P{}^{QR}\big]\Gamma^{NP}\epsilon \\ &\quad + \frac{1}{3}\big[e^{-2\phi}\nabla_M\left(e^{2\phi}\Sigma_{NPQ}\right) - \frac{3}{2}H_{MNS}\Sigma_{PQ}{}^S\big]\Gamma^{MNPQ}\epsilon\,.\end{aligned} \tag{5.5}$$

The left-hand sides of (5.2) and (5.4) still vanish because of the supersymmetry variations, but now the analysis of the right-hand sides is more involved. For zero condensate, after imposing the Bianchi identity, the only terms left are $\mathcal{E}^0_{NM}$ and $\mathcal{B}^0_{NM}$ multiplying one gamma matrix, and they must vanish separately because of their symmetry properties. In presence of condensate, the extra terms $A(\Sigma)$ and $B(\Sigma)$ in (5.2) and (5.4) contain several terms involving different numbers of gamma matrices, which, in ten dimensions, are not independent and hence cannot be set to zero separately.

We describe an alternative approach to study the relation between supersymmetry variations and equations of motion that generalisese Lichnerowicz theorem. Let us consider first the case when the condensate is zero. The starting point is the Bismut-Lichnerowicz identity. Provided, the Bianchi identity is satisfied, the Bismut-Lichnerowicz identity allows to write the bosonic Lagrangian as

$$\frac{1}{4}\mathcal{L}_b\epsilon + (\alpha^2) = D^0_M D^{0\,M}\epsilon - \not{D}^0\not{D}^0\epsilon + \frac{\alpha}{16}\left(\mathrm{Tr}\ \not{F}\not{F}\epsilon - \mathrm{Tr}\ \not{R}^-\not{R}^-\epsilon\right) - 2\nabla^M\phi D^0_M\epsilon\,, \tag{5.6}$$

where D^0_M and $\not{D}^0$ are the gravitino and modified dilatino equations, with zero condensate. R^- is the curvature two-form derived from the torsionful connection ∇^- (note that the gravitino variation involves ∇^+). Multiplying (5.6) by $e^{-2\phi}\epsilon^\dagger$ and integrating it, gives the action

$$\begin{aligned} S_b = -4\int_{M_{10}}\sqrt{-g}e^{-2\phi}\Big[&\epsilon^\dagger\not{D}^0\not{D}^0\epsilon - \epsilon^\dagger D^0_M D^{0\,M}\epsilon \\ &+ 2\nabla^M\phi\,\epsilon^\dagger D^0_M\epsilon - \frac{\alpha}{16}\left(\mathrm{Tr}\ \epsilon^\dagger\not{F}\not{F}\epsilon - \mathrm{Tr}\ \epsilon^\dagger\not{R}^-\not{R}^-\epsilon\right)\Big] + (\alpha^2)\,, \end{aligned} \tag{5.7}$$

where we assumed that $\epsilon^\dagger\epsilon = 1$.

We would like to write the action as a BPS-squared expression, since its variations will give the equations of motion in terms of the supersymmetry variations. If the theory were Euclidean, we could integrate (5.7) by parts, and end up with such an action. Unfortunately, when the metric has Lorentzian signature it is not possible, in general, to reconstruct the supersymmetry operators D^0_M and $\not{D}^0$ after the integration by parts. Since the problematic terms always involve components of the fields with one leg along the time direction, one can restrict to solutions where none of the fields has components of this type. This means

$$H_{0MN} = \Sigma_{0MN} = F_{0M} = 0\,, \tag{5.8}$$

and, for the metric

$$\delta s^2_{10} = -e^{2A}\delta t^2 + g_{mp}\delta x^m\delta x^p\,, \tag{5.9}$$

where $A = A(x^m)$, $g = g(x^m)$ and $\{x^m, x^n, ..\}$ denote spatial coordinates. Note also that, in order to perform the integration by parts, we assume that the fields vanish at infinity where the metric is flat. Under these assumptions the action can be integrated by parts to give

$$\begin{aligned} \int_{M_{10}} e^{-2\phi}\mathcal{L}_b =& 4\int_{M_{10}} e^{-2\phi}[(\not{D}^0\epsilon)^\dagger\not{D}^0\epsilon - (D^0_M\epsilon)^\dagger D^{0\,M}\epsilon \\ &+ \frac{\alpha}{16}\left(\mathrm{Tr}\ \epsilon^\dagger\not{F}\not{F}\epsilon - \mathrm{Tr}\ \epsilon^\dagger\not{R}^-\not{R}^-\epsilon\right)] + (\alpha^2)\,. \end{aligned} \tag{5.10}$$

The variations of (5.10) with respect to the metric, dilaton and B-field give the corresponding equations of motion written in terms of the supersymmetry conditions. Thus solutions of the supersymmetry constraints also automatically solve the equations of motion. The variation of the second line in (5.10) also vanishes. Its variation is proportional to

$$\not{R}^-_{MN}\,\epsilon = R^-_{PQMN}\Gamma^{PQ}\epsilon = R^+_{MNPQ}\Gamma^{PQ}\epsilon + (\alpha) = (\alpha)\,,$$

which vanishes on supersymmetric solutions to the appropriate order in α.

For non-trivial condensate the same analysis gives, up to (α^2) terms,

$$S_b = BPS^2 + \frac{\alpha}{8}\int_{M_{10}} \sqrt{-g}e^{-2\phi}\Big[(\not{D}\epsilon)^\dagger \not{\Sigma}\epsilon + (\not{\Sigma}\epsilon)^\dagger \not{D}\epsilon$$
$$+ \frac{1}{2}\left((D_M\epsilon)^\dagger \not{\Sigma}\Gamma^M\epsilon + (\not{\Sigma}\Gamma^M\epsilon)^\dagger D_M\epsilon\right) - H\lrcorner\Sigma\Big]\,, \tag{5.11a}$$

where $H\lrcorner\Sigma = \frac{1}{3!}H^{MNP}\Sigma_{MNP}$, and BPS^2 denotes the part of the action that can be written as the square of the supersymmetry variations

$$BPS^2 = \int_{M_{10}} \sqrt{-g}e^{-2\phi}\Big[(\not{D}\epsilon)^\dagger \not{D}\epsilon - (D_M\epsilon)^\dagger D^M\epsilon + \frac{\alpha}{16}\left(\mathrm{Tr}\ \epsilon^\dagger \not{F}\not{F}\epsilon - \mathrm{Tr}\ \epsilon^\dagger \not{R}^- \not{R}^-\epsilon\right)\Big]\,. \tag{5.11b}$$

Note that because of the extra terms involving Σ on the right-hand side of (5.11a), the action can no longer be written as a BPS-squared expression. The equations of motion for g_{MN}, ϕ and B_{MN} obtained by varying (5.11a) will contain a term coming from its BPS part, which vanishes for supersymmetric configurations, and terms in Σ, which will provide additional constraints to be imposed on the supersymmetric solution. These extra terms have a relatively simple structure and in particular do not contain the curvatures F and R^-. More concretely, the extra terms in the dilaton and B-field equations of motion are respectively

$$H_{MNP}\Sigma^{MNP} + e^{2\phi}\nabla^M\left(e^{-2\phi}\epsilon^\dagger\Gamma_{MNPQ}\Sigma^{NPQ}\epsilon\right) = 0 + (\alpha)\,, \tag{5.12a}$$
$$e^{2\phi}\nabla^M\left(e^{-2\phi}\,\epsilon^\dagger\{\Gamma_{MPQ},\not{\Sigma}\}\epsilon\right) = 0 + (\alpha)\,. \tag{5.12b}$$

Both equations are only linear in derivatives. Note the resemblance between (5.12b) and the usual flux equation of motion. The metric equation of motion evaluated on a supersymmetric solutions gives the equation

$$e^{2\phi}\nabla_N\left[e^{-2\phi}\left(\epsilon^\dagger\Gamma^{N(P}\not{\Sigma}\Gamma^{M)}\epsilon - 2g^{MP}\epsilon^\dagger\Gamma^N\not{\Sigma}\epsilon + 2g^{N(M}\epsilon^\dagger\Gamma^{P)}\not{\Sigma}\epsilon\right)\right] = \nabla^{(M}\phi\,\epsilon^\dagger\Gamma^{P)}\not{\Sigma}\epsilon$$
$$+ 2\,g^{MP}H\lrcorner\Sigma - \ H_{RS}{}^{(M}\Sigma^{P)RS} - 3\,H_{NR}{}^{(M}\epsilon^\dagger\Gamma^{P)N}\not{\Sigma}\Gamma^R\epsilon + (\alpha)\,, \tag{5.13}$$

We also need to vary the vielbeine in the gamma matrices as $\delta\Gamma^M = \delta e_A{}^M\,\Gamma^A$ and in the gaugino bilinear

$$\Sigma_{MNP} = \overline{\chi}\Gamma_{MNP}\chi = \overline{\chi}\Gamma_{ABC}\chi e_M{}^A e_N{}^B e_P{}^C\,. \tag{5.14}$$

The equations above can can be further simplified, using the fact that

$$H = \delta\phi = 0 + (\alpha)\,. \tag{5.15}$$

This follows from imposing that the fields must vanish at infinity. Indeed, substracting 1/4 of the trace we find

$$\frac{1}{2}\nabla^2\phi - (\nabla\phi)^2 + \frac{1}{4}\,H^2 = 0 + (\alpha)\,, \tag{5.16a}$$

or, alternatively,

$$e^{2\phi}\nabla^M\left(e^{-2\phi}\nabla_M\phi\right) + \frac{1}{2}\,H^2 = 0 + (\alpha)\,. \tag{5.16b}$$

Multiplying by $e^{-2\phi}$ and integrating over spacetime then gives

$$\int_{M_{10}} e^{-2\phi}H^2 = 0 + (\alpha)\,. \tag{5.17}$$

As the integrand is positive, it has to vanish point-wise, which implies (5.15). Integrating (5.16a), we also find that $\delta\phi = 0 + (\alpha)$. Using this result, the equations (5.12a) -(5.13) become (modulo (α) terms)

$$\tilde{\Psi}^{MNPQ}\,\nabla_{[M}\Sigma_{NPQ]} = 0\,, \tag{5.18a}$$

$$\nabla^P\Sigma_{PMN} + \tilde{\Psi}_{PQR[M}\nabla^P\Sigma^{QR}{}_{N]} - \frac{1}{2}\tilde{\Psi}_{MNPQ}\nabla^R\Sigma_R{}^{PQ} = 0\,, \tag{5.18b}$$

$$\tilde{\Psi}_{MPQ(R}\nabla^M\Sigma_{N)}{}^{PQ} - \frac{1}{6}\tilde{\Psi}^{TPQ}_{(R}\nabla_{N)}\Sigma_{TPQ} = 0\,, \tag{5.18c}$$

where, for simplicity of notation, we have defined $\tilde{\Psi}_{MNPQ} = \epsilon^\dagger\Gamma_{MNPQ}\epsilon$.

Poincaré invariance in four dimensions forces the three-form flux H and the three-form condensate to have only non-trivial components in the internal space. We are still taking a ten-dimensional expectation value, $\langle\Sigma_{MNP}\rangle$. We should also remember that the ten-dimensional gauginos are in the adjoint of $E_8\times E_8$ or $SO(32)$. Denoting the external four-dimensional gauge group as G, and the internal group as H (G is the stabilizer of H in the ten-dimensional group), we may decompose the ten-dimensional product representation as $\mathbf{496}\otimes\mathbf{496} \to \sum_i (R(G)_i, R(H)_i)$. Of course the details very much depend on the choice of G and H, but in general the ten-dimensional trace over fermion bilinears will break into a sum of many terms:

$$\begin{aligned}
\langle\Sigma\rangle_{mnp} &= \langle \mathrm{tr}\overline{\chi}_{(10)}\Gamma_{mnp}\chi_{(10)}\rangle \\
&= \sum_i \langle \mathrm{tr}_{R(G)_i}\overline{\chi}^{(4)}_+\chi^{(4)}_- \cdot \mathrm{tr}_{R(H)_i}\chi^\dagger_+\gamma_{mnp}\chi_-\rangle - \sum_i \langle \mathrm{tr}_{R(G)_i}\overline{\chi}^{(4)}_-\chi^{(4)}_+ \cdot \mathrm{tr}_{R(H)_i}\chi^\dagger_-\gamma_{mnp}\chi_+\rangle \\
&= -2\sum_i \mathrm{Re}\langle\Lambda_i\Sigma^i_{mnp}\rangle\,.
\end{aligned} \tag{5.19}$$

Here we have defined internal three-forms Σ^i_{mnp} as

$$\Sigma^i_{mnp} = \mathrm{tr}_{R(H)_i}\chi^\dagger_-\gamma_{mnp}\,\chi_+\,, \tag{5.20}$$

and a four-dimensional condensate *vector* Λ_i as

$$\Lambda_i = \mathrm{tr}_{R(G)_i}\overline{\chi}^{(4)}_-\chi^{(4)}_+\,. \tag{5.21}$$

From now on we shall suppress all the traces and the G and H representation indices. To simplify the notation, in the rest of this section we will set $\langle\Sigma\rangle = \hat{\Sigma} = -2\mathrm{Re}(\Lambda\Sigma)$. The ten-dimensional supersymmetry equations can be written as a set of conditions on the forms Ω and J defining the $SU(3)$ structure.

6 The Effective Field Theory of Heterotic Vacua

We are interested in heterotic vacua that realise $\mathcal{N}$=1 supersymmetric field theories in $\mathbb{R}^{3,1}$. At large radius, these take form $\mathbb{R}^{3,1}\times\mathcal{X}$ where $\mathcal{X}$ is a compact smooth complex three-fold with vanishing first Chern class. We study the $E_8{\times}E_8$ heterotic string, and so there is a holomorphic vector bundle $\mathcal{E}$ with a structure group $\mathcal{H}\subset E_8\times E_8$ and a d=4 spacetime gauge symmetry given by the commutant $\mathfrak{G} = [E_8{\times}E_8, \mathcal{H}]$. The bundle $\mathcal{E}$ has a connection A, with field strength F

satisfying the hermitian Yang-Mills equation. The field strength F is related to a gauge-invariant three-form H and the curvature of $\mathcal{X}$ through anomaly cancellation. The triple $(\mathcal{X}, \mathcal{E}, H)$ forms a heterotic structure, and the moduli space of these structures is described by what we call heterotic geometry. In this paper, we compute the contribution of fields charged under the spacetime gauge group $\mathfrak{G}$ to the heterotic geometry.

The challenge in studying heterotic vacua is the complicated relationship between H, the field strength F and the geometry of $\mathcal{X}$. Supersymmetry relates the complex structure J and Hermitian form ω of $\mathcal{X}$ to the gauge invariant three-form H:

$$H \;=\; \mathrm{d}^c\omega \;, \qquad \mathrm{d}^c\omega \;=\; \frac{1}{2} J_{m_1}{}^{n_1} J_{m_2}{}^{n_2} J_{m_3}{}^{n_3} (\partial_{n_1}\omega_{n_2 n_3})\, \mathrm{d}x^{m_1}\mathrm{d}x^{m_2}\mathrm{d}x^{m_3} \;. \tag{6.1}$$

where x^m are real coordinates on $\mathcal{X}$. Green–Schwarz anomaly cancellation gives a modified Bianchi identity for H

$$\mathrm{d}H = -\frac{\alpha^\backprime}{4}\left(\mathrm{Tr}\, F^2 - \mathrm{Tr}\, R^2\right) \;, \tag{6.2}$$

where in the second of these equations R is the curvature two-form computed with respect to a appropriate connection with torsion proportional to H. This means the tangent bundle $\mathcal{T}_\mathcal{X}$ has torsion if H is non-zero. Unless one is considering the standard embedding — in which $\mathcal{E}$ is identified with $\mathcal{T}_\mathcal{X}$ the tangent bundle to $\mathcal{X}$ — the right hand side of (6.2) is non-zero even when $\mathcal{X}$ is a Calabi-Yau manifold at large radius. This means that H is generically non-vanishing, though subleading in $\alpha^\backprime$, and so even for large radius heterotic vacua $\mathcal{X}$ is non-Kähler. Torsion is inescapable.

The effective field theory of the light fields for these vacua are described by a Lagrangian with $\mathcal{N}=1$ supersymmetry, whose bosonic sector is of the form

$$\mathcal{L} \;=\; \frac{1}{2\kappa_4^2}\sqrt{-G_4}\left(\mathcal{R}_4 - \frac{1}{4}\mathrm{Tr}\,|F_\mathfrak{g}|^2 - 2G_{A\overline{B}}\widehat{\mathcal{D}}_e\Phi^A\widehat{\mathcal{D}}_e\Phi^{\overline{B}} - V(\Phi,\overline{\Phi}) + \cdots\right) \;. \tag{6.3}$$

Here κ_4 is the four-dimensional Newton constant, $\mathcal{R}_4$ the four-dimensional Ricci-scalar, $F_\mathfrak{g}$ is the spacetime gauge field strength, the Φ^A range over the scalar fields of the field theory and their kinetic term comes with a metric G_A. The fields Φ^A may be charged under $\mathfrak{g}$, the algebra of the gauge group $\mathfrak{G}$, with an appropriate covariant derivative $\widehat{\mathcal{D}}_e$. Finally $V(\Phi,\overline{\Phi})$ is the bosonic potential for the scalars.

When $\mathcal{E} \cong \mathcal{T}_\mathcal{X}$ the moduli space of the heterotic theory reduces to that of a Calabi-Yau manifold, and is described by special geometry. The unbroken gauge group in spacetime is E_6, and the charged matter content consists of fields charged in the $\mathbf{27}$ and $\overline{\mathbf{27}}$ representations. The Yukawa couplings were calculated in supergravity. The effective field theory of this compactification was described in a beautiful paper, in which relations between the Kähler potential and superpotential were computed using string scattering amplitudes, $(2,2)$ supersymmetry and Ward identities. The Kähler and superpotential were shown to be related to each other and in fact, were both determined in terms of a pair of holomorphic functions. These are known as the special geometry relations. A key question is how these relations generalise to other choices of bundle $\mathcal{E}$.

We work towards answering this question by computing the effective field theory couplings correct to first order in $\alpha^\backprime$. In a previous paper [1] we commenced a study of heterotic geometry using $\alpha^\backprime$-corrected supergravity. This is complementary to a series of papers who identified the parameter space with certain cohomology groups. In the context of effective field theory (6.3), one of the results of [1] was to calculate the contribution of the bosonic moduli fields to the metric G_A. In this paper, we compute the contribution of the matter sector to the metric G_A, and the Yukawa couplings, correct to order $\alpha^{\backprime}$. We describe an ansatz for the superpotential and Kähler potential for effective field theory:

$$\begin{aligned} \mathcal{K} &= -\log\left(\frac{4}{3}\int\omega^3\right) - \log\left(\mathrm{i}\int\Omega\,\overline{\Omega}\right) + G_{\xi\overline{\eta}}\mathrm{Tr}\,C^{\xi}C^{\overline{\eta}} + G_{\rho\overline{\tau}}\mathrm{Tr}\,D^{\tau}D^{\overline{\rho}}, \\ \mathcal{W} &= -\mathrm{i}\sqrt{2}e^{-\mathrm{i}\phi}\int\Omega\Big(H - \mathrm{d}^c\omega\Big)\ . \end{aligned} \tag{6.4}$$

The superpotential is normalised by comparing with the Yukawa couplings computed in the dimensional reduction using the conventions of Wess–Bagger.

The moduli have a metric

$$\begin{aligned} \mathrm{d}s^2 &= 2G_{\alpha\overline{\beta}}\,\mathrm{d}y^{\alpha}\otimes\mathrm{d}y^{\overline{\beta}}\ , \\ G_{\alpha\overline{\beta}} &= \frac{1}{4V}\int\Delta_{\alpha}{}^{\mu}\star\Delta_{\overline{\beta}}{}^{\nu}\ g_{\mu\overline{\nu}} + \frac{1}{4V}\int\mathcal{Z}_{\alpha}\star\mathcal{Z}_{\overline{\beta}} + \\ &\quad + \frac{\alpha^{\backprime}}{4V}\int\mathrm{Tr}\Big(D_{\alpha}A\star D_{\overline{\beta}}A\Big) - \frac{\alpha^{\backprime}}{4V}\int\mathrm{Tr}\Big(D_{\alpha}\Theta\,\star\,D_{\overline{\beta}}\Theta^{\dagger}\Big)\ , \end{aligned} \tag{6.5}$$

where $\mathcal{Z}_{\alpha} = \mathcal{B}_{\alpha} + \mathrm{i}\partial_{\alpha}\omega$ is the $\alpha^{\backprime}$-corrected, gauge invariant generalisation of the complexified Kahler form $\delta B + \mathrm{i}\delta\omega$, the χ_{α} form a basis of closed $(2,1)$-forms, and the the last line is the Kobayashi metric, extended to the entire parameter space, including deformations of the spin connection on $\mathcal{T}_X$. The metric expressed this way is an inner product of tensors corresponding to complex structure Δ_{α}, hermitian moduli $\mathcal{Z}_{\alpha}$, and bundle moduli $D_{\alpha}A$. The role of the spin connection $D_{\alpha}\theta$ is presumably determined in terms of the other moduli as they do not correspond to independent physical fields. The tensors depend on parameters holomorphically through

$$\Delta_{\alpha}{}^{\overline{\nu}} = 0, \qquad \mathcal{Z}_{\overline{\alpha}} = \mathcal{B}_{\overline{\alpha}} + \mathrm{i}\partial_{\overline{\alpha}}\omega = 0\ , \quad \mathcal{D}_{\overline{\alpha}}A^{0,1} = 0\ , \qquad \mathcal{D}_{\overline{\alpha}}\theta^{0,1} = 0\ . \tag{6.6}$$

de la Ossa and Svanes [14] show that there exists a choice of basis for the parameters in which each of the tensors in the metric are in an appropriate cohomology, hence, the moduli space metric (7.2) is the natural inner product (Weil–Peterson) on cohomology classes.

The matter fields are C^{ξ} and D^{τ} and appear in the Kähler potential trivially, as they do in special geometry. The matter metric is the Weil-Petersson inner product of corresponding cohomology elements

$$G_{\tau\overline{\sigma}} = \frac{\alpha^{\backprime}}{4V}\int_X\psi_{\tau}\,\star\,\psi_{\overline{\sigma}}\ , \qquad G_{\xi\overline{\eta}} = \frac{\alpha^{\backprime}}{4V}\int_X\phi_{\xi}\,\star\,\phi_{\overline{\eta}}\ , \tag{6.7}$$

where ϕ_{ξ}, ψ_{ρ} are $(0,1)$-forms valued in a sum over representations of the structure group $\mathcal{H}$.

In some sense it was remarkable that one was able to find a compact closed expression for the Kähler potential for the moduli metric. This was not a priori obvious, especially given the non-linear PDEs relating parameters in the anomolous Bianchi identity and supersymmetry relations (6.1)-(6.2). Indeed, it turned out that the Kähler potential for the moduli in (6.4) is of the same in form as that of special geometry, except where one has replaced the Kähler form by the hermitian form ω. At first sight this is confusing as the only fields appearing in the Kähler potential are ω and Ω. Nonetheless, the Kähler potential still depends on bundle moduli in precisely the right way through a non-trivial analysis of the supersymmetry and anomaly conditions. The hermitian form ω contains, hidden within, information about both the bundle and hermitian moduli.

Expand the fields ϕ and ψ in a harmonic basis for $H^1(\mathcal{X}, \mathcal{E}_{\boldsymbol{r}})$ and $H^1(\mathcal{X}, \mathcal{E}_{\overline{\boldsymbol{r}}})$ respectively:

$$\phi \;=\; \sum_\xi \phi_\xi\, C^\xi \;\in (\boldsymbol{r}, \overline{\boldsymbol{R}})~, \qquad \psi \;=\; \sum_\tau \psi_\tau\, D^\tau \;\in (\overline{\boldsymbol{r}}, \boldsymbol{R})~, \tag{6.8}$$

where $\phi_\xi \in H^1(\mathcal{X}, \mathcal{E}_{\boldsymbol{r}})$ and $\psi_\tau \in H^1(\mathcal{X}, \mathcal{E}_{\boldsymbol{r}})$ are harmonic forms

$$\phi_\xi = \phi_{\xi\,\overline{\mu}}\,\mathrm{d}x^{\overline{\mu}} \in \boldsymbol{r}~, \qquad \psi_\tau = \psi_{\tau\,\overline{\mu}}\,\mathrm{d}x^{\overline{\mu}} \in \overline{\boldsymbol{r}}~. \tag{6.9}$$

while C^ξ and D^τ are valued in $\overline{\boldsymbol{R}}$ and $\boldsymbol{R}$ respectively.

For example, consider the standard embedding. Then, $\mathcal{E}_{\boldsymbol{3}} = \mathcal{T}_{\mathcal{X}}^{1,0}$ and $\phi_\xi \in H^1(\mathcal{X}, \mathcal{T}_{\mathcal{X}}^{1,0})$; $\mathcal{E}_{\overline{\boldsymbol{3}}} = \mathcal{T}_{\mathcal{X}}^{0,1}$ with the $\psi_\tau \in H^1(\mathcal{X}, \mathcal{T}_{\mathcal{X}}^{0,1})$. C^ξ and D^τ are in the $\overline{\boldsymbol{R}} = \overline{\boldsymbol{27}}$ and $\boldsymbol{R} = \boldsymbol{27}$.

We need to satisfy the reality condition $\Phi^\dagger = -\Psi$, which forces $\phi^\dagger = -\psi$ and so in terms of the (ϕ_ξ, ψ_τ) basis:

$$\Phi \;=\; \sum_\xi \phi_\xi\, C^\xi - \sum_{\overline{\tau}} \psi_{\overline{\tau}}\, D^{\overline{\tau}}~, \qquad \Psi \;=\; \sum_\tau \psi_\tau\, D^\tau - \sum \phi\, C~. \tag{6.10}$$

We denote conjugation through the barring of the indices. For example, $\phi_{=}(\phi_\xi)^\dagger$ is a $(1,0)$-form valued in $\overline{\boldsymbol{r}}$ of $\mathfrak{h}$ and $C^{=}(C^\xi)^\dagger$ is in the $\boldsymbol{R}$ of $\mathfrak{g}$.

The spirit of KK reduction is to promote the coefficients to spacetime fields: $Y^\alpha(X)$, $C^\xi(X)$, $D^\tau(X)$, and integrate over the six-dimensional manifold to get an effective four-dimensional theory. With the conventions of [1], the $D = 10$ $\mathfrak{e}_8$ Yang-Mills field contribution to the $D = 4$ effective field theory is:

$$\mathcal{L}_F \;=\; -\,\frac{\alpha^\backprime}{4V}\int_X \mathrm{d}^6x\sqrt{g}\,\mathrm{Tr}\,|\delta F_{\mathfrak{e}_8}|^2~, \quad |F|^2 = \frac{1}{2}F_{MN}F^{MN}~. \tag{6.11}$$

We dimensionally reduce, doing a background field expansion. A small fluctuation of the field strength is given, and so

$$\begin{aligned}\mathrm{Tr}\,|\delta F_{\mathfrak{e}_8}|^2 \;=\;& \mathrm{Tr}\,\left(\mathrm{d}_A(\delta A)\,\star\,\mathrm{d}_A(\delta A)\right) + \mathrm{Tr}\,\left(\mathrm{d}_{A+B}\Phi\,\star\,\mathrm{d}_{A+B}\Psi\right)\\ &+ \mathrm{Tr}\,\left(\mathrm{d}_{A+B}\Psi\,\star\,\mathrm{d}_{A+B}\Phi\right) + \mathrm{Tr}\,\left(\mathrm{d}_B\delta B\,\star\,\mathrm{d}_B\delta B\right)~,\end{aligned} \tag{6.12}$$

The first term involves just the bundle moduli, contributing to the moduli metric considered in [1]. The middle two terms involve the matter fields and the last term gives rise to the kinetic term for the D=4 spacetime gauge field. The terms involving the matter fields are:

$$\begin{aligned}\mathrm{d}_{A+B}\Phi \;&=\; (\partial_e\Phi + \Phi B_e)\,\mathrm{d}X^e + (\partial_M\Phi_N + A_M\Phi_N)\,\mathrm{d}x^M\mathrm{d}x^N\\ &=\; \widehat{\mathcal{D}}_e\Phi\,\mathrm{d}X^e + \mathrm{d}_A\Phi~,\\ \mathrm{d}_{A+B}\Psi \;&=\; \widehat{\mathcal{D}}_e\Psi\,\mathrm{d}X^e + \mathrm{d}_A\Psi~,\end{aligned} \tag{6.13}$$

where $\widehat{\mathcal{D}}_e$ is the spacetime $\mathfrak{g}$-covariant derivative and d_A the $\mathfrak{h}$-covariant derivative. Hence, using $\mathrm{Tr}\,|\delta F|^2 = \frac{1}{2}\mathrm{Tr}\,\delta F_{MN}\delta F^{MN} = 2\mathrm{Tr}\,\delta F_{e\mu}\delta F^{e\mu}$, where $\mathrm{Tr}\,(\delta F_{e\mu}\delta F^{e\mu}) = \mathrm{Tr}\,(\delta F_{e\overline{\nu}}\delta F^{e\overline{\nu}})$, and ignoring the moduli fields for the moment, we find the kinetic terms for the matter fields come from middle two terms in (6.12) and are

$$\mathrm{Tr}\,|\delta F_{\mathfrak{e}_8}|^2 \;=\; -\,2\mathrm{Tr}\,\left(\widehat{\mathcal{D}}_e\Phi_{\overline{\mu}}\,\widehat{\mathcal{D}}^e\Phi^{\dagger\,\overline{\mu}}\right) - 2\mathrm{Tr}\,\left(\widehat{\mathcal{D}}_e\Psi_{\overline{\mu}}\,\widehat{\mathcal{D}}^e\Psi^{\dagger\,\overline{\mu}}\right)\ . \tag{6.14}$$

It is convenient to introduce the indices denoting representations of $\mathfrak{h}\oplus\mathfrak{g}$ with $i,\overline{\jmath} = 1,\cdots,r$ for representations $\boldsymbol{r}$ of $\mathfrak{h}$; $a, = 1,\cdots,R$ for representations $\boldsymbol{R}$ of $\mathfrak{g}$. We have used the reality condition $\Phi^\dagger = -\Psi$. The matter fields have a KK anstaz, given by (6.10), which when substituted into each of the above terms gives

$$\begin{aligned}
\mathrm{d}^6x g^{\frac{1}{2}}\mathrm{Tr}\,\left(\widehat{\mathcal{D}}_e\Phi_{\overline{\mu}}\,\widehat{\mathcal{D}}^e\Phi^{\dagger\,\overline{\mu}}\right) \;&=\; \left(\widehat{\mathcal{D}}_e C^{\xi}\,(X)\phi^{\,i}_{\xi\,\overline{\mu}}(x)\right)\left(\widehat{\mathcal{D}}^e C^{\overline{\eta}\,M}(X)\phi^{\overline{\mu}\overline{\jmath}}_{\overline{\eta}}(x)\right)\delta_{i\overline{\jmath}}\delta_M\,(\star\,1)\\
&=\; \left(\widehat{\mathcal{D}}_e C^{\overline{\eta}\,M}(X)\right)\,\left(\widehat{\mathcal{D}}^e C^{\xi}\,(X)\right)\,\left(\phi^{\,i}_{\xi}(x)\star\phi^{\overline{\jmath}}_{\overline{\eta}}(x)\right)\,\delta_{i\overline{\jmath}}\delta_M\ ,\\
\mathrm{d}^6x g^{\frac{1}{2}}\mathrm{Tr}\,\left(\widehat{\mathcal{D}}_e\Psi_{\overline{\mu}}\,\widehat{\mathcal{D}}^e\Psi^{\dagger\,\overline{\mu}}\right) &=\; \left(\widehat{\mathcal{D}}_e D^{\sigma\,M}(X)\psi^{\overline{\jmath}}_{\sigma\,\overline{\mu}}(x)\right)\left(\widehat{\mathcal{D}}^e D^{\overline{\tau}\,N}(X)\right)\psi^{\overline{\mu}i}_{\overline{\tau}}(x)\delta_{i\overline{\jmath}}\delta_N\,(\star\,1)\\
&=\; \left(\widehat{\mathcal{D}}_e D^{\tau M}(X)\right)\,\left(\widehat{\mathcal{D}}^e D^{\overline{\sigma}}(X)\right)\,\left(\psi^{\overline{\jmath}}_{\sigma}(x)\star\psi^{\,i}_{\overline{\tau}}(x)\right)\,\delta_{i\overline{\jmath}}\delta_M\ ,
\end{aligned} \tag{6.15}$$

where indices for the representation $\boldsymbol{R}$ and $\boldsymbol{r}$ are explicit. The trace projects onto invariants constructed by the Krönecker delta functions $\delta_{i\overline{\jmath}}$ and δ_M. In the following we will suppress the indices and delta symbols where confusion will not arise.

Substituting (6.14) and (6.15) into $\mathcal{L}_F$ in (6.11), reintroducing the moduli contribution, calculated in [1], we find a kinetic term for both the matter fields and the moduli fields:

$$\mathcal{L}_F \;=\; -\,2G_{\alpha\overline{\beta}}\partial_e Y^{\alpha}\,\partial^e Y^{\overline{\beta}} - 2G_{\xi\overline{\eta}}\,\widehat{\mathcal{D}}_e C^{\xi}\,\widehat{\mathcal{D}}^e C^{\overline{\eta}} - 2G_{\sigma\overline{\tau}}\,\widehat{\mathcal{D}}_e D^{\overline{\tau}}\,\widehat{\mathcal{D}}^e D^{\sigma}\ , \tag{6.16}$$

from which we may identify the moduli space metric and matter field metric.

7 The Moduli, Matter Metrics and Yukawa Couplings

The effective field theory has $\mathcal{N}=1$ supersymmetry, with a gravity multiplet and a gauge symmetry $\mathfrak{g}$. The $\mathcal{N}=1$ chiral multiplets consist of

- $\mathfrak{g}$-neutral scalar fields Y^{α} and fermions $\mathcal{Y}^{\alpha}$ corresponding to moduli;
- $\mathfrak{g}$-charged bosons C^{ξ} and fermions $\mathcal{C}^{\xi}$ in the $\overline{\boldsymbol{R}}$ of $\mathfrak{g}$;
- $\mathfrak{g}$-charged bosons D^{ρ} and fermions $\mathcal{D}^{\rho}$ in the $\boldsymbol{R}$ of $\mathfrak{g}$;

The final result is expressed as a Lagrangian with normalisation conventions matching

$$\begin{aligned}
\mathcal{L}\;&= -2G_{\xi\overline{\eta}}\partial_e Y^{\alpha}\,\partial^e Y^{\overline{\eta}} - 2G_{\xi\overline{\eta}}\,\widehat{\mathcal{D}}_e C^{\xi}\,\widehat{\mathcal{D}}^e C^{\overline{\eta}} - 2G_{\sigma\overline{\tau}}\,\widehat{\mathcal{D}}_e D^{\overline{\tau}}\,\widehat{\mathcal{D}}^e D^{\sigma} - \frac{\mathrm{i}\alpha^{\backprime}}{2}\mathrm{Tr}\,_{\mathfrak{g}}\left({}^{\prime}_{\dagger_{\mathfrak{g}}}\,\overline{\sigma}^e\,\hat{D}_e\,\zeta^{\prime}_{\dagger_{\mathfrak{g}}}\right)\\
&\qquad -\,2G_{\alpha\overline{\beta}}\,\mathrm{i}^{\,\overline{\beta}}\,\overline{\sigma}^e\,\partial_e\,\mathcal{Y}^{\alpha} - 2G_{\xi\overline{\tau}}\,\mathrm{i}\,\mathcal{C}^{\xi}\,\sigma^e\,\hat{D}_e{}^{\,\overline{\eta}} - 2G_{\tau\overline{\sigma}}\,\mathrm{i}^{\,\overline{\sigma}}\,\overline{\sigma}^e\,\hat{D}_e\,\mathcal{D}^{\tau}\\
&-\left(e^{\mathcal{K}/2}m_{\alpha\beta}\,(\mathcal{Y}^{\alpha}\mathcal{Y}^{\beta})2e^{\mathcal{K}/2}m_{\xi\tau}\,(\mathcal{C}^{\xi}\mathcal{D}^{\tau}) + \mathrm{c.c.}\right) - \left(4e^{\mathcal{K}/2}\mathcal{Y}_{\xi\alpha\tau}(\mathcal{C}^{\xi}Y^{\alpha}\mathcal{D}^{\tau} + \mathcal{C}^{\xi}\mathcal{Y}^{\alpha}D^{\tau} + C^{\xi}\mathcal{Y}^{\alpha}\mathcal{D}^{\tau})\right.\\
&\left.+\,2e^{\mathcal{K}/2}\mathcal{Y}_{\xi\eta\pi}({}^{\xi}C^{\eta}\mathcal{C}^{\pi}) + 2e^{\mathcal{K}/2}\mathcal{Y}_{\rho\sigma\tau}({}^{\rho}D^{\sigma}\mathcal{D}^{\tau}) + 2e^{\mathcal{K}/2}\mathcal{Y}_{\alpha\beta\gamma}\left({}^{\alpha}\,Y^{\beta}\,\mathcal{Y}^{\gamma}\right) + \mathrm{c.c.}\right)\ .
\end{aligned} \tag{7.1}$$

The kinetic terms for fields contain metrics. The metric for fermions and bosons are identical, consistent with supersymmetry. The moduli metric, derived in [1], is:

$$
\begin{aligned}
\mathrm{d}s^2 &= 2G_{\alpha\overline{\beta}}\,\mathrm{d}y^\alpha \otimes \mathrm{d}y^{\overline{\beta}}\ , \\
G_{\alpha\overline{\beta}} &= \frac{1}{4V}\int \Delta_\alpha{}^\mu \star \Delta_{\overline{\beta}}{}^\nu\ g_{\mu\overline{\nu}} + \frac{1}{4V}\int \mathcal{Z}_\alpha \star \mathcal{Z}_{\overline{\beta}} + \\
&\quad + \frac{\alpha^{\backprime}}{4V}\int \mathrm{Tr}\left(D_\alpha A \star D_{\overline{\beta}} A\right) - \frac{\alpha^{\backprime}}{4V}\int \mathrm{Tr}\left(D_\alpha \Theta \star D_{\overline{\beta}}\Theta^\dagger\right)\ .
\end{aligned}
\tag{7.2}
$$

The metric terms for the fermionic superpartners to moduli $\mathcal{Y}^\alpha$ are fixed by supersymmetry from the the bosonic result. The matter field metrics are given

$$
\begin{aligned}
G_{\xi\overline{\eta}} &= \frac{\mathrm{i}\alpha^{\backprime}}{8V}\int_X \omega^2\,\mathrm{Tr}\ \phi_\xi\,\phi_{\overline{\eta}}\ , \qquad \phi_\xi \in H^1(\mathcal{X},\mathcal{E}_{\boldsymbol{r}})\ , \\
G_{\tau\overline{\sigma}} &= \frac{\mathrm{i}\alpha^{\backprime}}{8V}\int_X \omega^2\,\mathrm{Tr}\ \psi_\sigma\,\psi_{\overline{\tau}}\ , \qquad \psi_\sigma \in H^1(\mathcal{X},\mathcal{E}_{\overline{\boldsymbol{r}}})\ .
\end{aligned}
\tag{7.3}
$$

We have used $\omega^2 = \frac{1}{2}\star\omega$ to rewrite these metrics in a form analogous to that in (3.6). This makes explicit that the metrics are of the same form, being the natural inner product analogous to the Weil–Petersson metric.

The mass terms written vanish $m_{\alpha\beta} = m_{\xi\tau} = 0$. The Yukawa non-zero couplings are

$$
\mathcal{Y}_{\xi\eta\pi} = \frac{\mathrm{i}\alpha^{\backprime}\,e^{-\mathrm{i}\phi}}{2\sqrt{2}}\int_X \Omega\mathrm{Tr}\left(\phi_\xi\{\phi_\eta,\phi_\pi\}\right)\ , \qquad \mathcal{Y}_{\tau\sigma\rho} = \frac{\mathrm{i}\alpha^{\backprime}\,e^{-\mathrm{i}\phi}}{2\sqrt{2}}\int_X \Omega\mathrm{Tr}\left(\psi_\tau\{\psi_\sigma,\psi_\rho\}\right)\ . \tag{7.4}
$$

The similarity of the Yukawa couplings and mass terms suggests a unification through a suitable master index incorporating the moduli and matter fields. Using the covariant derivatives of fields as a basis for a Kaluza–Klein reduction, with the harmonic gauge fixing, gives the moduli space metric. It is Kähler after taking into account the second order relations between fields. This observation can be generalised to account for the charged matter fields and their fermionic superpartners in order to give the matter field metric as derived in [24]. This normalises physical Yukawa couplings.

The effective field theory has $\mathcal{N}=1$ supersymmetry in $\mathbb{R}^{3,1}$, and so the couplings ought to be derivable from a superpotential and Kähler potential. The Kähler potential for the moduli metric couplings was proposed in [1], and checked against a dimensional reduction of the $\alpha^{\backprime}$-corrected supergravity action. It is

$$
\mathcal{K}_{\,moduli} = -\log\left(\frac{4}{3}\int\omega^3\right) - \log\left(\mathrm{i}\int\Omega\,\overline{\Omega}\right)\ . \tag{7.5}
$$

in which ω is the hermitian form of $\mathcal{X}$. The $\alpha^{\backprime}$-corrections preserved the form of the special geometry Kähler potential, and the second term remains classical.

The Kähler potential for the matter field metric is trivial and given by

$$
\mathcal{K}_{\ matter} = G_{\xi\overline{\eta}} C^{\xi\,M} C^{\overline{\eta}}\delta_M + G_{\rho\overline{\tau}}\delta_M D^{\tau M} D^{\overline{\rho}}\ , \tag{7.6}
$$

where $a,b = 1,\ldots,R$ label the $\boldsymbol{R}$ representation and the trace is taken with respect to the delta function.

The F-term couplings for the $D=4$ chiral multiplets are described by a superpotential. In the language of $D=4$ effective field theory, this superpotential takes the general form

$$\mathcal{W}(Y^\alpha, C^\xi, D^\tau) \;=\; \frac{1}{3}\mathcal{Y}_{\xi\eta\pi}\,\mathrm{Tr}\,C^\xi C^\eta C^\pi + \frac{1}{3}\mathcal{Y}_{\rho\tau\sigma}\,\mathrm{Tr}\,D^\rho D^\tau D^\sigma + \cdots \;, \tag{7.7}$$

where the Tr projects onto the appropriate $\boldsymbol{R}$-invariant and we are to view these as chiral multiplets in $N=1$ $D=4$ superspace in the usual way. The omitted terms are the quartic and higher order couplings and non-perturbative corrections. It is important that $\mathcal{W}$ gives no singlet couplings, and this means all parameter derivatives of $\mathcal{W}$ vanish.

We would like to study a superpotential in a similar vein to the Kähler potential proposal (7.5). As ten-dimensional fields $A_{\mathfrak{e}_8}$ and H depend on both parameters and matter fields. The fields $\mathrm{d}^c\omega$ and Ω are valued on $\boldsymbol{X}$ and depend only on moduli fields. The spirit of the dimensional reduction is to promote the parameters to $D=4$ fields. In this vein define a superpotential

$$\mathcal{W}(Y^\alpha, C^\xi, D^\tau) \;=\; -\,\mathrm{i}\sqrt{2}e^{-\mathrm{i}\phi}\int \Omega\Big(H - \mathrm{d}^c\omega\Big)\;, \tag{7.8}$$

in which the fields are regarded as functionals of the $D=4$ chiral multiplets. The couplings in the effective field theory are specified by differentiating $\mathcal{W}$ and evaluating the integral after fixing the parameters $y=y_0$.

The rules for differentiating fields in the expressions for $\mathcal{K}$ and $\mathcal{W}$ with respect to parameters have been described in [1], which is complicated by virtue of $\mathfrak{h}$ gauge transformations being parameter and coordinate dependant. These transformations are, however, independent of matter fields, and so the rule for matter field differentiation is simple

$$\partial_\xi A_{\mathfrak{e}_8} \;=\; \frac{\partial A_{\mathfrak{e}_8}}{\partial C^\xi} \;=\; \phi_\xi \;.$$

It is important that we have written the ten-dimensional $\mathfrak{e}_8$ gauge field $A_{\mathfrak{e}_8}$, and not $A_{\mathfrak{h}}$, as this is the functional of the matter fields – C^ξ, D^τ – as illustrated in, for example (**??**) and (6.10). The integrand in $\mathcal{W}$ is a functional of the ten-dimensional H so that it depends on matter fields. The rule is to differentiate as noted above, and then evaluate the integral on the fields' vacuum expectation values (VEV). Note that it is the VEV of H that satisfies $\mathrm{d}^c\omega = H$, and the matter fields VEVs vanish $C^\xi = D^\tau = 0$.

For example, the tadpole matter and moduli couplings for a vacuum at the point $y=y_0$ are

$$\begin{aligned}
(\partial_\xi\mathcal{W})|_{y=y_0} \;&\sim\; \int \Omega\,\partial_\xi H|_{y=y_0} \;\sim\; \int \Omega\mathrm{Tr}\,F\phi_\xi|_{y=y_0} \;=\; 0\;,\\
(\partial_\alpha\mathcal{W})|_{y=y_0} \;&\sim\; \int\Big((\chi_\alpha - k_\alpha\Omega)(H-\mathrm{d}^c\omega) + \Omega\left(\overline{\partial}(\mathcal{B}_\alpha^{0,2} + \mathrm{i}\mathcal{D}_\alpha\omega^{0,2})\right)\Big)|_{y=y_0} \;=\; 0\;.
\end{aligned} \tag{7.9}$$

where we use $\partial_\alpha H$ and $\partial_\alpha\mathrm{d}^c\omega$, and we evaluate them on some fixed $y=y_0$.

As an ansatz $\mathcal{W}$ must satisfy a number of tests: it must be a section of a line bundle over the moduli space; any derivative with respect to parameters must vanish viz. $\partial_\alpha\partial_\beta\partial_\gamma\cdots\mathcal{W}=0$; be a holomorphic function of chiral fields; tadpole and mass terms for the matter fields must vanish; capture the F-term couplings derived through dimensional reduction in this paper. The expression (7.8) passes these tests.

The argument clearly extends to higher order. Consider the kth derivative

$$\Big(\partial_{\alpha_1}\cdots\partial_{\alpha_k}\mathcal{W}\Big)|_{y=y_0} \;=\; \int\Big((\partial_{\alpha_1}\cdots\partial_{\alpha_k}\Omega)\,(H-\mathrm{d^c}\omega)+ \\ +\,k\partial_{\{\alpha_2}\cdots\partial_{\alpha_k}\Omega\,\partial_{\alpha_1\}}(H-\mathrm{d^c}\omega)+\cdots\Big)|_{y=y_0} \;=\; 0\ .$$

This vanishes on any supersymmetric background: $\mathcal{W}$ is independent of moduli fields, and so $\mathcal{W}$ does not give rise to any singlet couplings in agreement with the dimensional reduction.

An analogous argument, together with Ω being holomorphic, shows that despite neither H nor $\mathrm{d}^c\omega$ being holomorphic, $\mathcal{W}$ is a holomorphic function of fields. For example, the first order derivative is

$$\partial_{\overline{\alpha}}\,\frac{1}{\Omega_0}\int\Big(\Omega(H-\mathrm{d^c}\omega)\Big) \;=\; \frac{1}{\Omega_0}\int\overline{\partial}\Big(\Omega(\mathcal{B}^{0,2}_{\overline{\alpha}}-\mathrm{i}\mathcal{D}_{\overline{\alpha}}\omega^{0,2})\Big)|_{y=y_0} \;=\; 0.$$

Using all higher order anti-holomorphic derivatives of $\Omega(H-\mathrm{d^c}\omega)$ vanish. It is also the case that $(\partial^n_{)}\mathcal{W}=0$ for all $n\geq 1$. So, $\mathcal{W}$ is a holomorphic function of chiral fields.

The expression for the masses can be written as derivatives of W

$$m_{\alpha\beta} \;=\; \partial_\alpha\partial_\beta W \;=\; 0, \qquad m_{\xi\tau} \;=\; \partial_\xi\partial_\tau W \;=\; 0\ , \tag{7.10}$$

where for the second term we use that $\mathrm{d}^c\omega,\Omega$ do not depend on C^ξ, D^τ while $\partial_\xi\partial_\tau H$ is given with $D_aA\to\partial_\xi A=\phi_\xi$. As A depends linearly on the matter fields, all second derivatives vanish.

The Yukawa couplings $\mathcal{Y}$ are also all derived from $\mathcal{W}$. Using, we find agreement with the functional forms in (**??**), of which the non-vanishing terms are

$$\mathcal{Y}_{\xi\eta\pi} \;=\; \frac{1}{2}\partial_\xi\partial_\eta\partial_\pi\mathcal{W}\ ,\quad \mathcal{Y}_{\sigma\tau\rho} \;=\; \frac{1}{2}\partial_\xi\partial_\alpha\partial_\tau\mathcal{W}\ . \tag{7.11}$$

Even though the singlet couplings vanish, one can check that their functional form is correctly derivable from $\mathcal{W}$. The fact of $1/2$ is in order to agree with the convention given in the literature. It is satisfying that the superpotential consistently captures the couplings derived in the dimensional reduction, both involving moduli and matter fields. Furthermore, it manifestly does not give rise to any singlet couplings.

Having seen how the infinitesimal deformations work, at least up to second order in deformations of the superpotential, it is interesting to consider higher order deformations of the theory. Generically, it is known that not all infinitesimal deformations can be integrated to finite deformations. The barriers to doing so are known as obstructions in the mathematics literature. For a holomorphic structure $\overline{D}$, the condition for the deformations to be unobstructed is that they are in the kernel of the obstruction map

$$\kappa\ :\ H^{(0,1)}()\to H^{(0,2)}(Q)\ ,$$

often also referred to as the Kuranishi map. The true moduli of the theory are thus the ones in the kernel of this map, while deformations not in this kernel will be obstructed. These obstructions are known to correspond to higher order Yukawa couplings in the four-dimensional effective theory. To show exactly how this works requires us to do higher order deformations of the superpotential, and show how these Yukawa couplings correspond to obstructions in the deformation theory of $\overline{D}$. This is quite involved and we leave the full treatment for future work. Instead we only investigate a couple of features of the obstructions here, in particular for compactifications where X_0 is Calabi-Yau. It should also be noted that obstructions and their correspondence to Yukawa couplings have been considered at length in the literature before.

In terms of holomorphic structures defined by an extension sequence, it can be shown that the obstruction maps in the corresponding long exact sequence commute with the other induced maps in cohomology

$$\begin{array}{c} ... \to H^0(E) \xrightarrow{\mathrm{H}_0} H^{(0,1)}(T^*X) \to H^{(0,1)}() \to H^{(0,1)}(E) \\ \qquad\qquad \downarrow \kappa_{T^*X} \qquad\quad \downarrow \kappa \qquad\quad \downarrow \kappa_E \\ \xrightarrow{\mathrm{H}} H^{(0,2)}(T^*X) \to H^{(0,2)}() \xrightarrow{\rho} H^{(0,2)}(E) \to 0 \, , \end{array} \tag{7.12}$$

where the last zero follows from the slope-zero stability of T^*X. The obstruction map κ_E can further be sandwiched between obstruction maps of the bundle and base as

$$\begin{array}{c} 0 \to H^{(0,1)}(\mathrm{End}(TX)) \oplus H^{(0,1)}(\mathrm{End}(V)) \to H^{(0,1)}(E) \to H^{(0,1)}(TX) \\ \downarrow \kappa_{\mathrm{End}(TX)} \qquad\quad \downarrow \kappa_{\mathrm{End}(V)} \qquad\quad \downarrow \kappa_E \qquad\quad \downarrow \kappa_{TX} \\ \xrightarrow{+} H^{(0,2)}(\mathrm{End}(TX)) \oplus H^{(0,2)}(\mathrm{End}(V)) \xrightarrow{\rho_E} H^{(0,2)}(E) \to H^{(0,2)}(TX) \to ... \, , \end{array} \tag{7.13}$$

where we have named the map ρ_E as it will appear in the following computations.

Let us begin by performing a second order deformation of W at the supersymmetric locus $W = \delta W = 0$. We take δ_1 to be a generic deformation while δ_2 is massless deformation. According to the above discussion, we need that

$$\begin{aligned} \delta_2\delta_1 W|_0 = & \int_X \delta\tau_1 \wedge \delta_2\Omega + \int_X \frac{\alpha}{2} \left(\mathrm{Tr}\ (\delta_1 A \wedge \delta_2(F \wedge \Omega)) - \mathrm{Tr}\ (\delta_1\Theta \wedge \delta_2(R \wedge \Omega))\right) \\ & + \int_X \delta_2(H + i\delta\omega) \wedge \delta_1\Omega + \int_X (H + i\delta\omega) \wedge \delta_2\delta_1\Omega = 0 \, , \end{aligned} \tag{7.14}$$

for all deformations δ_1 if the deformation δ_2 is to be massless. Here the zero in $\delta_2\delta_1 W|_0$ denotes we are imposing the ten-dimensional supersymmetry conditions found in the previous section.

We next consider the second line of (7.14). Writing the first term out, we get

$$\int_X \delta_2(H + i\delta\omega) \wedge \delta_1\Omega = \int_X \left(\frac{\alpha}{2}(\mathrm{Tr}\ \alpha_2 \wedge F - \mathrm{Tr}\ \kappa_2 \wedge R) + \delta\tau_2\right) \wedge \delta_1\Omega \, .$$

The second term of the second line of (7.14) is given by

$$\int_X (H + i\delta\omega) \wedge \delta_2\delta_1\Omega = 2i \int_X \partial\omega \wedge \delta_2\delta_1\Omega \, .$$

Noting that $H + i\,\omega = 2\,i\,\partial\omega$, it is clear that only the $(1,2)$-part of $\delta_2\delta_1\Omega$ contributes

$$(\delta_2\delta_1\Omega)^{(1,2)} = \Delta_1^{\,a} \wedge \chi_{2ab\bar{c}}\,\delta z^{b\bar{c}} = \Delta_1^{\,a} \wedge \Delta_2^{\,b}\,\Omega_{abc}\,\mathrm{d}x^c \, .$$

Using this, we can rewrite

$$
\begin{aligned}
2i\int_X \partial\omega\wedge\delta_2\delta_1\Omega = 2i\int_X &\partial_{[a}\omega_{b]\bar{c}}\,\mathrm{d}z^{\bar{c}}\ \wedge\Delta_1^d\,\Omega_{def}\ \wedge\Delta_2^{[e}\wedge\mathrm{d}z^{f]ab}\\
&= -2i\int_X \partial_{[a}\omega_{b]\bar{c}}\,\mathrm{d}z^{\bar{c}}\ \wedge\Delta_1^d\,\Omega_{def}\ \wedge\Delta_2^{[a}\wedge\mathrm{d}z^{b]ef}\\
&= -4i\int_X \Delta_2^a\wedge\partial_{[a}\omega_{b]\bar{c}}\,\delta z^{b\bar{c}}\wedge\chi_1\ ,
\end{aligned}
$$

where in the second line we have used

$$
0 = 2\,\Delta^{[e}\wedge\mathrm{d}z^{abf]} = \Delta^{[e}\wedge\mathrm{d}z^{f]ab} + \Delta^{[a}\wedge\mathrm{d}z^{b]ef}\ .
$$

Putting it all together, and requiring $\delta_1\Omega$ generic, we find that we need

$$
-4\Delta_2^a\wedge i\partial_{[a}\omega_{b]\bar{c}}\,\delta z^{b\bar{c}} + \frac{\alpha}{2}\big(\mathrm{Tr}\ (\alpha_2\wedge F) - \mathrm{Tr}\ (\kappa_2\wedge R)\big) + \partial\tau_2^{(0,2)} + \tau_2^{(1,1)} = 0\ .
$$

The first equation imply that $\tau^{(0,2)}$ is -exact, that is

$$
\tau_2^{(0,2)} = \beta^{(0,1)},
$$

for some $(0,1)$-form $\beta^{(0,1)}$. The second equation then gives the following condition

$$
-\,4\Delta_2^a\wedge i\partial_{[a}\omega_{b]\bar{c}}\,\delta z^{b\bar{c}} + \frac{\alpha}{2}\big(\mathrm{Tr}\ (\alpha_2\wedge F) - \mathrm{Tr}\ (\kappa_2\wedge R)\big) + \tau_2^{(1,1)} - \partial\beta^{(0,1)} = 0\ , \tag{7.15}
$$

which can be rewritten as

$$
-\,\mathrm{H}(x_2)_a\delta z^a = \frac{1}{2}\left(\tau_2^{(1,1)} - \partial\beta^{(0,1)}\right)\ , \tag{7.16}
$$

where $\mathcal{H}$ is the map defined by

$$
\mathrm{H} = \hat{H} + + \ :\quad \Omega^{(p,q)}(E)\to\Omega^{(p,q+1)}(T^*X), \tag{7.17}
$$

and

$$
\begin{aligned}
\mathrm{H}(x)_b &= \Delta^a\wedge\hat{H}_{ab\bar{c}}\,\delta z^{\bar{c}} - \frac{\alpha}{4}\Big(\mathrm{Tr}\ (\alpha\wedge F_{b\bar{c}}\delta z^{\bar{c}}) -\ \mathrm{Tr}\ (\kappa\wedge R_{b\bar{c}}\delta z^{\bar{c}})\Big)\ ,\\
\hat{H}_{ab\bar{c}}\,\delta z^{\bar{c}} &= 2i\partial_{[a}\omega_{b]\bar{c}}\,\delta z^{\bar{c}} = H^{(2,1)}_{ab\bar{c}}\,\delta z^{\bar{c}}\ .
\end{aligned}
$$

We have extended the definition of the maps and to forms $x = (\kappa,\alpha,\Delta)$ with values in E. In fact, and are understood as acting on both TX-valued forms as before, and on γ-valued forms by the trace on the endomorphism bundles. That is, we are extending the definition of these maps so that

$$
\begin{aligned}
\mathcal{F}_b(\alpha) &= \ \ \frac{\alpha}{4}\,\mathrm{Tr}\,(F_{b\bar{c}}\,\mathrm{d}z^{\bar{c}}\wedge\alpha)\ ,\quad \alpha\in\Omega^{(p,q)}(\mathrm{End}(V))\ ,\\
\mathcal{R}_b(\kappa) &= -\frac{\alpha}{4}\,\mathrm{Tr}\,(R_{b\bar{c}}\,\mathrm{d}z^{\bar{c}}\wedge\kappa)\ ,\quad \kappa\in\Omega^{(p,q)}(\mathrm{End}(TX))\ .
\end{aligned}
$$

Alternatively the pre-factors $\pm\frac{\alpha}{4}$ could be pulled into a re-definition of the trace on γ. Note the different sign of the action of relative to the action of . Altogether, these maps act as follows

$$
(x) = \begin{bmatrix} (\kappa)\\ {}_a(\alpha)\\ (\Delta)\end{bmatrix} = \begin{bmatrix} 0\\ \frac{\alpha}{4}\,\mathrm{Tr}\,(F_a\wedge\alpha)\\ {}_{a\bar{b}}\,\mathrm{d}x^{\bar{b}}\wedge\Delta^a\end{bmatrix}\ ,\qquad (x) = \begin{bmatrix} {}_a(\kappa)\\ (\alpha)\\ (\Delta)\end{bmatrix} = \begin{bmatrix} -\frac{\alpha}{4}\,\mathrm{Tr}\,(R_a\wedge\alpha)\\ 0\\ R_{a\bar{b}}\,\mathrm{d}x^{\bar{b}}\wedge\Delta^a\end{bmatrix}\ ,
$$

where $F_a = F_{a\bar{b}}\delta z^{\bar{b}}$ and $R_a = R_{a\bar{b}}\delta z^{\bar{b}}$. We will see below that the map $\mathcal{H}$ is in fact a map between cohomologies. Hence, we see that the equation for moduli (7.16) for x_2 can be equivalently stated as $x_2 = \ker\,(\mathcal{H})$. This of course is in agreement with what was found from the ten-dimensional supergravity perspective in [13, 14].

The anomaly cancellation condition induces a holomorphic structure $\overline{D}$ on $\mathcal{Q}$ and the moduli of the Strominger/Hull system is then given by the elements of the cohomology $H^1_{\overline{D}}(\mathcal{Q})$. The extension Q_2 of Q_1 by $\operatorname{End}\mathcal{T}_X$ is necessary to enforce the connection on the tangent bundle appearing in the anomaly cancelation condition to be an instanton. In fact, this is needed to satisfy the equations of motion. We do not give here the derivation of the holomorphic structure $\overline{D}$ on $\mathcal{Q}$ nor the derivation of the cohomology groups corresponding to the moduli space. The result however is that the moduli for the heterotic structure correspond to elements of the cohomology group

$$H^1_{\overline{D}}(\mathcal{Q}) = H^1(\mathcal{X}, \mathcal{T}_X^*) \oplus \ker\mathcal{H} \ , \qquad \ker\mathcal{H} \subseteq H^1(\mathcal{X}, Q_2) \ , \tag{7.18}$$

where

$$H^1(\mathcal{X}, Q_2) = H^1(\mathcal{X}, \operatorname{End}\mathcal{T}_X) \oplus H^1(X, \operatorname{End}E) \oplus (\ker\mathcal{F} \cap \ker\mathcal{R}) \ . \tag{7.19}$$

The first factor in (7.18) corresponds to complexified α'-corrected hermitian moduli. The second factor contains a map $\mathcal{H} : H^1(\mathcal{X}, Q_2) \to H^2(\mathcal{X}, \mathcal{T}_X^*)$ defined by

$$\mathcal{H}_\mu(\alpha, \kappa, \Delta) = H^{(2,1)}_{\mu\nu\bar{\rho}}\, \mathrm{d}x^{\bar{\rho}} \wedge \Delta^\nu - \frac{\alpha'}{4}\Big(\mathrm{Tr}\,(F_{\mu\bar{\nu}}\mathrm{d}x^{\bar{\nu}} \wedge \alpha) - \mathrm{Tr}\,(R_{\mu\bar{\nu}}\mathrm{d}x^{\bar{\nu}} \wedge \kappa)\Big) \ ,$$

where α is a $(0,1)$-form with values in $\operatorname{End}\mathcal{E}$ and κ is a $(0,1)$-form with values in $\operatorname{End}\mathcal{T}_X$. There is a subtlety in that the parameters in (7.19) corresponding to $H^1(\mathcal{X}, \operatorname{End}\mathcal{T}_X)$ are not physical and can be removed by field redefinitions [15]. The map $\mathcal{R}$ in (7.19) is the Atiyah map appropriate for the deformations of the holomorphic tangent bundle. Finally, we remark that the same results for the moduli problem of heterotic structures was obtained from first and second order deformations of a heterotic superpotential.

Armed with these complex coordinates on the space of $\mathcal{N} = 1$ structures, we can expand the superpotential in them around a supersymmetric point where $W = \delta W = 0$. As the superpotential in $\mathcal{N} = 1$ supergravity must be a holomorphic function of the complex scalar fields, we have that the variation of the superpotential is exactly equal to its holomorphic variation. Thus we expand

$$\begin{aligned} W + \Delta W = \Delta W &= \int_X (H + \mathrm{i}\,\mathrm{d}\omega + \mathrm{d}(\Delta B + \mathrm{i}\,\Delta\omega)) \wedge (\Omega + \Delta\Omega) \\ &= 2\int_X \Big(\mu^d \wedge \bar{\partial} x_d + \tfrac{1}{2}\mu^d \wedge \mu^e \wedge H_{de} \\ &\qquad + \mu^d \wedge \mu^e \wedge \partial_d x_e - \tfrac{1}{2}\mu^d \wedge \partial_d \tilde{b}\Big) \wedge \Omega \end{aligned} \tag{7.20}$$

where we have suppressed the anti-holomorphic form indices. Strictly though, we have solved a different problem to the original moduli problem, as we introduced extra degrees of freedom to the theory. The true moduli space will then be a subspace of the moduli space found in this way, on which the additional gauge field is fixed to be the Hull connection. How best to describe these additional constraints remains an open problem.

8 Conclusion

We have calculated the effective field theory of heterotic vacua of the form $\mathbb{R}^{3,1} \times X$ at large radius, correct to order α'. The field theory is specified by a Kähler potential and superpotential. Supersymmetry forbids $\mathcal{W}$ from being corrected perturbatively in α', but is in general corrected non-perturbatively in α'. For $\mathcal{E}$ obtained by deforming $\mathcal{T}_X$, some of these non-perturbative corrections have been computed as functions of moduli using linear sigma models. One can now use the results obtained here and those in [1] to determine the normalised quantum corrected Yukawa couplings, in examples that may be of phenomenological interest. Although the Kähler potential is corrected perturbatively in α', it was conjectured in [1] that the form of the Kähler potential does not change to all orders in perturbation theory, and that the α'-corrections are contained within the hermitian form ω. This conjecture is consistent with the work in [14, 15] and it would be very interesting to prove this conjecture, at least to second order in α'. It should however be noted that this need not have an effect on the physical matter spectrum. Indeed, the authors of in the current research suggest to use the hidden E_8 -bundle to stabilize complex structure moduli in more phenomenology oriented models. In this case one only lifts bundle moduli corresponding to deformations of the hidden bundle, and hence the physical spectrum important for phenomenology is unaffected. Finally, we collect some useful facts about geometries where the large volume limit is a compact Calabi-Yau. In particular, recall that a sufficient condition for a complex manifold (X, J) to satisfy the ∂-lemma is the existence of a Kähler form compatible with J. Since the complex structure does not change under α-corrections, and since there must exist a Kähler form ω_0 corresponding to the zeroth order Calabi-Yau geometry X_0, it follows that the corrected geometry X satisfies the ∂-lemma. Moreover, as the Dolbeault operator remains unchanged under α-corrections, we can conclude that the Hodge-diamond of X does not change either. Indeed, as the Dolbeault cohomologies of a Calabi-Yau manifold are topological, and as we have seen X admits a Kähler metric, any change to this at higher orders in α implies topological changes of X which contradicts the assumptions of the α-expansion. Note that a similar statement need not hold for bundle valued cohomologies, as the connections on the given bundles can potentially receive corrections, and the bundles need no longer be holomorphic in general. Of course, an enormous amount of work remains before such torsional compactifications are fully understood and potentially able to lead to fully realistic low energy phenomenology. An obvious omission is our present lack of knowledge of the Kähler potential, although this is the subject of current work. It may be hoped that, given the holomorphic structures discussed, the Kähler potential will take a fairly simple and elegant form. Indeed, holomorphic structures usually come equipped with some form of Weil-Peterson metric on their moduli space, and one can speculate that the Kähler metric one obtains upon dimensional reduction corresponds to such a metric. However, for this present time this remains an open question. The story with Yukawa couplings is also far from complete. In particular, the connection between higher order deformations of the superpotential and obstructions has not yet been made explicit and it would be interesting to see how the details of this emerge. Knowledge of the Yukawa couplings is also very important for phenomenological purposes as well. It would also be very interesting to study explicit examples of compactifications with torsion. Compactifications where a large volume Calabi-Yau locus exists are fairly easy to construct once the zeroth order Kähler geometry is known, and it would be interesting to investigate further what effects the generated torsion has for lifting further moduli. Studying examples where no zeroth order limit exist is more challenging. Examples of this kind found in the literature have been shown to negate some of the assumptions we make. It is hence less clear counts the true moduli for these types of compactifications, but it can be taken as a conjecture. In the longer term, it may even be hoped that there is the possibility of constructing examples with all moduli either removed from the low energy theory, or otherwise stabilised with phenomenologically acceptable masses. Investigations of other aspects of the low energy phenomenology, the number of Standard Model generations, exotic matter present, may also be possible and interesting.

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