The General Aspects of Heterotic Superstring/F-Theory Duality Correspondence of Moduli Superspaces

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Abstract

In the present article we aim to broaden the consideration of background geometry of manifolds/bundles arising in heterotic compactifications with an aim towards extending the validity and understanding of heterotic/F-theory duality. In particular, we will focus on elliptically fibered Calabi-Yau geometries arising in heterotic theories in the context of the so-called Fourier Mukai transforms of vector bundles on elliptically fibered manifolds. The duality between the Heterotic and F-theory is a powerful tool in gaining more insights into F-theory description of low-energy chiral multiplets. We propose a generalization of heterotic/F-theory duality and in order to complete the translation, the dictionary of the heterotic/F-theory duality has to be refined in some aspects. The precise map of spectral surface and complex structure moduli is obtained, and with the map, we find that divisors specifying the line bundles correspond precisely to codimension singularities in F-theory.

1 Introduction

Heterotic/F-theory duality has proven to be a robust and useful tool in the determination of F-theory effective physics as well a remarkable window into the structure of the string landscape. The seminal work on F-theory appealed to heterotic theories and ever since, many new developments and tools have been built on, or inspired by, the duality. Despite the important role that this duality has played however, it has remained at some level limited by the geometric assumptions that have been frequently placed on the background geometries in both the heterotic and F-theory compactifications. We generalize the available computational tools to explicitly construct the Fourier-Mukai transforms of vector bundles on elliptically fibered geometries. That is, given an explicit vector bundle constructed on an elliptic threefold for example built using the monad construction or as an extension bundle, we provide an algorithm to produce the spectral data which is key ingredient in determining an explicit F-theory dual of a chosen heterotic background. It is the goal of this work to investigate Fourier-Mukai transforms of vector bundles over elliptically fibered manifolds not in Weierstrass form as a necessary first step in extending heterotic/F-theory duality beyond the form considered. It has been argued that from the point of view of F-theory, Weierstrass models are the natural geometric point in which to consider/define the theory. In order to make sense of the axio-dilaton from a type IIB perspective, we require the existence of a section to the elliptic fibration, and for all reducible components of fibers not intersecting this zero section to be blown-down to zero size. This choice provides a unique value of the axio-dilaton for every point in the base geometry. Once it is further demanded that the torus fibration admits a section, it is guaranteed that the Weierstrass models are available and obtainable form the originally chosen geometry via birational morphisms. In the context of heterotic/F-theory duality, a range of possible geometries are possible in the elliptic and $K3$-fibered manifolds appearing in (1) and (2) (with many possible Hodge numbers, Picard groups, etc appearing). However, thanks to the work of Nakayama, the existence of an elliptic fibration guarantees the existence of a particular minimal form for the dual CY geometries – the so-called Weierstrass form in which all reducible components of the fiber not intersecting the zero-section have been blown down. As a result of the above decomposition, it is clear that the topology (i.e. Chern classes), cohomology (i.e. $H^*(X_3, V)$) and stability structure (i.e. stable regions within Kähler moduli space) of a stable, holomorphic bundle V on an elliptically fibered manifold can depend on these "extra" divisors (and elements of $h^{1,1}(X_3)$) which are not present in Weierstrass form. In addition, if X_n contains either a higher rank Mordell-Weil group or fibral divisors, the associated Weierstrass model is singular, leading to natural questions as to how to interpret the data of gauge fields/vector bundles over such spaces. As a result, in the processing of attempting to map the heterotic CY manifold into Weierstrass form, important topological and quasi-topological information – and its ensuing physical consequences – could be lost. To this end, the work of Friedman, Morgan and Witten introduced the tools of Fourier-Mukai Transforms into heterotic theories. In this context, the data of a rank N, holomorphic, slope-stable vector bundle $\pi : V \to X$ is presented by its so-called "spectral data", loosely described as a pair $(S, \mathcal{L}_S)$ consisting of an N-sheeted cover, S, of the base B_{n-1} (the "spectral cover") and a rank-1 sheaf $\mathcal{L}_S$ over it. Very loosely, this encapsulates the restriction of the bundle to each fiber (given by the N points on the elliptic curve over each point in the base) and the data of a line bundle, $\mathcal{L}_S$ encapsulating the "twisting" of this decomposition over the manifold. To begin, it should be recalled that compactifications of the $E_8 \times E_8$ heterotic theory on an elliptically fibered Calabi-Yau n-fold,

$$\pi_h : X_n \xrightarrow{\mathbb{E}} B_{n-1} \quad , \tag{1}$$

will lead to the same effective physics as F-theory compactifications on a $K3$-fibered Calabi-Yau $n+1$-fold,

$$\pi_f : Y_{n+1} \xrightarrow{K3} B_{n-1} \quad . \tag{2}$$

Here the base manifold B_{n-1} appearing in (1) and (2) is the same Kähler manifold (thus inducing a duality fiber by fiber over the base from the 8-dimensional correspondence. Within the heterotic theory, the geometry of the slope stable, holomorphic vector bundle, $\pi : V \to X_n$, must also be taken into account. In particular, to be understood in the context of the fiber-wise duality (induced from 8-dimensional correspondence), the data of the vector bundle must also be presented "fiber by fiber" in X_n over the base B_{n-1}. More precisely a Fourier-Mukai transform is a relative integral functor acting on the bounded derived category of coherent sheaves $\Phi : D^b(X) \to D^b(\hat{X})$ (where $\hat{X}$ is the Altman-Kleian compactification of the relative Jacobian of X). This functorial/category-theoretic viewpoint will prove a powerful tool as we examine and define the concepts above more carefully in the Sections to come and consider their generalizations. The basic idea for establishing a duality between two theories in lower dimensions is to use the adiabatic principle. Therefore one has first to establish a duality between two theories, then one variies slowly the parameters of these two theories over a common base space, and one expects that the duality holds on the lower dimensional space. The heterotic string compactified on a $n-1$-dimensional elliptically fibered Calabi-Yau $Z \to B$ together with a vector bundle V on Z is conjectured to be dual to F-theory compactified on a n-dimensional Calabi-Yau $X \to B$, fibered over the same base B with elliptic $K3$ fibers. A duality between the two theories involves the comparision of the moduli spaces on both sides. In the following we will be interested in $n = 2, 3, 4$. Now, on the level of parameter counting one gets 16 complex parameters coming from the Wilson lines, and additional 2 complex parameters from the complex structure modulus U and the Kähler modulus T of T^2. These 18 parameters parametrize the moduli superspace

$$\mathbb{M}_{het} = SO(18, 2; \mathbf{Z}) \backslash SO(18, 2)/SO(18) \times SO(2) \tag{3}$$

further, one has to take into account the heterotic coupling constant, which is parametrized by a positive real number λ^2, so we have 18 complex + 1 real parameters. It was argued on the level of parameter counting that the heterotic superstring on T^2 in the presence of Wilson lines is dual to F-theory compactified on $K3$ given by

$$y^2 = x^3 + g_2(z)x + g_3(z) \tag{4}$$

where the equation describes a hypersurface in a $\mathbf{P}^2$ bundle over $\mathbf{P}^1$. In particular one has $(9 + 13 - 3 - 1) = 18$ parameters, where $9 + 13$ coming from specifying g_2 and g_3, then mod out by $SL(2,\mathbf{C})$ action on $\mathbf{P}^1$ means just subtracting 3 and an additional 1 for overall rescalings. Furthermore, the remaining real parameter (the heterotic coupling) can be identified with the size of the $\mathbf{P}^1$.

$$\mathbb{M}_F = SO(18,2;\mathbf{Z})\backslash SO(18,2)/SO(18)\times SO(2) \tag{5}$$

We get moduli fields from hyper and tensor multiplets. Therefore one expects the moduli superspace to be in the form

$$\mathbb{M} = \mathbb{M}_H \times \mathbb{M}_T \tag{6}$$

where $\mathbb{M}_H$ is a quaternionic Kähler manifold and $\mathbb{M}_T$ is a Riemannian manifold, their dimensions are given. Since the supergravity is chiral, there are constaints on the allowed spectrum, due to gauge and gravitational anomaly cancellation conditions. The number of vector multiplets is given by the dimension of the adjoint representation of the gauge group, since the vector multiplets belong to the adjoint representation. To determine the number of charged hypermultiplets, we consider an H-bundle V with fibre in an irreducible representation R_i of the structure group. We compare the number of moduli in heterotic and F-theory which lead to $N = 1$ neutral chiral multiplets. Note that this is possible as long as we assume that no four-flux is turned on which otherwise would imply that we have to take into account the twistings appearing in the spectral cover construction of V . The twistings lead to a multi-component structure of the bundle moduli superspace. If the F-theory geometry also admits a $K3$-fibration then the choice of Weierstrass form described above also imposes the expected form of the heterotic ellitpically fibered geometry in the stable degeneration limit. As a result, in much of the literature to date, it has simply been assumed that the essential procedure of heterotic/F-theory duality must be to place both CY geometries, X_n and Y_{n+1} into Weierstrass form from the start.

We review the basic tools and key results of Fourier-Mukai transforms and spectral cover bundles in the case of Weierstrass models. We then generalize these results to the case of elliptically fibered manifolds with fibral divisors and geometries with additional sections to the elliptic fibration. We provide explicit examples of Fourier-Mukai transforms by beginning with a bundle defined via some explicit construction (e.g. a monad or extension bundle) and then computing its spectral data directly. We apply our new results to the problem of chirality changing small instanton transitions. We illustrate the distinctions and obstructions that can arise between smooth and singular spectral covers. Finally we summarize this work and briefly discuss future directions. This article contain a set of well-known but useful mathematical results on the topics of derived categories and Fourier-Mukai functors. Although the material contained there is well-established in the mathematics literature, it is less commonly used by physicists and we provide a small overview in the hope that readers unfamiliar with these tools might find a brief and self-contained summary of these results useful. There are some motivations to develop theoretical tools to extract physics out of G_2-holonomy compactification of 11-dimensional supergravity or Calabi–Yau four-fold compactification of F-theory. The identification of quarks and leptons, or of chiral matter multiplets in general, has been such a challenging problem in F-theory, because an intrinsic formulation of F-theory has not been fully developed yet. The elementary degrees of freedom in F-theory can be described by (p, q) strings or $M2$-branes of 11-dimensional supergravity. It may be possible, to identify chiral matter multiplets on 3+1 dimensions with some of their fluctuation modes. In practice, however, it is extremely difficult to disentangle complicated geometry of triple intersection of (p, q) 7-branes, or to maintain distinction between left-handed and right-handed fermions in Calabi–Yau 4-fold compactification of 11-dimensional supergravity down to 2+1 dimensions. Instead, the duality between the Heterotic string and F-theory will be the most powerful tool in studying F-theory. This article is along the line of this approach, the Heterotic string theory and the Heterotic–F-theory duality are used to study F-theory. The Heterotic string theory compactified on an elliptically fibered Calabi–Yau 3-fold $\pi_Z : Z \to B_2$ is dual to F-theory compactified on an elliptically fibered Calabi–Yau 4-fold $\pi_X : X \to B_3$ whose base 3-fold B_3 is a $\mathbb{P}^1$ fibration over B_2. The various matter multiplets in low-energy effective theory are identified with $H^1(Z; \rho(V))$ in Heterotic string description, where $\rho(V)$ is a vector bundle V in representation ρ. Cohomology groups on a fibered space can be calculated first on the fiber geometry, and later on the base geometry; except for certain cases (which will be covered in section 4),

$$H^1(Z; \rho(V)) \simeq H^0(B_2; R^1\pi_Z{*}\rho(V)), \tag{7}$$

and the direct images $R^1\pi_{Z*}\rho(V)$ have their support only on curves in B_2. In the Heterotic/F-theory duality, these support curves correspond to 7-brane intersections, and the sheaves on the curves should be those on the 7-brane intersection curves. Chiral matter multiplets are identified with global holomorphic sections of such sheaves (except for certain cases). Thus, by using the Heterotic–F duality, we can obtain the sheaves whose sections are identified with quarks and leptons. Direct images $R^1\pi_{Z*}\rho(V)$ are, therefore, the information we would like to obtain from the Heterotic superstring theory. Direct images of bundles in the fundamental representation $\rho(V) = V$ were obtained in 1990's [16,17]. Those of bundles in the anti-symmetric representation $\rho(V) = \wedge^2 V$ have not been clearly described as sheaves so far in the last decade, apart from some developments in [18, 19] in the context of Heterotic theory compactification. Calculation of the direct images of $\wedge^2 V$, therefore, is one of the central themes in this article. This is by no means a minor problem. Both $\bar{\mathbf{5}}$ and $\bar{H}(\bar{\mathbf{5}})$ multiplets arise from $\wedge^2 V$ of an SU(5) bundle V, and $H(\mathbf{5})$ from $\wedge^2 V^\times$, where $V^\times$ is the dual bundle of V. Without understanding the geometry associated with $R^1\pi_{Z*}\wedge^2 V$ and $R^1\pi_{Z*}\wedge^2 V^\times$, there is no way to understand the Yukawa couplings of quarks and leptons in F-theory. We introduce a new notion, covering matter curve, roughly speaking in order to deal with singularities that appear along matter curves. The direct image $R^1\pi_{Z*}\wedge^2 V$ is represented as a pushforward of a locally free rank-1 sheaf $\widetilde{\mathcal{F}}_{\wedge^2 V}$ on the covering matter curve for all the cases we have study, i.e. for rank $V = 3, 4, 5, 6$. We should also note here that a minor assumption is made on structure of $R^1\pi_{Z*}\wedge^2 V$ around a particular type of singularity for the rank $V = 4$ case. Divisors determining the locally free rank-1 sheaves are determined in terms of data defining spectral surfaces. Further, consistent F-theory compactification requires a number of space-time filling threebranes, which should turn, under duality, into heterotic fivebranes, which wrap- ping the elliptic fiber. Another problem which begs further analysis is to work out a refined heterotic/F-theory matter dictionary, which requires an improved understanding of intersecting seven-branes.

2 The Review of Vector Bundles over Weierstrass Elliptic Fibrations and Fourier-Mukai Transforms

In this section we provide a brief review of some of the necessary existing tools and standard results of Fourier-Mukai transforms arising in elliptically fibered Calabi-Yau geometry. Since the literature on this topic is vast, we make no attempt at a comprehensive review, but instead aim for a curated survey of some of the tools that will prove most useful. Moreover, we hope that this review is of use in making the present paper somewhat self-contained. For more information about the applications of Fourier-Mukai functors in studying the moduli space of stable sheaves over elliptically fibered manifolds, the interested reader is referred to the research literature.

2.1 Irreducible smooth elliptic curve

To set notation and introduce the necessary tools let us begin by considering the case of $n = 1$ in (1), a one (complex) dimensional Calabi-Yau manifold – that is X is a smooth elliptic curve, E. In the case of a smooth elliptic curve, there is a classical result due to Atiyah (which can generalized to abelian varieties) which states that any (semi)stable coherent sheaf, $\mathcal{E}$, of rank N and degree zero over E is S-equivalent to a direct sum of general degree zero line bundles,

$$\mathcal{E} \sim \bigoplus_i \mathcal{L}_i^{\oplus N_i}, \qquad \Sigma_i N_i = N, \quad deg(\mathcal{L}_i) = 0. \tag{8}$$

In the context then of the moduli space of semi-stable sheaves on an elliptic curve, one can introduce an integral functor

$$\Phi^{\mathcal{P}}_{E\to E} : D^b(E) \longrightarrow D^b(E) \tag{9}$$

(note that here $\hat{E}$ the Jacobian of E is simply isomorphic to E and thus we do not make the distinction). This functor admits a canonical kernel, $\mathcal{P}$, the so-called *Poincare sheaf*,

$$\mathcal{P} := \mathcal{I}_\Delta \otimes \pi_1^* \mathcal{O}_E(p_0) \otimes \pi_2^* \mathcal{O}_E(p_0) \tag{10}$$

where π_1, π_2 are the projection of $E \times E$ to the first and second factor respectively, p_0 is the divisor corresponding to the zero element of the abelian group on the elliptic curves, and Δ is the diagonal divisor in $E \times E$ (and also δ is the diagonal morphism). It is not hard to prove that $\mathcal{P}$ satisfies the conditions due to Orlov and Bondal that guarantee that $\Phi^{\mathcal{P}}_{E\to E}$ is indeed a Fourier-Mukai transform (i.e. it is an equivalence of derived categories).

To illustrate how this specific Fourier-Mukai functor acts on coherent sheaves of degree zero, it is useful to highlight its specific behavior in several explicit cases. To begin, consider the simplest possible case of $\mathcal{E} = \mathcal{O}_E(p - p_0)$, i.e. a generic degree zero line bundle over E. Here,

$$\Phi^{\mathcal{P}}_E(\mathcal{O}_E(p - p_0)) = R\pi_{2*}(\pi_1^* \mathcal{O}_E(p - p_0) \otimes \mathcal{P})$$

To compute this explicitly, consider the following short exact sequence induced by the morphism $\delta : E \longrightarrow E \times E$,

$$0 \longrightarrow \mathcal{P} \longrightarrow \pi_1^*\mathcal{O}_E(p_0) \otimes \pi_2^*\mathcal{O}_E(p_0) \longrightarrow \delta_*\mathcal{O}_E(2p_0) \longrightarrow 0. \tag{11}$$

Twistin the sequence above with $\mathcal{O}_E(p-p_0)$, and then applying the (left exact) functor $R\pi_*$ to that yields the following long exact sequence,

$$\begin{aligned} 0 &\longrightarrow \Phi^0(\mathcal{O}_E(p-p_0)) \longrightarrow (R^0\pi_{2*}\pi_1^*\mathcal{O}_E(p)) \otimes \mathcal{O}_E(p_0) \longrightarrow \mathcal{O}_E(p_0) \otimes \mathcal{O}_E(p) \\ &\hookrightarrow \Phi^1(\mathcal{O}_E(p-p_0)) \longrightarrow (R^1\pi_{2*}\pi_1^*\mathcal{O}_E(p)) \otimes \mathcal{O}_E(p_0) \longrightarrow 0. \end{aligned} \tag{12}$$

To determine the the FM transform, it is necessary to understand the sheaves appearing in the middle column, and to that end, it is possible to apply the base change formula for flat morphisms,

$$\begin{array}{ccc} E \times E & \xrightarrow{\pi_2} & E \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle P} \\ E & \xrightarrow{P} & p \end{array}$$

$$R\pi_{2*}\pi_1^* \simeq P^* R P_*, \tag{13}$$

where P is just a projection to a point. Therefore,

$$R\pi_{2*}\pi_1^*\mathcal{O}_E(p) = P^* R\Gamma(E, \mathcal{O}_E(p)) = \mathcal{O}_E. \tag{14}$$

Consequently, it follows that $\mathcal{O}_E(p-p_0)$ must be a WIT_1, and it is supported on p,

$$\Phi^{\mathcal{P}}(\mathcal{O}_E(p-p_0)) = \mathcal{O}_p[-1]. \tag{15}$$

In summary, the Fourier-Mukai transform of any direct sum degree zero line bundles on an elliptic curve, is a direct sum of torsion sheaves supported on the corresponding points of the Jacobian.

As another simple example, consider the non-trivial extension of two trivial line bundles,

$$0 \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{O}_E \longrightarrow 0. \tag{16}$$

Applying Φ on this short exact sequence yields

$$\begin{aligned} 0 &\longrightarrow \Phi^0(\mathcal{O}_E) \longrightarrow \Phi^0(\mathcal{E}_2) \longrightarrow \Phi^0(\mathcal{O}_E) \\ &\hookrightarrow \Phi^1(\mathcal{O}_E) \longrightarrow \Phi^1(\mathcal{E}_2) \longrightarrow \Phi^1(\mathcal{O}_E) \longrightarrow 0. \end{aligned} \tag{17}$$

From the previous discussion we have reviewed that $\Phi^{\mathcal{P}}(\mathcal{O}_E) = \mathcal{O}_{p_0}[-1]$, so the first row must be zero (i.e. $\Phi^0(\mathcal{E}_2) = 0$), and

$$0 \longrightarrow \mathcal{O}_{p_0} \longrightarrow \Phi^1(\mathcal{E}_2) \longrightarrow \mathcal{O}_{p_0} \longrightarrow 0, \tag{18}$$

but this cannot be a non-trivial extension of the torsion sheaves, and one concludes,

$$\Phi^{\mathcal{P}}(\mathcal{E}_2) = (\mathcal{O}_{p_0} \oplus \mathcal{O}_{p_0})[-1]. \tag{19}$$

Note that $\mathcal{E}_2$ is S-equivalent to $\mathcal{O}_E^{\oplus 2}$ but not equal, however, Fourier-Mukai of both of them is the same.

2.2 Weierstrass elliptic fibration

With the results above in hand for a single elliptic curve, they can now be extended fiber-by-fiber for a smooth elliptic fibration. We begin with the simplest case, that of a smooth Weierstrass elliptic fibration $\pi : X \longrightarrow B$. This fibration admits a holomorphic section $\sigma : B \to X$ and every fiber $X_b = \pi^{-1}(b)$ is integral, and generically smooth for $b \in B$. Note that from here onward we will mainly work with smooth Calabi-Yau threefolds and since there exists an isomorphism, $\hat{X} \simeq X$, we will ignore the distinction between X and its relative Jacobian.

In general, on a fibered space, it is possible to define a relative integral functor Φ in almost the same way it was defined for a trivial fibration i.e. $E \times B$. So for any $\mathcal{E}^\bullet \in D^b(X)$ there exists the following:

$$\begin{array}{ccccc} & & X \times_B X & & \\ & \swarrow^{\pi_1} & \downarrow \rho & \searrow^{\pi_2} & \\ X & & B & & X \end{array}$$

$$\Phi(\mathcal{E}^\bullet) := R\pi_{2*}(\pi_1^*\mathcal{E}^\bullet \otimes^L \mathcal{K}^\bullet), \tag{20}$$

with $X \times_B X$ is the fiber product and the kernel is chosen as $\mathcal{K}^\bullet \in D^b(X \times_B X)$. In the case at hand, the kernel is required to be the “relative” Poincare sheaf,

$$\mathcal{P} := \mathcal{I}_\Delta \otimes \pi_1^*\mathcal{O}_X(\sigma) \otimes \pi_2^*\mathcal{O}_X(\sigma) \otimes \rho^* K_B^*, \tag{21}$$

where $\mathcal{I}_\Delta$ is the ideal sheaf of the relative diagonal divisor,

$$0 \longrightarrow \mathcal{I}_\Delta \longrightarrow \mathcal{O}_{X \times_B X} \longrightarrow \delta_*\mathcal{O}_X \longrightarrow 0,$$

$$\delta : X \hookrightarrow X \times_B X, \tag{22}$$

and K_B is the canonical bundle of the base B which is chosen to make the restriction

$$\mathcal{P}|_{\pi_1^*\sigma_1} \simeq \mathcal{O}_X \tag{23}$$

and similarly for σ_2. From this relative integral functor, it is possible to define “absolute” integral functor with kernel $j_*\mathcal{P}$, where $j : X \times_B X \hookrightarrow X \times X$ is a closed immersion. Note that

$$\Phi(\mathcal{E}^\bullet) \simeq \Phi^{j_*\mathcal{P}}_{X \to X}(\mathcal{E}^\bullet) \tag{24}$$

for any $\mathcal{E}^\bullet$. It can be proved that this kernel is indeed strongly simple, so the corresponding integral functor is fully faithful. Moreover, since X is a smooth Calabi-Yau manifold, it follows that this integral functor is indeed an equivalence, i.e. a Fourier-Mukai functor. It should also be noted that there exist simple formulas for base change compatibility, and it can be readily verified that the restriction of this Fourier Mukai functor over a generic smooth elliptic fiber is the same as the absolute integral functor that was reviewed briefly.

3 Spectral Cover Construction and Direct Images

Heterotic string theory compactified on an elliptic Calabi–Yau 3-fold is dual to F-theory compactified on $K3$-fibered elliptic Calabi–Yau 4-fold. Once massless chiral multiplets are described in Heterotic string theory, the description can be passed on to F-theory, using the duality. In this section, we will review a powerful way to describe them that is known since late 1990's, mainly for the purpose of setting up notations used in this article. Heterotic string theory has an F-theory dual description, if it is compactified on an elliptically fibered manifold. We consider an elliptic fibered Calabi–Yau 3-fold Z

$$\pi_Z : Z \to B_2 \tag{25}$$

over a base 2-fold, so that $\mathcal{N} = 1$ supersymmetry is left in low-energy effective theory below the Kaluza–Klein scale. An elliptic fibration Z over B_2 is given by a Weierstrass equation,

$$y^2 = x^3 + f_0 x + g_0. \tag{26}$$

Here, f_0 and g_0 are sections of line bundles $\mathcal{L}_H^{\otimes 4}$ and $\mathcal{L}_H^{\otimes 6}$ on B_2, respectively, and $\mathcal{L}_H \simeq \mathcal{O}(-K_{B_2})$ for Z to be a Calabi–Yau 3-fold. The coordinates (x, y) transform as sections of $\mathcal{L}_H^{\otimes 2}$ and $\mathcal{L}_H^{\otimes 3}$, respectively. The zero section $\sigma : B_2 \hookrightarrow Z$ maps B_2 to the locus of infinity points, $(x, y) = (\infty, \infty)$.

3.1 Spectral Cover Construction

Compactification of the Heterotic $E_8 \times E_8'$ string theory involves a pair of vector bundles (V_0, V_∞) on a Calabi–Yau 3-fold Z. Spectral cover construction [12, 20, 21] describes vector bundles on an elliptic fibered Calabi-Yau 3-fold Z. Let us consider a rank-N vector bundle V on Z. Spectral surface $C_V \in |N\sigma + \pi_Z^*\eta|$ is a smooth hypersurface of Z that is a degree N cover over B_2, where η is a divisor on B_2. When a line bundle $\mathcal{N}_V$ on C_V is given, a rank-N vector bundle V is given by the Fourier–Mukai transform

$$V = p_{2*}(p_1^*(\mathcal{N}_V) \otimes \mathcal{P}_{B_2}), \tag{27}$$

where $p_{1,2}$ are maps associated with a fiber product

$$\begin{array}{ccccc} & & C_V \times_{B_2} Z & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ C_V & & & & Z \\ & \searrow_{\pi_C} & & \swarrow_{\pi_Z} & \\ & & B_2 & & \end{array} \tag{28}$$

$q := \pi_C \circ p_1 = \pi_Z \circ p_2$, and $\mathcal{P}_{B_2}$ is the Poincaré line bundle $\mathcal{O}_{C_V \times_{B_2} Z}(\Delta - \sigma_1 - \sigma_2 + q^* K_{B_2})$ with $\sigma_1 = \sigma \times Z, \sigma_2 = Z \times \sigma$ and Δ is a diagonal divisor of $Z \times Z$ restricted on $C_V \times_{B_2} Z$. The data $(C_V, \mathcal{N}_V)$, i.e. the spectral surface and a line bundle on it, determines a vector bundle V.

The characteristic classes of vector bundles constructed that way are expressed in terms of spectral data $(C_V, \mathcal{N}_V)$. The first Chern class of the vector bundle V is given by [12]

$$c_1(V) = \pi_Z^* \pi_{C*} \left(c_1(\mathcal{N}_V) - \frac{1}{2} r \right) \tag{29}$$

where $r := \omega_{C/B_2} := K_{C_V} - \pi_C^* K_{B_2}$ is the ramification divisor on C_V of $\pi_C : C_V \to B_2$, and $c_1(V)$ is a pullback of a 2-form on the base 2-fold B_2. With the notation

$$\gamma := c_1(\mathcal{N}_V) - \frac{1}{2} r, \tag{30}$$

we have

$$c_1(V) = \pi_Z^* \pi_{C*} \gamma. \tag{31}$$

We will sometimes use $c_1(V)$ in the sense of $\pi_{C*}\gamma$. The second Chern character is

$$\mathrm{ch}_2(V) = -\sigma \cdot \eta + \pi_Z^* \omega, \tag{32}$$

where ω is some 4-form on B_2 [12].

We do not restrict our attention to cases with vanishing first Chern class $c_1(V)$. By considering vector bundles V whose structure group is $\mathrm{U}(N)$, rather than $\mathrm{SU}(N)$, we will be able to perform a consistency check in calculating $R^1\pi_{Z*}(\wedge^2 V)$ by examining $c_1(V)$ dependence. We maintain our discussion to be valid for $\mathrm{U}(N)$ bundles also because there are some phenomenological motivations to think of Heterotic string compactification with a bundle whose structure group is within $\mathrm{U}(N_1) \times \mathrm{U}(N_2) \subset \mathrm{SU}(5)$ [1].

We now present a few technical remarks about the nature of vector bundles given by spectral cover construction. Such bundles cannot be completely generic $\mathrm{U}(N)$ bundles. For example, the first Chern class $c_1(V) = c_1(\det V)$ is always given by a pullback of a 2-form on B_2 to Z (see (31)). In other words, the first Chern class of $\det V$ is trivial in the fiber direction. This is not a serious limitation when we are analysing Heterotic compactification in an attempt to understand F-theory better. In Heterotic string compactification, vector bundles have to be stable, and the stability condition (Donaldson–Uhlenbeck–Yau equation) is

$$\int_Z c_1(V) \wedge J \wedge J = 0 \tag{33}$$

at tree level, where J is the Kähler form of Z. When the T^2-fiber is small, description in the Heterotic theory becomes less reliable, but a dual F-theory description becomes better. This is the situation we are interested in. In such a limit, the size of T^2-fiber becomes much smaller than that of the base, and the dominant contribution of (33) is from two J's in the two directions along B_2, and $c_1(V)$ in the fiber direction. Thus, the sole dominant contribution has to vanish, and hence stable vector bundles should not have non-vanishing $c_1(V)$ along the fiber direction. Spectral cover construction, therefore, is fine for our purpose in this article, although it cannot describe a bundle with a non-vanishing first Chern class along the T^2-fiber direction.

For $\mathrm{U}(N)$ bundles given by spectral cover construction, $\det V$ are actually trivial along the elliptic fiber direction, not just degree zero. The spectral surface $C_V \hookrightarrow Z$ is (on a local patch of B_2) defined by the zero locus of an equation

$$s = a_0 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + \cdots + a_N \left(x^{N/2} \text{ or } x^{(N-3)/2} y \right) = 0, \tag{34}$$

where a_r are sections of $\mathcal{O}(\eta) \otimes \mathcal{L}_H^{\otimes(-r)} \simeq \mathcal{O}(rK_{B_2} + \eta)$ on B_2. The last term is $x^{N/2}$ or $x^{(N-3)/2}y$ depending on whether N is even or odd. On a given fiber $E_b := \pi_Z^{-1}(b)$, s determines an elliptic function, with N zero points $\{p_i\}_{i=1,\cdots,N}$ (for $\mathrm{U}(N)$ bundles) and a rank-N pole at e_0, zero section σ on E_b. Since the group-law sum of the zero points of an elliptic function is the same as that of the poles,

$$\boxplus_i p_i = e_0, \tag{35}$$

where $\boxplus$ stands for the summation according to the group law of an elliptic curve.

3.2 Direct Images and Matter Curves

If an SU(N) vector bundle V is turned on within one of E_8 gauge group of the Heterotic $E_8 \times E_8'$ string theory, symmetry group is reduced to $H \subset E_8$ that commutes with the SU(N) in effective theory below the Kaluza–Klein scale. The chiral multiplets in low-energy effective theory are identified with $H^1(Z; \rho(V))$. The correspondence between the representations $\rho(V)$ of V and those of the unbroken symmetry group H is summarized.

For a Calabi–Yau 3-fold Z that is an elliptic fibration over a 2-fold B_2, cohomology groups $H^1(Z; \rho(V))$ can be calculated by Leray spectral sequence. One calculates the cohomology in the fiber direction first, $R^i \pi_{Z*} \rho(V)$ $(i = 0, 1)$, and then the cohomology in the base directions. If $R^0 \pi_{Z*} \rho(V)$ vanishes everywhere on B_2, which is often the case, then

$$H^1(Z; \rho(V)) \simeq H^0(B_2; R^1 \pi_{Z*} \rho(V)). \tag{36}$$

If one is interested only in the net chirality, i.e. the difference between the number of chiral multiplets and anti-chiral multiplets in a given representation,

$$\begin{aligned} \chi(\rho(V)) &:= h^1(Z; \rho(V)) - h^1(Z; \rho(V)^\times), \\ &= h^1(Z; \rho(V)) - h^2(Z; \rho(V)), \\ &= -\chi(\rho(V)^\times), \end{aligned} \tag{37}$$

then one has

$$\begin{aligned} \chi(\rho(V)) &= -\chi(Z; \rho(V)), \\ &= -\chi(B_2; R^0 \pi_{Z*} \rho(V)) + \chi(B_2; R^1 \pi_{Z*} \rho(V)), \qquad (38) \\ &\to \chi(B_2; R^1 \pi_{Z*} \rho(V)) \qquad \text{(if } R^0 \pi_{Z*} \rho(V) = 0). \qquad (39) \end{aligned}$$

Suppose that the vector bundle V is given by spectral cover construction from $(C_V, \mathcal{N}_V)$. Let us consider a Fourier–Mukai transform of $\rho(V)$:

$$R^1 p_{1*} \left[p_2^* (\rho(V)) \otimes \mathcal{P}_B^{-1} \otimes \mathcal{O}(-q^* K_{B_2}) \right], \tag{40}$$

which is a sheaf on Z, and p_1 and p_2 here are maps in

$$\begin{array}{ccccc} & & Z \times_{B_2} Z & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ Z & & & & Z \\ & \searrow_{\pi_Z} & & \swarrow_{\pi_Z} & \\ & & B_2 & & \end{array} \tag{41}$$

and $q = \pi_Z \circ p_1 = \pi_Z \circ p_2$. This sheaf is supported only on a codimension-1 subvariety $C_{\rho(V)}$. Unless $C_{\rho(V)}$ contains the zero section σ as an irreducible component, $\rho(V)$ does not contain a trivial bundle when it is restricted on a fiber E_b of a generic point $b \in B_2$. Thus, $R^1\pi_Z{*}\rho(V)$ vanishes on a generic point on B_2; it survives only along a curve

$$\bar{c}_{\rho(V)} = C_{\rho(V)} \cdot \sigma. \tag{42}$$

in B_2. Curves $\bar{c}_{\rho(V)}$ for various representations $\rho(V)$ are called matter curves, because cohomology groups are localized.

The localization of cohomology groups (or matter multiplets that appear in low-energy effective theory) on matter curves is not just an artifact of mathematical calculation. It also has physics meaning. For small elliptic fiber, where F-theory description becomes better, zero modes of Dirac equation in a given representation $\rho(V)$ have Gaussian profile around a locus where Wilson lines in the elliptic fiber directions vanish, just like the case explained for the T^3-fibration in [9]. Localized massless matter multiplets in Heterotic theory description correspond to those on 7-brane intersection curves in Type IIB / F-theory description.

Suppose that the sheaf (40) on Z is given by a pushforward of a sheaf $\mathcal{N}_{\rho(V)}$ on $C_{\rho(V)}$:

$$R^1 p_{1*}\left[p_2^*(\rho(V)) \otimes \mathcal{P}_B^{-1} \otimes \mathcal{O}(-q^* K_{B_2})\right] = i_{C_{\rho(V)}*}(\mathcal{N}_{\rho(V)}), \tag{43}$$

where $i_{C_{\rho(V)}} : C_{\rho(V)} \hookrightarrow Z$. Then, the direct images $R^1\pi_{Z*}\rho(V)$ are given by pushforwards of sheaves on matter curves [16–18]:

$$\begin{aligned} R^1\pi_{Z*}(\rho(V)) &= i_{\rho(V)*}\, \mathcal{F}_{\rho(V)}, &(44)\\ \mathcal{F}_{\rho(V)} &= j^*_{\rho(V)}\mathcal{N}_{\rho(V)} \otimes \mathcal{O}(i^*_{\rho(V)} K_{B_2}); &(45) \end{aligned}$$

here, $i_{\rho(V)} : \bar{c}_{\rho(V)} = \sigma \cdot C_{\rho(V)} \hookrightarrow \sigma \simeq B_2$, and $j_{\rho(V)} : \bar{c}_{\rho(V)} = \sigma \cdot C_{\rho(V)} \hookrightarrow C_{\rho(V)}$. Chiral multiplets in low-energy effective theory are characterized as global holomorphic sections of the sheaves $\mathcal{F}_{\rho(V)}$ on the matter curves:

$$H^1(Z; \rho(V)) \simeq H^0(B_2; R^1\pi_{Z*}\rho(V)) \simeq H^0(\bar{c}_{\rho(V)}; \mathcal{F}_{\rho(V)}). \tag{46}$$

The net chirality (38) is now expressed by Euler characteristic on the matter curve:

$$\chi(\rho(V)) = \chi(B_2; R^1\pi_{Z*}\rho(V)) = \chi(\bar{c}_{\rho(V)}; \mathcal{F}_{\rho(V)}). \tag{47}$$

3.3 Matter from Bundles in the Fundamental Representation

In the above discussion we have assumed that the sheaf (40) on Z is given by a pushforward of a sheaf on $C_{\rho(V)}$. This is actually a highly non-trivial statement. Even if a sheaf $\mathcal{E}$ on an algebraic variety X is supported on a closed subvariety $i_Y : Y \hookrightarrow X$, it is not true in general that $\mathcal{E}$ is a pushforward of a sheaf $\mathcal{F}$ on Y; $\mathcal{E} = i_{Y*}\mathcal{F}$. It is true that $\mathcal{E} = i_{Y*}\mathcal{F}$ for some $\mathcal{F}$ on Y as a sheaf of Abelian group, but not necessarily as a sheaf of $\mathcal{O}_X$-module. Thus, the discussion after (43) is not necessarily applied immediately for bundles in any representation.

For bundles V in the fundamental representation, however, their Fourier–Mukai transforms are pushforward of the original line bundles $\mathcal{N}_V$. Thus, the discussion all the way down to (47) is applicable. The matter curves $\bar{c}_V = C_V \cdot \sigma$ belong to a topological class

$$\bar{c}_V \in |NK_{B_2} + \eta| \tag{48}$$

because $C_V \in |N\sigma + \pi_Z^*\eta|$, and $\sigma \cdot \sigma = -\sigma \cdot c_1(\mathcal{L}_H) = \sigma \cdot K_{B_2}$ [12].

$R^1\pi_{Z*}V$ is given by a pushforward of a sheaf on $\bar{c}_V$

$$\mathcal{F}_V = j_V^*\mathcal{N} \otimes i_V^*\mathcal{O}(K_{B_2}) = \mathcal{O}\left(i_V^*K_{B_2} + \frac{1}{2}j_V^*r + j_V^*\gamma\right) \tag{49}$$

as a sheaf of $\mathcal{O}_{B_2}$-module. Since the canonical divisor K_{C_V} is also the divisor $C_V|_{\bar{c}_V}$ in a Calabi–Yau 3-fold,

$$\begin{aligned} i_V^*K_{B_2} + \frac{1}{2}j_V^*r &= i_V^*K_{B_2} + \frac{1}{2}j_V^*\left(K_{C_V} - \pi_C^*K_{B_2}\right) = \frac{1}{2}\left(i_V^*K_{B_2} + C_V|_{\bar{c}_V}\right) \\ &= \frac{1}{2}\left(i_V^*K_{B_2} + N_{\bar{c}_V|B_2}\right) = \frac{1}{2}K_{\bar{c}_V}, \end{aligned} \tag{50}$$

where adjunction formula was used for $i_V : \bar{c}_V \hookrightarrow B_2$ [16]. Thus, the sheaf can be rewritten as

$$\mathcal{F}_V = \mathcal{O}\left(\frac{1}{2}K_{\bar{c}_V} + j_V^*\gamma\right), \tag{51}$$

$$\mathcal{F}_{V^\times} = \mathcal{O}\left(\frac{1}{2}K_{\bar{c}_V} - j_V^*\gamma\right); \tag{52}$$

here we determined $\mathcal{F}_{V^\times}$ by replacing γ by $-\gamma$ [17]. It is easy to see that these sheaves satisfy

$$\mathcal{F}_{V^\times} = \mathcal{O}(K_{\bar{c}_V}) \otimes \mathcal{F}_V^{-1}. \tag{53}$$

Massless chiral multiplets from the bundles V and $V^\times$ are now given by independent global holomorphic sections of $\mathcal{F}_V$ and $\mathcal{F}_{V^\times}$, respectively. If one is interested only in the difference between the numbers of those chiral multiplets, the net chirality is obtained by Riemann–Roch theorem [16, 17]:

$$\chi(V) = \chi(\bar{c}_V; \mathcal{F}_V), \tag{54}$$

$$= [1 - g(\bar{c}_V)] + \deg\left(K_{B_2} + \frac{1}{2}j_V^*r\right) + \int_{\bar{c}_V} j_V^*\gamma, \tag{55}$$

$$= [1 - g(\bar{c}_V)] + \frac{1}{2}\deg K_{\bar{c}_V} + \int_{\bar{c}_V} j_V^*\gamma, \tag{56}$$

$$= \int_{\bar{c}_V} j_V^*\gamma = \bar{c}_V \cdot \gamma. \tag{57}$$

It is reasonable that the final result is proportional to γ, because we know that $\chi(V) = -\chi(V^\times)$, and $V \leftrightarrow V^\times$ corresponds to $\gamma \leftrightarrow -\gamma$ and $\mathcal{P}_B \leftrightarrow \mathcal{P}_B^{-1}$ [17].

4 Bundles Trivial in the Fiber Direction

In this section we briefly discuss the cohomology groups $H^i(Z;\pi_Z^* E)$, where Z is an elliptic fibration $\pi_Z : Z \to B_2$, and we consider a bundle given by a pullback of a bundle E on B_2. Bundles given by π_Z^* are trivial in the fiber direction, and hence $R^0\pi_{Z*}(\pi_Z^* E)$ on B_2 does not vanish, and $R^1\pi_{Z*}(\pi_Z^* E)$ is not supported on a curve in B_2, either. Thus, the cohomology groups of the bundles $\pi_Z^* E$ are not described in the same way as those of such bundles as V, $\wedge^2 V$ and $\wedge^3 V$. We need to express $H^i(Z;\pi_Z^* E)$ $(i = 1, 2)$ in terms of cohomology groups of $R^p\pi_{Z*}(\pi_Z^* E)$ $(p = 0, 1)$, so that those expressions are interpreted in F-theory.

This issue has been discussed in the footnote 13 of [14]. Here, we add a minor comment to the description given there.

First, note that

$$R^0\pi_{Z*}(\pi_Z^* E) \simeq E, \tag{58}$$
$$R^1\pi_{Z*}(\pi_Z^* E) \simeq E \otimes \mathcal{L}_H^{-1} \simeq E \otimes \mathcal{O}(K_{B_2}), \tag{59}$$

where the Calabi–Yau condition of $\pi_Z : Z \to B_2$ is used in the last equality. Thus,

$$H^0(Z;\pi_Z^* E) \simeq H^0(B_2;E), \tag{60}$$
$$\left[H^0(Z;\pi_Z^* E^\times)\right]^\times \simeq H^3(Z;\pi_Z^* E) \simeq H^2(B_2;E\otimes\mathcal{O}(K_{B_2})) \simeq [H^0(B_2;E^\times)]^\times. \tag{61}$$

Since these cohomology groups correspond to massless gauginos at low energy, one can assume that those groups are trivial when one is concerned with matter multiplets. Using the spectral sequence, one can see that the two other cohomology groups $H^r(Z;\pi_Z^* E)$ $(r = 1, 2)$ satisfy

$$0 \to H^1(B_2;E) \to H^1(Z;\pi_Z^* E) \to H^0(B_2;E\otimes\mathcal{O}(K_{B_2})) \to H^2(B_2;E), \tag{62}$$
$$H^0(B_2;E\otimes\mathcal{O}(K_{B_2})) \to H^2(B_2;E) \to H^2(Z;\pi_Z^* E) \to H^1(B_2;E\otimes\mathcal{O}(K_{B_2})) \to 0. \tag{63}$$

In the spectral sequence calculation of cohomology groups, $E_2^{p,q} = H^p(B_2;R^q\pi_{Z*}\pi_Z^* E)$, and $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$ for $(p,q) = (0,1)$ determines the map

$$d_2 : H^0(B_2;E\otimes\mathcal{O}(K_{B_2})) \to H^2(B_2;E) \simeq \left[H^0(B_2;E^\times\otimes\mathcal{O}(K_{B_2}))\right]^\times \tag{64}$$

used in (62, 63).

It thus follows that

$$h^1(Z;\pi_Z^* E) = h^1(B_2;E) + \ker d_2, \tag{65}$$
$$h^2(Z;\pi_Z^* E) = h^1(B_2;E\otimes\mathcal{O}(K_{B_2})) + \text{coker } d_2, \tag{66}$$
$$= h^1(B_2;E^\times) + \text{coker } d_2, \tag{67}$$

where d_2 is the one in (64). If d_2 is trivial (including cases where either $h^0(B_2; E \otimes \mathcal{O}(K_{B_2})) = 0$ or $h^0(B_2; E^\times \otimes \mathcal{O}(K_{B_2})) = 0$), the results in [14] follow:

$$
\begin{aligned}
h^1(Z; \pi_Z^* E) &= h^1(B_2; E) + h^0(B_2; E \otimes \mathcal{O}(K_{B_2})), && (68)\\
h^1(Z; \pi_Z^* E^\times) = h^2(Z; \pi_Z^* E) &= h^1(B_2; E \otimes \mathcal{O}(K_{B_2})) + h^2(B_2; E), && (69)\\
&= h^1(B_2; E^\times) + h^0(B_2; E^\times \otimes \mathcal{O}(K_{B_2})). && (70)
\end{aligned}
$$

For a general d_2, (65, 67) are the right expressions for the number of massless matter multiplets from $\pi_Z^* E$. This means that some degrees of freedom in $H^0(B_2; E \otimes \mathcal{O}(K_{B_2}))$ and $H^0(B_2; E^\times \otimes \mathcal{O}(K_{B_2}))$ are paired up and do not remain in the low-energy spectrum. One might phrase this phenomenon as those degrees of freedom having "masses." It should be noted that all the degrees of freedom in $H^1(B_2; E)$ and $H^1(B_2; E^\times)$ do not have such "masses." We do not study the detail of the map d_2 based on explicit examples. Such "masses" may be understood as a kind of obstruction in geometry. We leave these interesting questions as open problems for the future.

The structure group of a bundle E can be chosen so that the unbroken symmetry H is reduced to whatever one likes, say $\mathrm{SU}(5)_{\mathrm{GUT}}$ or $\mathrm{SU}(3) \times \mathrm{SU}(2)$. The irreducible decomposition of $\mathbf{adj}.H$ under the structure group of E and the true unbroken symmetry may contain a pair of vector-like representations, $(\rho(E), repr.)$–$(\rho(E)^\times, repr.^\times)$. For such a pair, the net chirality is calculated by

$$
\begin{aligned}
\chi(\rho(E)) &:= h^1(Z; \pi_Z^* \rho(E)) - h^1(Z; \pi_Z^* \rho(E)^\times), && (71)\\
&= -\chi(Z; \pi_Z^* \rho(E)), && (72)\\
&= -\chi(B_2; \rho(E)) + \chi(B_2; \rho(E) \otimes \mathcal{O}(K_{B_2})), && (73)\\
&= -\int_{B_2} c_1(TB_2) \wedge c_1(\rho(E)). && (74)
\end{aligned}
$$

Rank of the map d_2 in (64) does not matter here.

The chirality formula (74) can also be obtained from the discussion reviewed in the previous section [1]. The bundle $\pi_Z^* \rho(E)$ is regarded as a Fourier–Mukai transform of $(C_{\rho(E)}, \mathcal{N}_{\rho(E)}) = (\sigma, \rho(E))$. Thus, the matter curve is formally given by $C_{\rho(E)} \cdot \sigma$ which belongs to a class of K_{B_2}. Since the ramification divisor of $\pi_C : C \to B_2$ is trivial, one finds (i) from the argument in (50) that K_{B_2} is half the canonical divisor of the "matter curve" $\bar{c}_{\rho(E)} \sim K_{B_2}$ in B_2, and (ii) that $\mathcal{N}_{\rho(E)} \otimes \mathcal{O}(r/2)^{-1} = \rho(E)$. Therefore,

$$
\chi(\rho(E)) = \int_{K_{B_2}} c_1(E) = -\int_{B_2} c_1(TB_2) \wedge c_1(\rho(E)), \qquad (75)
$$

reproducing (74).

5 Analysis of $R^1 \pi_{Z*} \wedge^2 V$

Not all the chiral multiplets in low-energy effective theory are identified with cohomology groups of bundle V (or $V^\times$) in the fundamental (anti-fundamental) representation. In order to obtain description of all kinds of matter multiplets in F-theory, we also need to determine the sheaves

$R^1\pi_{Z*}\rho(V)$ for bundles associated with $\rho(V) = \wedge^2 V$ and $\wedge^3 V$. As we have emphasized in Introduction, the Higgs multiplets and $\bar{\mathbf{5}} = (\bar{D}, L)$ in the $\mathrm{SU}(5)_{\mathrm{GUT}}$-$\mathbf{5}$+$\bar{\mathbf{5}}$ representations originate from the bundle $\wedge^2 V$, and the Higgs multiplet in the SO(10)-$\mathbf{10}$ = **vec**. representation from $\wedge^2 V$. Thus, it is important to determine $R^1\pi_{Z*}\wedge^2 V$ in order to understand Yukawa couplings of quarks and leptons in F-theory language. For the bundles V (or $V^\times$), the generic element of a topological class of spectral surface $|N\sigma + \pi_Z^*\eta|$ is smooth, and the transverse coordinate of C_V in Z can be chosen at any points on C_V. This property can be used to show that the Fourier–Mukai transform of V is given by a pushforward of a sheaf on C_V as a sheaf of $\mathcal{O}_Z$ module. Furthermore, the rank of fiber of the Fourier–Mukai transform never jumps on C_V and the sheaf is the locally-free rank-1 sheaf $\mathcal{N}_V$ itself. For the bundles $\wedge^2 V$ (or $\wedge^2 V^\times$), on the other hand, $C_{\wedge^2 V}$ is not necessarily smooth, even if C_V is. Here, we denote by $C_{\wedge^2 V}$ the support of Fourier–Mukai transform (40) of $\rho(V) = \wedge^2 V$. Suppose that $C_V|_{E_b}$ for a point $b \in B_2$ consists of N points $\{p_i\}_{i=1,\cdots N}$. Then, $C_{\wedge^2 V}|_{E_b}$ is given by $\{p_i \boxplus p_j\}_{1\leq i<j\leq N}$. At a generic point $b \in B_2$, the $N(N-1)/2$ points $p_i \boxplus p_j$ $(i < j)$ in elliptic fiber E_b are all different, and $C_{\wedge^2 V}$ is a smooth degree $N(N-1)/2$ cover. For these points, the arguments can be used to show that there a locally free rank-1 sheaf $\mathcal{N}_{\wedge^2 V}$ exists on $C_{\wedge^2 V}$ (locally around smooth points in $C_{\wedge^2 V}$), and the Fourier–Mukai transform of $\wedge^2 V$ is represented as the pushforward of $\mathcal{N}_{\wedge^2 V}$ as a sheaf of $\mathcal{O}_Z$-module. But, on a codimension-1 locus of $C_{\wedge^2 V}$, $C_{\wedge^2 V}$ may become singular [10], and a little more attention must be paid. We will describe a rough sketch of how to determine $R^1\pi_{Z*}\wedge^2 V$ in this section, beginning with how to deal with such singularities. Details of $R^1\pi_{Z*}\wedge^2 V$ are deferred to the next section. Since some crucial aspects of $R^1\pi_{Z*}\wedge^2 V$ depend very much on the rank of V, we will provide detailed description of $R^1\pi_{Z*}\wedge^2 V$ for the rank of V between 2 and 6 in the next section. Once we see how to deal with $R^1\pi_{Z*}\wedge^2 V$, it is rather straightforward to find how to deal with $R^1\pi_{Z*}\wedge^3 V$.

5.1 Resolving Double-Curve Singularity of $C_{\wedge^2 V}$

$C_{\wedge^2 V}$ is described locally as $N(N-1)/2$ surfaces that $p_i \boxplus p_j$ $(i < j)$ scan. $C_{\wedge^2 V}$ has a double-curve singularity if $p_i \boxplus p_j$ $(i < j)$ and $p_k \boxplus p_l$ $(k < l,\ \{i,j\}\cap\{k,l\} = \phi)$ become equal. It is not obvious how to choose a coordinate in Z that is normal to $C_{\wedge^2 V}$ along the double-curve locus, and the argument is not readily applicable.

In a local neighborhood of the double curve, $C_{\wedge^2 V}$ consists of two irreducible components, $C_{(ij)}$ and $C_{(kl)}$, and their intersection is the double-curve singularity. $C_{(ij)}$ and $C_{(kl)}$ are surfaces scanned in Z by $p_i \boxplus p_j$ and $p_k \boxplus p_l$. $\rho(V) = \wedge^2 V$ can be regarded locally as direct sum of $\mathcal{O}(C_{(ij)} - \sigma)$, $\mathcal{O}(C_{(kl)} - \sigma)$ and others. Its Fourier–Mukai transform in (40) is given by a sum of the above two summands. The Fourier–Mukai transform of the two summands $\mathcal{O}(C_{(ij)} - \sigma)$ and $\mathcal{O}(C_{(kl)} - \sigma)$ is expressed locally as

$$R^1 p_{1*}\left[\mathcal{O}(C_{(ij)} - \sigma) \otimes \mathcal{P}_B^{-1} \otimes \mathcal{O}(-q^* K_{B_2})\right] = i_{C_{(ij)}*}\mathcal{O}_{C_{(ij)}}, \tag{76}$$

$$R^1 p_{1*}\left[\mathcal{O}(C_{(kl)} - \sigma) \otimes \mathcal{P}_B^{-1} \otimes \mathcal{O}(-q^* K_{B_2})\right] = i_{C_{(kl)}*}\mathcal{O}_{C_{(kl)}}. \tag{77}$$

Here, $i_{C_{\wedge^2 V}} : C_{\wedge^2 V} \hookrightarrow Z$ (which is different from previously defined $i_{\wedge^2 V} : \bar{c}_{\wedge^2 V} \hookrightarrow \sigma$), and

$$\nu_{C_{ij}} : C_{(ij)} \hookrightarrow C_{\wedge^2 V}, \qquad i_{C_{(ij)}} = i_{C_{\wedge^2 V}} \circ \nu_{C_{ij}}, \tag{78}$$

$$\nu_{C_{kl}} : C_{(kl)} \hookrightarrow C_{\wedge^2 V}, \qquad i_{C_{(kl)}} = i_{C_{\wedge^2 V}} \circ \nu_{C_{kl}}. \tag{79}$$

Therefore, the Fourier–Mukai transform of $\wedge^2 V$ is

$$R^1 p_{1*} \left[p_2^*(\wedge^2 V) \otimes \mathcal{P}_B^{-1} \otimes \mathcal{O}(-q^* K_{B_2}) \right] \simeq i_{C_{\wedge^2 V}*} \left(\nu_{C_{ij}*} \mathcal{O}_{C_{(ij)}} \oplus \nu_{C_{kl}*} \mathcal{O}_{C_{(kl)}} \right) \tag{80}$$

locally along a double-curve singularity. Thus, it is given by a pushforward of a sheaf $\mathcal{N}_{\wedge^2 V}$ on $C_{\wedge^2 V}$ as a sheaf of $\mathcal{O}_Z$-module. The sheaf $\mathcal{N}_{\wedge^2 V}$ is the object inside the parenthesis on the right hand side.

The sheaf $\mathcal{N}_{\wedge^2 V}$ is not locally free along the double-curve singularity. The rank of fiber jumps up there. But we already know that the sheaf $\mathcal{N}_{\wedge^2 V}$ is given by a pushforward of locally-free rank-1 sheaf via

$$\nu_{C_{\wedge^2 V}} : \widetilde{C}_{\wedge^2 V} = C_{(ij)} \coprod C_{(kl)} \to C_{(ij)} \cup C_{(kl)} = C_{\wedge^2 V}. \tag{81}$$

The map $\nu_{C_{\wedge^2 V}}$ is determined by $\nu_{C_{ij}} \coprod \nu_{C_{kl}}$. Note that $\widetilde{C}_{\wedge^2 V} := C_{(ij)} \coprod C_{(kl)}$ is the resolution of double-curve singularity in $C_{\wedge^2 V}$. Therefore, the discussion so far means that there exists a locally free rank-1 sheaf $\widetilde{\mathcal{N}}_{\wedge^2 V}$ on the resolved $\widetilde{C}_{\wedge^2 V}$ such that

$$\mathcal{N}_{\wedge^2 V} = \nu_{C_{\wedge^2 V}*} \widetilde{\mathcal{N}}_{\wedge^2 V}. \tag{82}$$

We have seen that the sheaf $\mathcal{N}_{\wedge^2 V}$ exists on $C_{\wedge^2 V}$ and (43) is satisfied as a sheaf of $\mathcal{O}_Z$ module. Thus, the discussion around equations (43–47) is applied for the bundles $\rho(V) = \wedge^2 V$ and $\wedge^2 V^\times$ as well. In particular, the sheaf on the matter curve $\bar{c}_{\wedge^2 V}$ is given by

$$\mathcal{F}_{\wedge^2 V} = j_{\wedge^2 V}^* \mathcal{N}_{\wedge^2 V} \otimes i_{\wedge^2 V}^* \mathcal{O}(K_{B_2}). \tag{83}$$

We introduce the notion of covering matter curve, which turns out to be very important in characterizing matter multiplets in F-theory. The covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$ is defined as the set-theoretic inverse image of the matter curve $\bar{c}_{\wedge^2 V}$ in $\widetilde{C}_{\wedge^2 V}$. That is, $\tilde{\bar{c}}_{\wedge^2 V} := \nu_{C_{\wedge^2 V}}^{-1}(\bar{c}_{\wedge^2 V})$. Since the matter curve $\bar{c}_{\wedge^2 V}$ is also regarded as a divisor $\sigma|_{C_{\wedge^2 V}}$ in $C_{\wedge^2 V}$, the covering matter curve is also regarded as a divisor $\nu_{C_{\wedge^2 V}}^*(\sigma)$ on $\widetilde{C}_{\wedge^2 V}$. Using a locally rank-1 sheaf $\widetilde{\mathcal{N}}_{\wedge^2 V}$ on $\widetilde{C}_{\wedge^2 V}$, a locally free rank-1 sheaf $\widetilde{\mathcal{F}}_{\wedge^2 V}$ can be defined on the covering matter curve:

$$\widetilde{\mathcal{F}}_{\wedge^2 V} = \tilde{j}_{\wedge^2 V}^* \widetilde{\mathcal{N}}_{\wedge^2 V} \otimes \tilde{i}_{\wedge^2 V}^* \mathcal{O}(K_{B_2}), \tag{84}$$

where $\tilde{j}_{\wedge^2 V} : \tilde{\bar{c}}_{\wedge^2 V} \hookrightarrow \widetilde{C}_{\wedge^2 V}$, $\nu_{\bar{c}_{\wedge^2 V}} := \nu_{C_{\wedge^2 V}}|_{\tilde{\bar{c}}_{\wedge^2 V}}$, and $\tilde{i}_{\wedge^2 V} := i_{\wedge^2 V} \circ \nu_{\bar{c}_{\wedge^2 V}} : \tilde{\bar{c}}_{\wedge^2 V} \hookrightarrow \sigma \simeq B_2$. The sheaf $\mathcal{F}_{\wedge^2 V}$ on the matter curve $\bar{c}_{\wedge^2 V}$ is given by

$$\mathcal{F}_{\wedge^2 V} = \nu_{\bar{c}_{\wedge^2 V}*} \widetilde{\mathcal{F}}_{\wedge^2 V}. \tag{85}$$

Although we have dealt with double-curve singularities on $C_{\wedge^2 V}$, there can still be other types of singularities on $C_{\wedge^2 V}$. For example, there may be codimension-2 singularities on $C_{\wedge^2 V}$. Thus, the argument in section 5.1 is not regarded as a complete proof of the existence of $\mathcal{N}_{\wedge^2 V}$

or the existence of $\widetilde{\mathcal{N}}_{\wedge^2 V}$ and its locally-free rank-1 nature. For practical purposes, however, we only need to know $R^1\pi_{Z*} \wedge^2 V$ along the matter curves. Codimension-1 singularities such as double curve on $C_{\wedge^2 V}$ may be encountered somewhere along the matter curve $\bar{c}_{\wedge^2 V}$ [10], but codimension-2 singularities of $C_{\wedge^2 V}$ are seldom exactly on the matter curve. Thus, an analysis of codimension-2 singularities of $C_{\wedge^2 V}$ is not required for the generic case. We will see, however, that codimension-2 singularities inevitably show up on the matter curve $\bar{c}_{\wedge^2 V}$ when rank $V = 4, 6$. We will deal with such exceptional cases separately.

5.2 Determining $\widetilde{\mathcal{N}}_{\wedge^2 V}$ in Terms of $\mathcal{N}_V$

Even after we find that a sheaf $\mathcal{N}_{\wedge^2 V}$ exists and (43) is satisfied as a sheaf of $\mathcal{O}_Z$ module, we still face a theoretical challenge. How is $\mathcal{N}_{\wedge^2 V}$ (or $\widetilde{\mathcal{N}}_{\wedge^2 V}$) expressed in terms of the original spectral data $(C_V, \mathcal{N}_V)$? Pioneering work was done in [18,19]. Our presentation in the following is basically along their idea,but we introduce a little modification for a couple of reasons. First, we will obtain sheaves $\tilde{\mathcal{N}}_{\wedge^2 V}$ and $\widetilde{\mathcal{F}}_{\wedge^2 V}$ on the covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$, instead of $\mathcal{F}_{\wedge^2 V}$ on the matter curve $\bar{c}_{\wedge^2 V}$. By doing so, much clearer description of the direct image $R^1\pi_{Z*} \wedge^2 V$ is obtained. The other reason for modification is that we are not assuming that $\mathcal{N}_V|_D$ is invariant under τ that flips the sign of the coordinate y. D is a curve on C_V; we will explain it later.

Since (43) for $\rho(V) = \wedge^2 V$ is the definition of $\mathcal{N}_{\wedge^2 V}$, it follows that

$$\mathcal{N}_{\wedge^2 V} = i^*_{C_{\wedge^2 V}} R^1 p_{1*}\left[p_2^*(\wedge^2 V) \otimes \mathcal{P}_{B_2}^{-1} \otimes \mathcal{O}(-q^* K_{B_2})\right]. \tag{86}$$

What we really need is its restriction on $\bar{c}_{\wedge^2 V}$, and hence

$$\begin{aligned} \mathcal{F}_{\wedge^2 V} &= \mathcal{N}_{\wedge^2 V}|_{\bar{c}_{\wedge^2 V}} \otimes i^*_{\wedge^2 V}\mathcal{O}(K_{B_2}), \\ &= (i_{C_{\wedge^2 V}} \circ j_{\wedge^2 V})^* R^1 p_{1*}\left[p_2^*(\wedge^2 V) \otimes \mathcal{P}_{B_2}^{-1} \otimes \mathcal{O}(-q^* K_{B_2})\right] \otimes i^*_{\wedge^2 V}\mathcal{O}(K_{B_2}), \\ &= (i_{C_{\wedge^2 V}} \circ j_{\wedge^2 V})^* R^1 p_{1*}\left[p_2^*(\wedge^2 V)\right], \\ &= R^1 p_{1Y*}\left[\wedge^2 V|_Y\right]; \end{aligned} \tag{87}$$

here, $Y := \bar{c}_{\wedge^2 V} \times_{B_2} Z = \pi_Z^{-1}(\bar{c}_{\wedge^2 V})$. In the third equality, we used the property that $\mathcal{P}_{B_2}$ is trivial when it is restricted to a zero section [12], and in the last equality the base change theorem associated with the commutative diagram

$$\begin{array}{ccccc} Y := \bar{c}_{\wedge^2 V} \times_{B_2} Z & \hookrightarrow & & & Z \times_{B_2} Z \\ \downarrow p_{1Y} & & & & \downarrow p_1 \\ \bar{c}_{\wedge^2 V} & \overset{j_{\wedge^2 V}}{\hookrightarrow} & C_{\wedge^2 V} & \overset{i_{C_{\wedge^2 V}}}{\hookrightarrow} & Z \end{array} . \tag{88}$$

This is the standard procedure used in [16–18].

The rank-N bundle $V|_Y$ is given by a Fourier–Mukai transform of $\mathcal{N}_V|_{C_V \cdot Y}$:

$$\begin{array}{ccccc} & & (C_V \cdot Y) \times_{\bar{c}_{\wedge^2 V}} Y & & \\ & \swarrow p_1 & & p_2 \searrow & \\ C_V \cdot Y & & & & Y, \\ & \pi_C|_{C_V \cdot Y} \searrow & & \swarrow \pi_Y & \\ & & \bar{c}_{\wedge^2 V} & & \end{array} \qquad V|_Y = p_{2*}\left(\mathcal{P}_{B_2} \otimes p_1^*(\mathcal{N}_V|_{C_V \cdot Y})\right). \tag{89}$$

The spectral curve $C_V \cdot Y$ is a degree-N cover over $\bar{c}_{\wedge^2 V}$.

Let $C_V|_{E_b}$ be a collection of N points $\{p_i\}_{i=1,\cdots,N}$. For a point $b \in \bar{c}_{\wedge^2 V} \subset C_{\wedge^2 V}$, some pairs of the N points, e.g., p_k and p_l, satisfy $p_k \boxplus p_l = e_0$. Such points in $C_V \cdot Y$ form a curve D, and others form a curve D'.

$$C_V \cdot Y = D + D'. \tag{90}$$

By the definition of D, the following diagram commutes [19],

$$\begin{array}{ccc} D & \xrightarrow{\tilde{\pi}_D} & \tilde{\bar{c}}_{\wedge^2 V} \\ & \searrow^{\pi_D} & \downarrow \nu_{\bar{c}_{\wedge^2 V}} \\ & & \bar{c}_{\wedge^2 V} \end{array} \tag{91}$$

and $\tilde{\pi}_D$ is a degree-2 cover, and π_D is a restriction of π_C on D. If $b \in \bar{c}_{\wedge^2 V} \hookrightarrow \sigma$ is on the double-curve singularity of $C_{\wedge^2 V}$, then there are four points $p_{i,j,k,l}$, satisfying $p_i \boxplus p_j = e_0$ and $p_k \boxplus p_l = e_0$. In the covering matter curve, the inverse image of b, that is, $\nu^{-1}_{\bar{c}_{\wedge^2 V}}(b)$, consists of two points. Two points $p_{i,j} \in D$ are mapped by $\tilde{\pi}_D$ to one of the two points in $\nu^{-1}_{\bar{c}_{\wedge^2 V}}(b)$, and $p_{k,l} \in D$ to the other. Although all the four points are mapped to $b \in \bar{c}_{\wedge^2 V}$ by π_D, $\tilde{\pi}_D$ remains strictly a degree-2 cover everywhere on $\tilde{\bar{c}}_{\wedge^2 V}$.

The Fourier–Mukai transform of $\mathcal{N}_V|_D$ on a degree-2 cover spectral curve $\tilde{\pi}_D : D \to \tilde{\bar{c}}_{\wedge^2 V}$ gives a rank-2 bundle W_2:

$$\begin{array}{ccccc} & & D \times_{\tilde{\bar{c}}_{\wedge^2 V}} \tilde{Y} & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ D & & & & \tilde{Y} \\ & \searrow_{\tilde{\pi}_D} & & \swarrow_{p_{1\tilde{Y}}} & \\ & & \tilde{\bar{c}}_{\wedge^2 V} & & \end{array}, \qquad W_2 = p_{2*}\left(\mathcal{P}_B|_{D \times_{\tilde{\bar{c}}_{\wedge^2 V}} Y} \otimes p_1^*(\mathcal{N}_V|_D)\right), \tag{92}$$

where $\tilde{Y} := \tilde{\bar{c}}_{\wedge^2 V} \times_{\bar{c}_{\wedge^2 V}} Y$. The pushforward of this rank-2 bundle W_2 through projection $\nu_Y : \tilde{Y} = \tilde{\bar{c}}_{\wedge^2 V} \times_{\bar{c}_{\wedge^2 V}} Y \to Y$ defines a subsheaf of $V|_Y$.

For a point $b \in \bar{c}_{\wedge^2 V} \subset C_{\wedge^2 V}$ that is not on the double-curve locus, $H^1(E_b; \wedge^2 V|_{E_b})$ comes from $H^1(E_b; \wedge^2(\nu_{Y*}W_2)|_{E_b}) = H^1(E_b; \wedge^2 W_2|_{E_b})$. For a point $b \in \bar{c}_{\wedge^2 V}$ on the double-curve singularity of $C_{\wedge^2 V}$, however, there are two independent contributions corresponding to $H^1(E_b; \wedge^2 W_2|_{E_{\tilde{b}}})$ for two points $\tilde{b} \in \nu^{-1}_{\bar{c}_{\wedge^2 V}}(b)$. We introduced the covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$ in order to resolve these two contributions. The locally free rank-1 sheaf $\widetilde{\mathcal{N}}_{\wedge^2 V}|_{\tilde{\bar{c}}_{\wedge^2 V}}$ (and $\widetilde{\mathcal{F}}_{\wedge^2 V}$, consequently) is obtained by assigning them to the corresponding two points $\tilde{b}$ on $\tilde{\bar{c}}_{\wedge^2 V}$. Therefore,

$$\begin{aligned} \widetilde{\mathcal{F}}_{\wedge^2 V} &= \widetilde{\mathcal{N}}_{\wedge^2 V}|_{\tilde{\bar{c}}_{\wedge^2 V}} \otimes \tilde{i}^*_{\wedge^2 V}\mathcal{O}(K_{B_2}), \\ &= R^1 p_{1\tilde{Y}*}\left[\wedge^2 W_2\right]. \end{aligned} \tag{93}$$

The line bundle $\wedge^2 W_2$ is trivial in the fiber direction of $p_{1\tilde{Y}}$. Thus, it is regarded as a Fourier–Mukai transform of $(C_{\wedge^2 W_2}, \mathcal{N}_{\wedge^2 W_2}) = (\sigma, \widetilde{\mathcal{N}}_{\wedge^2 V}|_{\tilde{\bar{c}}_{\wedge^2 V}})$. It then follows that

$$\wedge^2 W_2 = p^*_{1\tilde{Y}}(\mathcal{N}_{\wedge^2 W_2}). \tag{94}$$

Thus,

$$\widetilde{\mathcal{F}}_{\wedge^2 V} = \mathcal{N}_{\wedge^2 W_2} \otimes \mathcal{L}_H^{-1} = \mathcal{N}_{\wedge^2 W_2} \otimes \tilde{\imath}^*_{\wedge^2 V} \mathcal{O}(K_{B_2}). \tag{95}$$

Now it is useful to remember that the first Chern class of the line bundle $\wedge^2 W_2$ is

$$\begin{aligned} c_1(\wedge^2 W_2) = c_1(W_2) &= p^*_{1\tilde{Y}} \tilde{\pi}_{D*} \left(c_1(\mathcal{N}_V|_D) - \frac{1}{2} R \right), & (96) \\ &= p^*_{1\tilde{Y}} \tilde{\pi}_{D*} \left(\gamma|_D + \frac{1}{2}(r|_D - R) \right), & (97) \end{aligned}$$

just like in (29). Here, $R := K_D - \tilde{\pi}^*_D K_{\tilde{\bar{c}}_{\wedge^2 V}}$ is the ramification divisor on D associated with the projection $\tilde{\pi}_D : D \to \tilde{\bar{c}}_{\wedge^2 V}$. Thus, by dropping $p^*_{1\tilde{Y}}$ from (94) and (97),

$$\widetilde{\mathcal{F}}_{\wedge^2 V} = \mathcal{N}_{\wedge^2 W_2} \otimes \tilde{\imath}^*_{\wedge^2 V} \mathcal{O}(K_{B_2}) = \mathcal{O}\left(\tilde{\imath}^*_{\wedge^2 V} K_{B_2} + \tilde{\pi}_{D*} \left(\gamma|_D + \frac{1}{2}(r|_D - R) \right) \right). \tag{98}$$

Since $\wedge^2 V = V^\times \otimes \det V$, and $\det V = \pi_Z^* \mathcal{O}(\pi_{C*}\gamma)$, it is straightforward to obtain the sheaf $\mathcal{F}_{\wedge^2 V}$ on the matter curve $\bar{c}_{\wedge^2 V} = \bar{c}_V$. Applying $\otimes \mathcal{O}(\pm \pi_{C*}\gamma)$ to $\mathcal{F}_{V^\times}$ and $\mathcal{F}_V$,

$$\begin{aligned} \mathcal{F}_{\wedge^2 V} &= \mathcal{O}\left(i^* K_{B_2} + \frac{1}{2} j^* r - j^* \gamma + i^* \pi_{C*}\gamma \right), & (99) \\ \mathcal{F}_{\wedge^2 V^\times} &= \mathcal{O}\left(i^* K_{B_2} + \frac{1}{2} j^* r + j^* \gamma - i^* \pi_{C*}\gamma \right), & (100) \end{aligned}$$

where $i : \bar{c}_{\wedge^2 V} = \bar{c}_V \hookrightarrow B_2$, and $j : \bar{c}_{\wedge^2 V} = \bar{c}_V \hookrightarrow C_V$. It is thus unnecessary to use the idea presented in section 5 in determining the $\mathcal{F}_{\wedge^2 V}$ for rank-3 bundles V. In the rest of section **??**, however, we use the idea to reproduce this result, so that we get accustomed to using the idea in practice.

In the fiber E_b of an arbitrary point $b \in \bar{c}_V = \bar{c}_{\wedge^2 V} \subset B_2$, $C_V|_{E_b}$ consists of three points, one in the zero section $p_i = e_0 = \sigma \cdot E_b$ and two others satisfying $p_j \boxplus p_k = e_0$. Thus, the irreducible decomposition (90) becomes

$$C_V \cdot Y = D + \bar{c}_V. \tag{101}$$

The curve D is already a degree-2 cover on $\bar{c}_V = \bar{c}_{\wedge^2 V}$, and we do not need to introduce a covering curve $\tilde{\bar{c}}_{\wedge^2 V}$ for rank-3 bundles V.

Among various components of divisors specifying the rank-1 sheaf $\mathcal{F}_{\wedge^2 V}$ in (98), $\pi_{D*}\gamma$ and $\pi_{D*}(r|_D - R)/2$ can be treated separately. Because of the irreducible decomposition we have seen above,

$$\pi_{D*}\gamma = i^* \pi_{C*}\gamma - j^*\gamma, \tag{102}$$

and hence the γ-dependent part of (99) is reproduced.

The remaining task is to examine $\pi_{D*}(r|_D - R)/2$. Because the spectral surface C_V is ramified over σ whenever $D \subset C_V$ is on $\bar{c}_{\wedge^2 V}$, we begin with classifying the intersection points of the two divisors r and D in C_V. For rank-3 bundles V, there are two types of r–D intersection points on C_V. We show the behavior of the spectral surface C_V around a D–r intersection point of type.

Once the sheaf (and in particular, line bundle) for $\wedge^2 V$ is obtained, its net chirality follows immediately. Using the Riemann–Roch theorem on the matter curve $\bar{c}_{\wedge^2 V} = \bar{c}_V$,

$$\begin{aligned} \chi(\wedge^2 V) &= 1 - g(\bar{c}_V) + \deg \mathcal{F}_{\wedge^2 V}, && (103)\\ &= 1 - g(\bar{c}_V) + \deg \mathcal{F}_{V^\times} + \deg i^* \pi_{C*}\gamma && (104)\\ &= -\chi(V) + (\pi_{C*}\gamma)\cdot(3K_B + \eta). && (105) \end{aligned}$$

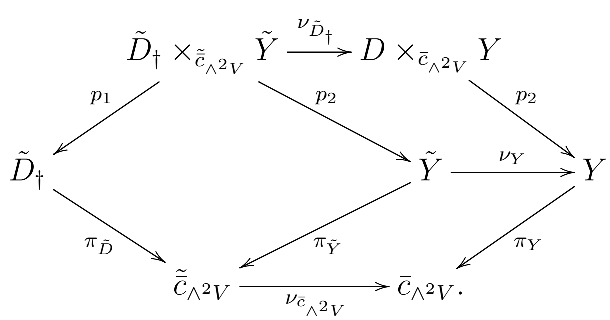
This calculation confirms, using only the sheaves on the matter curves, that a consistency relation between $\chi(V)$ and $\chi(\wedge^2 V)$ is satisfied.

Let us now study the structure of $R^1\pi_{Z*}\wedge^2 V$ in a local neighborhood of a zero point of $R^{(4)}$. We assume that $R^1\pi_{Z*}\wedge^2 V$ is written as $i_{\wedge^2 V*}\mathcal{F}$ for some sheaf $\mathcal{F}$ on $\bar{c}_{\wedge^2 V}$. Then,

$$\begin{aligned} \mathcal{F} &\cong i^*_{\wedge^2 V*} R^1\pi_{Z*}\wedge^2 V && (106)\\ &\cong R^1\pi_{Y*}(\wedge^2 V|_Y), && (107) \end{aligned}$$

where we used the base change formula in the second isomorphism.

Let $b \in \bar{c}_{\wedge^2 V}$ be a zero locus of $R^{(4)}$. Locally (in the analytic topology) around b, the curve D is decomposed into a disjoint union of D_+ and D_-. We consider the following diagram,



(108)

Here $\tilde{Y} = \tilde{\bar{c}}_{\wedge^2 V} \times_{\bar{c}_{\wedge^2 V}} Y$ (as we have already introduced in section 5), and $\tilde{D}_\dagger = \tilde{\bar{c}}_{\wedge^2 V} \times_{\bar{c}_{\wedge^2 V}} D$. We have the decompositions,

$$\tilde{D}_\dagger = \tilde{D}_+ \coprod \tilde{D}_-, \quad \tilde{D}_\pm = \tilde{D}_\pm^{(1)} \cup \tilde{D}_\pm^{(2)},$$

where $\tilde{D}_\pm = \tilde{\bar{c}}_{\wedge^2 V} \times_{\bar{c}_{\wedge^2 V}} D_\pm$ and $\tilde{D}_\pm^{(i)}$ for $i = 1, 2$ are irreducible components of $\tilde{D}_\pm$. Note that each $\tilde{D}_\pm^{(i)}$ is a section of $\pi_{\tilde{Y}}$, and $\tilde{D}_\pm^{(1)}$, $\tilde{D}_\pm^{(2)}$ intersect at one point transversally, say $\tilde{p}_\pm \in \tilde{D}_\pm^{(1)} \cap \tilde{D}_\pm^{(2)}$. Moreover we may assume $\tilde{D}_+^{(1)} \boxplus \tilde{D}_-^{(2)}$ and $\tilde{D}_-^{(1)} \boxplus \tilde{D}_+^{(2)}$ are zero sections of $\pi_{\tilde{Y}}$.

Since $\nu_{\bar{c}_{\wedge^2 V}}$ is a Galois cover with Galois group $G = \mathbb{Z}/2\mathbb{Z}$, we have

$$R^1\pi_{Y_*}(\wedge^2 V|_Y) \cong \left(\nu_{\bar{c}_{\wedge^2 V}*}\nu^*_{\bar{c}_{\wedge^2 V}} R^1\pi_{Y_*}(\wedge^2 V|_Y)\right)^G.$$

By the base change formula, we have

$$\nu^*_{\bar{c}_{\wedge^2 V}} R^1\pi_{Y_*}(\wedge^2 V|_Y) \cong R^1\pi_{\tilde{Y}*}(\wedge^2 \nu_Y^*(V|_Y)), \qquad (109)$$

and

$$\begin{aligned} \nu_Y^* V|_Y &\cong p_{2*}(p_1^*\mathcal{N}_V|_{\tilde{D}_\dagger} \otimes \mathcal{P}_{\tilde{\bar{c}}_{\wedge^2 V}}) && (110)\\ &\cong p_{2*}(p_1^*\mathcal{N}_V|_{\tilde{D}_+} \otimes \mathcal{P}_{\tilde{\bar{c}}_{\wedge^2 V}}) \oplus p_{2*}(p_1^*\mathcal{N}_V|_{\tilde{D}_-} \otimes \mathcal{P}_{\tilde{\bar{c}}_{\wedge^2 V}}), && (111) \end{aligned}$$

Here $\mathcal{P}_{\tilde{\bar{c}}_{\wedge^2 V}}$, $\mathcal{N}_V|_{\tilde{D}_*}$ for $* = \dagger, \pm$ are pullbacks of $\mathcal{P}_B$, $N_V|_{\tilde{D}_*}$ via $\nu_{\tilde{D}_\dagger}$ and the second projection $\tilde{D}_* \to D$ respectively. Let us set $W_\pm = p_{2*}(p_1^* \mathcal{N}_V|_{\tilde{D}_\pm} \otimes \mathcal{P}_{\tilde{\bar{c}}_{\wedge^2 V}})$. We have

$$\nu^*_{\tilde{c}_{\wedge^2 V}} R^1 \pi_{Y_*}(\wedge^2 V|_Y) \cong R^1 \pi_{\tilde{Y}*}(W_+ \otimes W_-). \tag{112}$$

It is useful in calculating $R^1\pi_{\tilde{Y}*}(W_+ \otimes W_-)$ to note that

$$0 \to \mathcal{O}_{\tilde{D}_\pm} \to \mathcal{O}_{\tilde{D}^{(1)}_\pm} \oplus \mathcal{O}_{\tilde{D}^{(2)}_\pm} \to \mathcal{O}_{\tilde{p}_\pm} \to 0 \tag{113}$$

are exact. Applying $\otimes \mathcal{N}_V|_{\tilde{D}_\dagger}$ and Fourier-Mukai transforms, we obtain the exact sequences,

$$0 \to W_\pm \to W^{(1)}_\pm \oplus W^{(2)}_\pm \to W^{(0)}_\pm \to 0. \tag{114}$$

Here $W^{(*)}_\pm$ for $* = 0, 1, 2$ are Fourier–Mukai transforms,

$$W^{(*)}_\pm = p_{2*}(p_1^* \mathcal{N}_V|_{\tilde{D}^{(*)}_\pm} \otimes \mathcal{P}_{\tilde{\bar{c}}_{\wedge^2 V}}), \tag{115}$$

where $\tilde{D}^{(0)}_\pm = \tilde{p}_\pm$. Note that $W^{(*)}_\pm$ for $* = 1, 2$ is a line bundle on Y and $W^{(0)}_\pm$ is a line bundle on the fiber $\tilde{Y}_b = \pi^{-1}_{\tilde{Y}} \nu^{-1}_{\tilde{c}_{\wedge^2 V}}(b)$.

Applying $\otimes W^{(1)}_-$ to the above sequence yields the exact sequence,

$$0 \to W_+ \otimes W^{(1)}_- \to (W^{(1)}_+ \otimes W^{(1)}_-) \oplus (W^{(2)}_+ \otimes W^{(1)}_-) \to W^{(0)}_+ \otimes W^{(1)}_- \to 0. \tag{116}$$

$W^{(0)}_+ \otimes W^{(1)}_-$ is a trivial line bundle on $\tilde{Y}_b$, and $W^{(2)}_+ \otimes W^{(1)}_-$ is also a line bundle that is trivial in the elliptic fiber direction. Thus, by applying $R^i\pi_{\tilde{Y}*}$, we have the long exact sequence,

$$\begin{array}{l} 0 \longrightarrow R^0\pi_{\tilde{Y}*}(W_+ \otimes W^{(1)}_-) \longrightarrow \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}\left(\tilde{b} + \tilde{\pi}_{D*}\gamma\right) \longrightarrow \mathcal{O}_{\tilde{b}} \longrightarrow \\ \longrightarrow R^1\pi_{\tilde{Y}*}(W_+ \otimes W^{(1)}_-) \longrightarrow \mathcal{O}_{\tilde{b}} \oplus \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}\left(\tilde{b} + \tilde{\pi}_{D*}\gamma + \tilde{i}^*_{\wedge^2 V} K_{B_2}\right) \longrightarrow \mathcal{O}_{\tilde{b}} \longrightarrow 0, \end{array} \tag{117}$$

where $\tilde{b} := \nu^{-1}_{\tilde{c}_{\wedge^2 V}}(b)$, $\tilde{\imath}_{\wedge^2 V} := i_{\wedge^2 V} \circ \nu_{\tilde{c}_{\wedge^2 V}}$, and we used

$$R^0\pi_{\tilde{Y}*}(W^{(2)}_+ \otimes W^{(1)}_-) \cong \tilde{\pi}_{D_+*}(\mathcal{N}_V|_{D_+}) \otimes \tilde{\pi}_{D_-*}(\mathcal{N}_V|_{D_-}) \cong \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}(\tilde{b} + \tilde{\pi}_{D*}\gamma), \tag{118}$$

$$R^1\pi_{\tilde{Y}*}(W^{(2)}_+ \otimes W^{(1)}_-) \cong \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}(\tilde{b} + \tilde{\pi}_{D*}\gamma) \otimes \mathcal{L}_H^{-1} \cong \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}(\tilde{b} + \tilde{\pi}_{D*}\gamma + \tilde{i}^*_{\wedge^2 V} K_{B_2}); \tag{119}$$

the ramification divisor r on C_V intersects with $D_\pm$ at $p_\pm$, and $\tilde{\pi}_{D_\pm} = \tilde{\pi}_D|_{D_\pm} : D_\pm \to \tilde{\bar{c}}_{\wedge^2 V}$ maps $p_\pm$ to $\tilde{b} \in \tilde{\bar{c}}_{\wedge^2 V}$. This is why we have a divisor $\tilde{b}$ in (118). We thus conclude that

$$R^0\pi_{\tilde{Y}*}(W_+ \otimes W^{(1)}_-) \cong \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}(\tilde{\pi}_{D*}\gamma), \tag{120}$$

$$R^1\pi_{\tilde{Y}*}(W_+ \otimes W^{(1)}_-) \cong \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}(\tilde{b} + \tilde{\pi}_{D*}\gamma + \tilde{i}^*_{\wedge^2 V} K_{B_2}). \tag{121}$$

By the same argument, we also have the same results for $R^i\pi_{\tilde{Y}*}(W_+ \otimes W^{(2)}_-)$ $(i = 0, 1)$.

Finally we have the exact sequence,

$$0 \to W_+ \otimes W_- \to (W_+ \otimes W^{(1)}_-) \oplus (W_+ \otimes W^{(2)}_-) \to W_+ \otimes W^0_- \to 0. \tag{122}$$

Note that $W_+ \otimes W_-^0$ is a rank two degree-zero sheaf on an elliptic curve $\tilde{Y}_b$ given in [26]. Thus, we have the associated long exact sequence,

$$\begin{array}{c} \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}(\tilde{\pi}_{D*}\gamma) \oplus \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}(\tilde{\pi}_{D*}\gamma) \twoheadrightarrow \mathcal{O}_{\tilde{b}} \longrightarrow R^1\pi_{\tilde{Y}*}(W_+ \otimes W_-) \longrightarrow \\ \longrightarrow \oplus^2 \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}(\tilde{b} + \tilde{\pi}_{D*}\gamma + \tilde{\imath}^*_{\wedge^2 V} K_{B_2}) \longrightarrow \mathcal{O}_{\tilde{b}} \longrightarrow 0. \end{array} \tag{123}$$

Therefore, we obtain

$$\begin{aligned} R^1\pi_{\tilde{Y}*}(W_+ \otimes W_-) &\cong \operatorname{Ker}\left(\mathcal{O}(\tilde{b} + \tilde{\pi}_{D*}\gamma + \tilde{\imath}^*_{\wedge^2 V} K_{B_2}) \oplus \mathcal{O}(\tilde{b} + \tilde{\pi}_{D*}\gamma + \tilde{\imath}^*_{\wedge^2 V} K_{B_2}) \to \mathcal{O}_{\tilde{b}}\right), \\ &= \left\{(f,g) | f,g \in \mathcal{O}(\tilde{b} + \tilde{\pi}_{D*}\gamma + \tilde{\imath}^*_{\wedge^2 V} K_{B_2}), \quad f|_{\tilde{b}} = g|_{\tilde{b}}\right\} \end{aligned} \tag{124}$$

Under the above isomorphism, we can easily see that the action of G on $\nu_{\bar{c}_{\wedge^2 V}*} R^1\pi_{\tilde{Y}*}(W_+ \otimes W_-)$ is given by $(f(\tilde{u}), g(\tilde{u})) \mapsto (g(-\tilde{u}), f(-\tilde{u}))$, where $\tilde{u}$ is the local coordinate of $\tilde{\bar{c}}_{\wedge^2 V}$ around $\tilde{b}$. Hence we have

$$\mathcal{F}_{\wedge^2 V} \cong \left(\nu_{\bar{c}_{\wedge^2 V}*} R^1\pi_{\tilde{Y}*}(W_+ \otimes W_-)\right)^G \tag{125}$$

$$\cong \nu_{\bar{c}_{\wedge^2 V}*} \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}(\tilde{b} + \tilde{\pi}_{D*}\gamma + \tilde{\imath}^*_{\wedge^2 V} K_{B_2}). \tag{126}$$

Therefore, after making the assumption discussed, we find that $\mathcal{F}_{\wedge^2 V}$ on $\bar{c}_{\wedge^2 V}$ is given by a pushforward of a locally free rank-1 sheaf

$$\widetilde{\mathcal{F}}_{\wedge^2 V} = \mathcal{O}\left(\tilde{\imath}^*_{\wedge^2 V} K_{B_2} + \tilde{b}^{(c)} + \tilde{\pi}_{D*}\gamma\right) \tag{127}$$

on $\tilde{\bar{c}}_{\wedge^2 V}$ via $\nu_{\bar{c}_{\wedge^2 V}}$ everywhere on $\bar{c}_{\wedge^2 V}$. Here, $\tilde{b}^{(c)}$ denotes a divisor $\nu^{-1}_{\bar{c}_{\wedge^2 V}} b^{(c)}$, collecting all the points that we have denoted as $\tilde{b}$ up to now.

Matter chiral multiplets from the $\wedge^2 V$ bundle are now identified with

$$H^1(Z; \wedge^2 V) \simeq H^0(\tilde{\bar{c}}_{\wedge^2 V}; \widetilde{\mathcal{F}}_{\wedge^2 V}) \simeq H^0(\bar{c}_{\wedge^2 V}; \mathcal{F}_{\wedge^2 V}). \tag{128}$$

$\mathcal{F}_{\wedge^2 V} = \nu_{\bar{c}_{\wedge^2 V}*} \widetilde{\mathcal{F}}_{\wedge^2 V}$ is a locally-free rank-2 sheaf (rank-2 vector bundle) on $\bar{c}_{\wedge^2 V}$.

The genus of the covering matter curve is given by

$$g(\tilde{\bar{c}}_{\wedge^2 V}) = 1 + 2(g(\bar{c}_{\wedge^2 V}) - 1) + \frac{1}{2}\deg\, b^{(c)}, \tag{129}$$

since $\nu_{\bar{c}_{\wedge^2 V}} : \tilde{\bar{c}}_{\wedge^2 V} \to \bar{c}_{\wedge^2 V}$ is a degree-2 cover with $(1/2)\deg\, b^{(c)}$ branch cuts. Thus, it is also expressed as

$$g(\tilde{\bar{c}}_{\wedge^2 V}) = 1 + (3K_{B_2} + \eta) \cdot (4K_{B_2} + \eta) + \frac{1}{2}(3K_{B_2} + \eta) \cdot (4K_{B_2} + 2\eta), \tag{130}$$

and it follows that

$$\deg\, K_{\tilde{\bar{c}}_{\wedge^2 V}} = 2 \times (3K_{B_2} + \eta) \cdot (6K_{B_2} + 2\eta). \tag{131}$$

On the other hand, one can also calculate the following independently:

$$\begin{aligned} \deg\ \left(\tilde{\imath}^*_{\wedge^2 V} K_{B_2} + \tilde{b}^{(c)}\right) &= 2(3K_{B_2}+\eta)\cdot K_{B_2} + (3K_{B_2}+\eta)\cdot(4K_{B_2}+2\eta), \\ &= (3K_{B_2}+\eta)\cdot(6K_{B_2}+2\eta) = \frac{1}{2}\deg\ K_{\tilde{\bar{c}}_{\wedge^2 V}}. \end{aligned} \tag{132}$$

Because of this non-trivial relation between the genus of the covering curve and the degree of the divisor above, we obtain through Riemann–Roch theorem that

$$\begin{aligned} \chi(\wedge^2 V) &= \chi(\tilde{\bar{c}}_{\wedge^2 V}; \widetilde{\mathcal{F}}_{\wedge^2 V}) = (1 - g(\tilde{\bar{c}}_{\wedge^2 V})) + \deg\ \left(\tilde{\imath}^*_{\wedge^2 V} K_{B_2} + \tilde{b}^{(c)}\right) + \int_{\tilde{\bar{c}}_{\wedge^2 V}} \tilde{\pi}_{D*}\gamma \\ &= \int_{\tilde{\bar{c}}_{\wedge^2 V}} \tilde{\pi}_{D*}\gamma = \int_{\bar{c}_{\wedge^2 V}} \pi_{D*}\gamma = (3K_{B_2}+\eta)\cdot \pi_{C*}\gamma. \end{aligned} \tag{133}$$

This chirality formula in terms of (covering) matter curve and γ was rather anticipated from the beginning. We know that $\chi(\wedge^2 V) = -\chi(\wedge^2 V^\times)$, and the difference between V and $V^\times$ comes from changing the sign of γ. For an SU(4) bundle V, $\wedge^2 V \simeq \wedge^2 V^\times$ and the net chirality should vanish. We can confirm this in the formula above, because $\pi_{C*}\gamma = 0$ for an SU(4) bundle V. For a U(4) bundle V, its chirality formula (133) agrees that is obtained without calculating direct images. All these consistency checks give us confidence that the locally free rank-1 sheaf (127) provides the right description for the matter multiplets from $\wedge^2 V$.

$R^{(5)}_{\rm mdfd}|_{\bar{c}_{\wedge^2 V}}$ has +2 fake contribution from every type (d) point of $\bar{c}_{\wedge^2 V}$. We are benifited from (2.71) of [53], in making an improvement in version 4 here. Since the authors of [53] assigned a scaling dimension r to a_r ($r = 5, 4, 3, 0$), the first three terms have all scaling dimension 9, whereas the last two term have higher dimensions. This is why the last two terms are missing in (2.71) of [53], whereas they are retained here.

To conclude, the locally free rank-1 sheaf $\widetilde{\mathcal{F}}_{\wedge^2 V}$ on $\tilde{\bar{c}}_{\wedge^2 V}$ is given by

$$\widetilde{\mathcal{F}}_{\wedge^2 V} = \mathcal{O}\left(\tilde{\imath}^*_{\wedge^2 V} K_{B_2} + \frac{1}{2}\tilde{b}^{(c)} + \tilde{\pi}_{D*}\gamma\right), \tag{134}$$

where $\tilde{\imath}_{\wedge^2 V} = i_{\wedge^2 V} \circ \nu_{\bar{c}_{\wedge^2 V}} : \tilde{\bar{c}}_{\wedge^2 V} \to \sigma$. We show a couple of examples of geometric data of the matter curves for different choice of the divisor η.

The covering matter curve is determined through

$$\begin{aligned} 2g(\tilde{\bar{c}}_{\wedge^2 V}) - 2 &= \deg\ K_{\tilde{\bar{c}}_{\wedge^2 V}}, \\ &= \deg\ K_{\bar{c}_{\wedge^2 V}} - 2 \times \#(\mathrm{d}), && (135) \\ &= \bar{c}_{\wedge^2 V}\cdot(11K_{B_2}+3\eta) - 2(5K_{B_2}+\eta)\cdot(3K_{B_2}+\eta), && (136) \\ &= 80K_{B_2}^2 + 47K_{B_2}\cdot\eta + 7\eta^2. && (137) \end{aligned}$$

We used the fact in the second equality that the Euler number (genus) of a curve increases by +2 (resp. −1) whenever a double point is blown up [27, 28]. On the other hand, one can calculate the following:

$$\begin{aligned} \deg\ \left(\tilde{\imath}^*_{\wedge^2 V} K_{B_2} + \frac{1}{2}\tilde{b}^{(c)}\right) &= \bar{c}_{\wedge^2 V}\cdot K_{B_2} + \frac{1}{2}D\cdot(\sigma + 2K_{B_2} + \eta), && (138) \\ &= 40K_{B_2}^2 + \frac{47}{2}K_{B_2}\cdot\eta + \frac{7}{2}\eta^2 = \frac{1}{2}\deg\ K_{\tilde{\bar{c}}_{\wedge^2 V}}. && (139) \end{aligned}$$

Thus, by applying Riemann–Roch theorem, the net chirality is given by

$$\begin{aligned}\chi(\wedge^2 V) &= \chi(\tilde{\bar{c}}_{\wedge^2 V}; \widetilde{\mathcal{F}}_{\wedge^2 V}) = [1 - g(\tilde{\bar{c}}_{\wedge^2 V})] + \deg \left(\tilde{\imath}^*_{\wedge^2 V} K_{B_2} + \frac{1}{2}\tilde{b}^{(c)}\right) + \int_{\tilde{\bar{c}}_{\wedge^2 V}} \tilde{\pi}_{D*}\gamma, \\ &= \int_{\tilde{\bar{c}}_{\wedge^2 V}} \tilde{\pi}_{D*}\gamma = D \cdot \gamma. \qquad (140)\end{aligned}$$

In a local neighborhood of a triple point, $C_{\wedge^2 V} \subset Z$ consists of three irreducible components, one for $C_{(ij)}$, one for $C_{(kl)}$ and the other for $C_{(mn)}$. Intersection of any two of the three irreducible components are double-curve singularity of $C_{\wedge^2 V}$, and the triple points are where three double-curve singularities collide. As this type of codimension-2 singularity inevitably appears on the zero section in the case of rank-6 bundles, we need to modify the argument that we presented in the article.

Only a straightforward generalization is required, however. We choose

$$\widetilde{C}_{\wedge^2 V} = C_{(ij)} \coprod C_{(kl)} \coprod C_{(mn)} \qquad (141)$$

locally around any triple points. $\nu_{C_{\wedge^2 V}}$ is defined around this codimension-2 singularity by

$$\nu_{C_{ij}} \coprod \nu_{C_{kl}} \coprod \nu_{C_{mn}} : C_{(ij)} \coprod C_{(kl)} \coprod C_{(mn)} \to C_{(ij)} \cup C_{(kl)} \cup C_{(mn)} = C_{\wedge^2 V}. \qquad (142)$$

By repeating almost the same argument as in section 5.1, one can see that i) $\mathcal{N}_{\wedge^2 V} = \nu_{C_{\wedge^2 V}*} \widetilde{\mathcal{N}}_{\wedge^2 V}$ exists, ii) (43) is satisfied as a sheaf of $\mathcal{O}_Z$ module, and iii) $\widetilde{\mathcal{N}}_{\wedge^2 V}$ on $\widetilde{C}_{\wedge^2 V}$ is a locally free rank-1 sheaf. Thus, (98) can be used for this case as well. The covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$ is defined as $\tilde{\bar{c}}_{\wedge^2 V} := \nu^{-1}_{C_{\wedge^2 V}}(\bar{c}_{\wedge^2 V})$ as before, and each triple point is resolved into three points in $\tilde{\bar{c}}_{\wedge^2 V}$, one in $C_{(ij)}$, one in $C_{(kl)}$ and the other in $C_{(mn)}$.

The classification of the D–r intersection goes exactly the same as in the case of a rank-5 bundle V. There are type (a), (b) and (c) D–r intersection points, and only the type (c) points contribute to $\tilde{\pi}_{D*}(r|_D - R)/2$ in (98).

$$\widetilde{\mathcal{F}}_{\wedge^2 V} = \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}} \left(\tilde{\imath}^*_{\wedge^2 V} K_{B_2} + \frac{1}{2}\tilde{b}^{(c)} + \tilde{\pi}_{D*}\gamma|_D\right), \qquad (143)$$

the same as in (134). $\deg \tilde{b}^{(c)}$ is given by (**??**), now with $N = 6$.

The covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$ has

$$\begin{aligned}2g(\tilde{\bar{c}}_{\wedge^2 V}) - 2 &= \deg K_{\tilde{\bar{c}}_{\wedge^2 V}}, \\ &= \deg K_{\bar{c}_{\wedge^2 V}} - 6 \times \#(\mathrm{e}), \qquad (144)\\ &= (15K_{B_2} + 4\eta) \cdot (16K_{B_2} + 4\eta) - 6(5K_{B_2} + \eta) \cdot (3K_{B_2} + \eta), \qquad (145)\\ &= 150K^2_{B_2} + 76K_{B_2} \cdot \eta + 10\eta^2. \qquad (146)\end{aligned}$$

In the second equality, we have used the fact that the genus of a curve reduces by 3 when a triple point is blown up; see [27, 28]. On the other hand,

$$\begin{aligned}\deg \left(\tilde{\imath}^*_{\wedge^2 V} K_{B_2} + \frac{1}{2}\tilde{b}^{(c)}\right) &= (15K_{B_2} + 4\eta) \cdot K_{B_2} + \frac{1}{2} D \cdot (2\sigma + 2K_{B_2} + \eta), \qquad (147)\\ &= 75K^2_{B_2} + 38K_{B_2} \cdot \eta + 5\eta^2 = \frac{1}{2} K_{\tilde{\bar{c}}_{\wedge^2 V}}. \qquad (148)\end{aligned}$$

Thus, using the Riemann–Roch theorem, the net chirality from the bundle $\wedge^2 V$ is given by

$$\chi(\wedge^2 V) = \chi(\tilde{\bar{c}}_{\wedge^2 V}; \widetilde{\mathcal{F}}_{\wedge^2 V}) = \int_{\tilde{\bar{c}}_{\wedge^2 V}} \tilde{\pi}_{D*}\gamma = D \cdot \gamma. \tag{149}$$

After studying the direct images $R^1\pi_{Z*} \wedge^2 V$ one by one for V of various ranks, we find the the net chirality from these bundles is given by the same expression, $\chi(\wedge^2 V) = D \cdot \gamma$. It will be clear that the rank-4 (133), rank-5 (140) and rank-6 (149) cases have this form of expression. In the rank-3 case, $\chi(\wedge^2 V) = \int_{\bar{c}_{\wedge^2 V}} -j^*\gamma + \pi_{C*\gamma} = D \cdot \gamma$, too. Thus, it is tempting to guess that

$$\chi(\wedge^2 V) = D \cdot \gamma \tag{150}$$

for any $\mathrm{U}(N)$ bundles given by spectral cover construction.

First, note that γ can be decomposed into

$$\gamma = \gamma_0 + \pi_Z^*\omega, \qquad \gamma_0 = \lambda(N\sigma - \eta + NK_{B_2}) \tag{151}$$

for some λ and a 2-form ω on B_2. Since $\pi_{C*}\gamma_0 = 0$ and $\pi_{C*}\pi_Z^*\omega = N\omega$, only the$\gamma_0$ part is allowed for SU(N) bundles [12]. For SU(N) bundles,

$$\chi(V) = \bar{c}_V \cdot \gamma_0 = -\lambda\eta \cdot (NK_{B_2} + \eta) \tag{152}$$

from (57), and

$$\begin{aligned}
\chi(\wedge^2 V) &= D \cdot \gamma_0 \\
&= [\sigma \cdot (N(N-1)K_{B_2} + 2(N-2)\eta) + \eta \cdot (3K_{B_2} + \eta)] \cdot \lambda(N\sigma - \eta - NK_{B_2}), \\
&= \lambda \left(-\eta \cdot (N(N-1)K_{B_2} + 2(N-2)\eta) + N\eta \cdot (3K_{B_2} + \eta)\right), \qquad (153) \\
&= -\lambda\eta \cdot (NK_{B_2} + \eta) \times (N-4) \qquad (154)
\end{aligned}$$

from (150). Thus, the expression (150) yields a result consistent with the relation $\chi(\wedge^2 V) = (N-4)\chi(V)$ in (**??**) for the case with $c_1(V) = 0$. It is also easy to show (through a similar calculation) that

$$D \cdot \pi_Z^*\omega = (N-4) \times \bar{c}_V \cdot \omega + (3K_{B_2} + \eta) \cdot (N\omega). \tag{155}$$

Therefore, the chirality formula (150) for $\mathrm{U}(N)$ bundles always yields a result consistent with the representation.

An idea was presented how to study $R^1\pi_{Z*} \wedge^2 V$. The same idea can be applied to $R^1\pi_{Z*} \wedge^3 V$ only with quite a natural generalization. The treatment allows us to obtain a locally free rank-1 sheaf $\widetilde{\mathcal{F}}_{\wedge^3 V}$ on a covering matter curve $\tilde{\bar{c}}_{\wedge^3 V}$, if the two conditions are satisfied: i) the Fourier–Mukai transform of $\wedge^3 V$ on Z is represented as a pushforward (as in (43)) as a sheaf of $\mathcal{O}_Z$-module, and ii) $\mathcal{N}_{\wedge^3 V}$ on $C_{\wedge^3 V}$ is given by a pushforward of a locally free rank-1 sheaf $\widetilde{\mathcal{N}}_{\wedge^3 V}$ on $\widetilde{C}_{\wedge^3 V}$, a resolution of $C_{\wedge^3 V}$. In the situation we have, the matter curve $\bar{c}_{\wedge^3 V}$ itself is a double curve in $C_{\wedge^3 V}$, but this double-curve singularity is resolved by blowing up Z with a center along the double-curve singularity, where $\wedge^2 V$ bundle for a rank-4 bundle V was discussed. We now have a covering curve $\tilde{\bar{c}}_{\wedge^2 V}$, which is a degree-2 cover of $\bar{c}_{\wedge^2 V}$. Furthermore, since $[A:B:C] = [P:Q:R]$ can be realized only on a codimension-2 locus in curve $\bar{c}_{\wedge^3 V}$, the degree-2 cover does not ramify for a generic choice of moduli parameters $a_{0,2,3,4,5,6}$. The covering matter curve is a disjoint union of two copies of $\bar{c}_{\wedge^3 V}$:

$$\tilde{\bar{c}}_{\wedge^3 V} = \tilde{\bar{c}}_{\wedge^3 V+} \coprod \tilde{\bar{c}}_{\wedge^3 V-}. \tag{156}$$

We have no reason to expect that singularities appear on these curves. Therefore, no extra complication arises other than the original double-curve singularity, and we have shown how to deal with double-curve singularity; thus, the idea in the article is now applicable to the analysis of $R^1\pi_{Z*} \wedge^3 V$ for a rank-6 bundle V.

Instead of a curve D in $Y = \pi_Z^{-1}(\bar{c}_{\wedge^2 V})$, a curve T in $Y = \pi_Z^{-1}(\bar{c}_{\wedge^3 V})$ is introduced. A triplet of points $\{p, p', p''\}$ in $C_V|_{E_b}$ $(b \in \bar{c}_{\wedge^3 V})$ satisfying $p \boxplus p' \boxplus p'' = e_0$ sweeps a curve in Y, and that is the definition of T. $\pi_T = \pi_Z|_T : T \to \bar{c}_{\wedge^3 V}$ is not necessarily a degree-3 cover, but a projection to the covering curve $\tilde{\pi}_{\tilde{T}} : \tilde{T} \to \tilde{\bar{c}}_{\wedge^3 V}$ is a degree-3 cover. $\tilde{T}$ is a resolution of T as we will explain it later. For the case of a rank-6 bundle V, the three solutions of $(Ay + Bx + C) = 0$ [resp. of $(Py + Qx + R) = 0$] form T_+ part [resp. T_- part] of $T = T_+ \cup T_-$, and $\tilde{T} = T_+ \coprod T_-$. $T_\pm$ is mapped to $\tilde{\bar{c}}_{\wedge^3 V\pm}$ separately.

A locally free rank-1 sheaf $\widetilde{\mathcal{F}}_{\wedge^3 V}$ on $\tilde{\bar{c}}_{\wedge^3 V}$ is given by

$$\widetilde{\mathcal{F}}_{\wedge^3 V} = \mathcal{O}\left(\tilde{i}^*_{\wedge^3 V} K_{B_2} + \tilde{\pi}_{\tilde{T}*}\left(\frac{1}{2}(r|_{\tilde{T}} - R_{(T)}) + \gamma|_T\right)\right), \tag{157}$$

a straightforward generalization of the discussion that has led to (98). A divisor $R_{(T)}$ is a ramification divisor of $\tilde{\pi}_{\tilde{T}} : \tilde{T} \to \tilde{\bar{c}}_{\wedge^3 V}$, and hence $R_{(T)} := K_{\tilde{T}} - \tilde{\pi}^*_{\tilde{T}} K_{\tilde{\bar{c}}_{\wedge^3 V}}$.

For the rank-6 case, the covering matter curve is a disjoint union of two curves, $\tilde{\bar{c}}_{\wedge^3 V\pm}$, and each curve has a locally free rank-1 sheaf

$$\widetilde{\mathcal{F}}_{\wedge^3 V\pm} = \mathcal{O}\left(\tilde{i}^*_{\wedge^3 V\pm} K_{B_2} + \tilde{\pi}_{T\pm *}\left(\frac{1}{2}(r|_{T_\pm} - R_{(T_\pm)}) + \gamma|_{T_\pm}\right)\right), \tag{158}$$

where $\tilde{\pi}_{T\pm} := \tilde{\pi}_{\tilde{T}}|_{T_\pm}$ maps $T_\pm$ to $\tilde{\bar{c}}_{\wedge^3 V\pm}$, $R_{(T_\pm)}$ their ramification divisors, and $r|_{T_\pm}$ a restriction on $T_\pm$ of a pullback of $r|_T$ to $\tilde{T}$. $\tilde{i}^*_{\wedge^3 V\pm}$ denotes pullback via either one of $\tilde{i}_{\wedge^3 V\pm} := (i_{\wedge^3 V} \circ \nu_{\bar{c}_{\wedge^3 V}})|_{\tilde{\bar{c}}_{\wedge^2 V}}$.

Therefore, we are now ready to write down the line bundles on the covering matter curves

$$\widetilde{\mathcal{F}}_{\wedge^3 V\pm} = \mathcal{O}\left(\tilde{i}^*_{\wedge^2 V\pm} K_{B_2} + \frac{1}{2}\tilde{b}^{(f)}_\pm + \tilde{\pi}_{T\pm *}\gamma|_{T_\pm}\right). \tag{159}$$

$\mathcal{F}_{\wedge^3 V}$ on the matter curve $\bar{c}_{\wedge^3 V}$ is given by a pushforward of the two line bundles $\widetilde{\mathcal{F}}_{\wedge^3 V\pm}$, and hence becomes a direct product of two line bundles. Massless chiral multiplets are identified with

$$H^1(Z; \wedge^3 V) \cong H^0(\tilde{\bar{c}}_{\wedge^3 V+}; \widetilde{\mathcal{F}}_{\wedge^3 V+}) \oplus H^0(\tilde{\bar{c}}_{\wedge^3 V-}; \widetilde{\mathcal{F}}_{\wedge^3 V-}). \tag{160}$$

It is now straightforward to see that

$$\deg\left(i^* K_{B_2} + \frac{1}{2}\tilde{b}^{(f)}_\pm\right) = (10K_{B_2} + 3\eta) \cdot \frac{1}{2}(2K_{B_2} + (9K_{B_2} + 3\eta)) = \frac{1}{2}\deg K_{\tilde{\bar{c}}_{\wedge^3 V\pm}}. \tag{161}$$

Therefore,

$$\begin{aligned}\chi(\wedge^3 V) &= \chi(\tilde{\bar{c}}_{\wedge^3 V+}; \widetilde{\mathcal{F}}_{\wedge^3 V+}) + \chi(\tilde{\bar{c}}_{\wedge^3 V-}; \widetilde{\mathcal{F}}_{\wedge^3 V-}), \\ &= \int_{\tilde{\bar{c}}_{\wedge^3 V+}} \tilde{\pi}_{T_+ *}\gamma + \int_{\tilde{\bar{c}}_{\wedge^3 V-}} \tilde{\pi}_{T_- *}\gamma = T_+ \cdot \gamma + T_- \cdot \gamma = \int_{\bar{c}_{\wedge^3 V}} \pi_{T*}\gamma. \end{aligned} \tag{162}$$

For physics application that we mentioned, $\wedge^3 V$ bundle of a rank-6 bundle is purely of SU(6) bundle V; even when a structure group of V is chosen to be $\mathrm{U}(6) \subset \mathrm{SO}(12)$, the bundle $\wedge^3 V$ is neutral under the U(1) symmetry in the structure group. Thus, $\pi_{C*}\gamma = 0$ should be use for the calculation of chirality here, and hence $\chi(\wedge^3 V) = 0$. This should be the case, since coming out of the bundle $\wedge^3 V$ are chiral multiplets in the doublet representation of an unbroken SU(2) gauge group, and there is no well-defined chirality associate with this representation (or gauge group). This serves as a consistency check, giving a confidence in the description of the bundles we have provided.

6 Four-Form Fluxes

In the Heterotic string theory description, matter multiplets are characterized in terms of spectral surfaces and line bundles on them. All these pieces of information are associated with the fiber elliptic curve, which is now found in the $z'_f = 0$ ($z_f = \infty$) locus of the dP_9 fibration. On the other hand, in the F-theory description, non-Abelian gauge field of the unbroken symmetry group are localized within the locus of enhanced singularity, which is found in $z_f = 0$ locus. Chiral matter multiplets are also supposed to be at the $z_f = 0$ locus. The spectral surface C_V in the Heterotic description only determines N points for an SU(N) bundle in a given elliptic fiber (which is at $z_f = \infty$), but each point corresponds to a line l^p belonging to I_8. The N lines specified by cover all the region of the base $\mathbb{P}^1$, including $z_f = Z/Z' = \infty$ and $z_f = 0$. Thus, in a description using dP_8 fibration (and dP_9 fibration), the information of spectral surface is not particularly localized at either end of the elliptic fibration over $\mathbb{P}^1$. In fact, the data $a_{0,2,\cdots,5}$ specifying the spectral surface controls the entire geometry of the del Pezzo fibration. More important in generating chiral matter spectrum in low-energy physics is the line bundle $\mathcal{N}_V$ on C_V, or to be more precise, γ determining $c_1(\mathcal{N}_V)$ through (30). Reference [15] introduced four-form flux $G_H^{(4)}$ in a description using dP_8 fibration, so that it plays the role of γ in the Heterotic theory. C_V is regarded locally as N copies of a local patch of B_2, and each copy corresponds to a point p_p for one of $p \in \{6^\flat, 6, 7, 8, 8^\sharp\}$ (in case of an SU(5) bundle) sweeping over B_2. γ on C_V is locally described by two forms on the each one of those copies. Suppose that a four-form flux $G_H^{(4)}$ is given in a dP_8-fibration $\pi_U : U \to B_2$. Then, γ on the copy of p_p, γ_p, is given by

$$\gamma_p = \int_{l^p} i^*_{l^p} G_H^{(4)}. \tag{163}$$

Because of this correspondence between γ and $G_H^{(4)}$, only topological aspects of $G_H^{(4)}$ in dP_8 matter.

When considering SU(N) vector bundle, γ on C_V has a constraint. The vanishing of the first Chern class $c_1(V) = 0$ means that

$$\pi_{C*}\gamma = 0. \tag{164}$$

This implies that the integration of $G_H^{(4)}$ over the five lines specified should vanish. Because of the topological relation satisfied by the five lines, the condition above is equivalent to

$$\int_{5x} G_H^{(4)} = 0, \qquad \text{and hence} \int_{x_8} G_H^{(4)} = 0. \tag{165}$$

Here, we assume that only $\mathrm{SU}(5)_{\rm GUT}$ preserving fluxes are introduced in the dP_8 fibration. Because of these constraints, $G_H^{(4)}$ can be expressed as

$$G_H^{(4)} \equiv \sum_{P=\tilde{8},6,7,-\theta} C_P \otimes \pi_Z^* \omega^P, \tag{166}$$

where ω^P's are 2-forms on B_2, and C_P's are 2-cycles—Poincaré dual of 2-forms—in $S = dP_8$. Fluxes proportional to x_8 should not be introduced.

We should be clear what we mean by (166). Four-form $G_H^{(4)}$ is classified by $H^4(U;\mathbb{Z})$, where $\pi_U : U \to B_2$ is a dP_8-fibration. Using Leray spectral sequence, one finds that $H^4(U;\mathbb{Z})$ has a filtration structure:

$$H^4(U;\mathbb{Z}) = F_0 \supset F_2 \supset F_4 \supset \{0\}, \tag{167}$$

with

$$F_4 \cong H^4(B_2; R^0\pi_{U*}\mathbb{Z}), \quad F_2/F_4 \cong H^2(B_2; R^2\pi_{U*}\mathbb{Z}), \quad F_0/F_2 \cong H^0(B_2; R^4\pi_{U*}\mathbb{Z}). \tag{168}$$

$G_H^{(4)}$ in (166) is understood as an element of F_2/F_4 modulo $F_4 = H^4(B_2;\mathbb{Z})$, and C_P as local generators of $R^2\pi_{U*}\mathbb{Z}$. Although the Poincare dual 2-forms of C_P's are well-defined in $H^2(U;\mathbb{Z})$ only modulo $H^2(B_2;\mathbb{Z})$, this ambiguity does not appear in (166) because $G_H^{(4)}$ is given in (166) only modulo $H^4(B_2;\mathbb{Z})$. Since x_8 is a cycle in the fiber direction, differential forms on B_2 is trivial when pulled back to x_8, and hence (165) cannot determine the F_4 part. Because of the same reason, however, (163) does not depend on the F_4 part either. Therefore, in describing vector bundles in Heterotic theory, it is sufficient to have a four-form $G_H^{(4)}$ in F_2/F_4, and leave the ambiguity in F_4 unfixed.

By using this explicit expression of $G_H^{(4)}$ and the intersection form

$$C^{ab;p=6^\flat,6,7,8,8^\sharp} \cdot C_{P=\tilde{8},6,7,-\theta} = \begin{pmatrix} -1 & & & \\ 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \\ & & & 1 \end{pmatrix}, \tag{169}$$

one can see explicitly that [15]

$$\pi_{C*}(\gamma \cdot \gamma) = \sum_p \gamma_p \wedge \gamma_p = \sum_{P,Q} C_{A_4\, PQ} \omega^P \wedge \omega^Q = -\pi_{U*} G_H^{(4)} \wedge G_H^{(4)}. \tag{170}$$

Although $G_H^{(4)}$ in (166) has the ambiguity $F_4 = H^4(B_2;\mathbb{Z})$, $G_H^{(4)} \wedge G_H^{(4)}$ does not depend on the ambiguity. The same is true for other $\mathrm{SU}(N)$ bundles with $N < 5$; $\omega^{P=\tilde{8}}$ is set to zero for $\mathrm{SU}(4)$ bundles, and $\omega^{P=6} = 0$ is further imposed for $\mathrm{SU}(3)$ bundles.

In the F-theory compactification, there is totally an independent condition for 4-form flux $G_F^{(4)}$ on a Calabi–Yau 4-fold compactification: the (2,2) part of the four-form flux has to be primitive in order to preserve $\mathcal{N}=1$ supersymmetry. Reference [15] observed that the condition (164) and the primitiveness condition

$$J \wedge G_F^{(4)} = 0 \tag{171}$$

are quite "similar," and certainly they are. On the other hand, $H^2(dP_9;\mathbb{Z})$ is larger than $H^2(dP_8;\mathbb{Z})$ by rank one. Thus, with only one constraint on $H^2(dP_8;\mathbb{Z})$ and one for $H^2(dP_9;\mathbb{Z})$, there should be no one-to-one correspondence between Heterotic and F-theory vacua. This gap has to be filled in order to complete the dictionary of the Heterotic–F theory duality.

The primitiveness condition (171) involves a two-form J on W and a four-form $G_F^{(4)}$ on W, where $\pi_W : W \to B_2$ is a dP_9 fibration. $H^4(W;\mathbb{Q})$, in which $G_F^{(4)}$ takes its value, has a filtration structure just like in (167):

$$H^4(W;\mathbb{Q}) = F_0 \supset F_2 \supset F_4, \tag{172}$$

with

$$F_4 \cong H^4(B_2;\mathbb{Q}), \quad F_2/F_4 \cong H^2(B_2; R^2\pi_{W*}\mathbb{Q}), \quad F_0/F_2 \cong H^0(B_2; R^4\pi_{W*}\mathbb{Q}); \tag{173}$$

notations $F_{0,2,4}$ are recycled here, as we expect little confusion. $\mathbb{Z}$ in (167) is replaced by $\mathbb{Q}$ here, because the four-form flux in F-theory is not necessarily quantized as integral value [33]. It is known that the four-form flux in F_0 has its two legs in the T^2-fiber directions of the dP_9, and results in non SO(3,1) Lorentz symmetric vacuum [51]. Thus, we only consider $G_F^{(4)}$ that belongs to F_2 in the following. $G_F^{(4)}$ being an element of F_2 is not a sufficient condition for the $\mathrm{SO}(3,1)$ Lorentz symmetry; we will elaborate on it later.

Similarly, the Kähler form J takes its value in $H^2(W;\mathbb{R})$, and this cohomology group also has a filtration structure:

$$H^2(W;\mathbb{R}) = E_0 \supset E_2, \quad E_2 \cong H^2(B_2;\mathbb{R}), \quad E_0/E_2 \cong H^0(B_2; R^2\pi_{W*}\mathbb{R}). \tag{174}$$

Thus, the Kähler form J is written as

$$J = \pi_W^* J_{B_2} + t_2 J_0; \tag{175}$$

projection of J into E_0/E_2 specifies a 2-form on dP_9, and J_0 is a representative of the class specified by the 2-form on dP_9. $t_2 \geq 0$ is a parameter.

In the Heterotic–F theory duality, moduli space is shared by the two theories, but one of the two theories provides a better description of some part of the moduli space, and the other of some other parts. The description in the Heterotic theory (without stringy excitations taken into account in calculations) becomes unreliable either when the Heterotic theory dilaton expectation value is large, or when the volume of the T^2 fiber becomes comparable to α'. In the first case, the base $\mathbb{P}^1$ manifold of $S' = dP_9$ has a large volume. Thus, whenever F-theory provides a better description, the volume of the base $\mathbb{P}^1$ of dP_9 is larger than that of the T^2 fiber. Therefore, in the F-theory limit, we can take it that the Kähler form on dP_9 specified by J or J_0 has a dominant contribution only from the the $\mathbb{P}^1$ base of dP_9, not from the T^2 fiber. Thus, J_0 (or J) regarded as a 2-form on dP_9 is a Poincaré dual of x_9.

The filtration structure of J and $G_F^{(4)}$ makes the analysis of the primitiveness condition (171) easier. The condition (171) takes its value in $H^6(W;\mathbb{R})$, and this group also has a filtration structure

$$H^6(W;\mathbb{R}) = G_2 \supset G_4 \supset \{0\}, \tag{176}$$

with

$$G_4 \cong H^4(B_2; R^2\pi_{W*}\mathbb{R}), \qquad G_2/G_4 \cong H^2(B_2; R^4\pi_{W*}\mathbb{R}). \tag{177}$$

We begin with the primitiveness condition in G_2/G_4, and we will come back later to the condition in the G_4 part. The G_2/G_4 part of the primitiveness condition receives contributions only from the wedge product of the E_0/E_2 part and the F_2/F_4 part, and we find that

$$t_2 x_9 \cdot G_F^{(4)} \equiv 0 \tag{178}$$

mod G_4.

The primitiveness condition (178) allows two types of local expressions for the four-form flux:

$$G_{F;\gamma}^{(4)} \equiv \sum_{I=1}^{8} C_I \otimes \omega^I, \tag{179}$$

$$G_{F;9}^{(4)} \equiv x_9 \otimes \omega^{I=9}, \tag{180}$$

here, we abuse the notation, and denote $\pi^*(C_I)$ of $H^2(dP_9;\mathbb{Z})$ as C_I, because the intersection form of $\pi^*(C_I)$'s are the same as those of C_I's in $H^2(dP_8;\mathbb{Z})$. The four-form flux $G_H^{(4)}$ (166) in the Heterotic theory description can be mapped into the first type of $G_F^{(4)}$:

$$G_{F;\gamma}^{(4)} \equiv \pi^* G_H^{(4)}; \tag{181}$$

everything is in modulo $F_4 = H^4(B_2;\mathbb{Q})$ here. We understand that the four-form flux G_γ in [10] belongs to this class modulo $F_4 = H^4(B_2;\mathbb{Q})$.

A little more attention has to paid in interpreting the other contribution (180). F-theory dual of a Heterotic compactification involves a Calabi–Yau 4-fold that is a $K3$-fibration on a base 2-fold. Although the $K3$ fiber becomes two dP_9 surfaces in the stable degeneration limit, $K3$ fiber, rather than two dP_9's, is better in understanding this aspect. As explained clearly in [51], out of 22 two-cycles of a $K3$ fiber, $2 \times 8 = 16$ two-cycles correspond to the C_I's in two dP_9's. Four-form fluxes associated with these two-cycles, like (179), satisfy the primitiveness condtion in the G_2/G_4 part. Fluxes associated with the zero section of the elliptic fibered $K3$ (like σ of dP_9) and with the T^2-fiber class (like x_9 of dP_9), on the other hand, do not either satisfy the primitiveness condition or preserve the $\mathrm{SO}(3,1)$ Lorentz symmetry. Thus, such fluxes should not be introduced. Two other (1,1) two-cycles remain, and the four-form fluxes associated with these two two-cycles as well as the (2,0) and (0,2) two-cycles of $K3$ fiber correspond to the three-form fluxes of the Type IIB string theory [51]. Therefore, when the three-form fluxes are set to zero,

$$\begin{aligned}\pi_{C*}(\gamma\wedge\gamma) &= -\pi_{U*}\left(G_H^{(4)}\wedge G_H^{(4)}\right), \\ &= -\pi_{W*}\left(G_{F;\gamma}^{(4)}\wedge G_{F;\gamma}^{(4)}\right) = -\pi_{W*}\left(G_F^{(4)}\wedge G_F^{(4)}\right). \end{aligned} \tag{182}$$

Once again, the $F_4 = H^4(B_2;\mathbb{Z})$ ambiguity in $G_F^{(4)}$ does not matter to the relation (182).

The correspondence between the number of $M5$-branes in the Heterotic theory and the number of 3-branes in F-theory is one of the most important clues of the Heterotic–F theory duality. The number of $M5$-branes wrapped on the elliptic fiber is given by [12]

$$n_5 = \int_{B_2}(c_2(TZ) - c_2(V_1)|_{\gamma_1=0} - c_2(V_2)|_{\gamma_2=0}) + \frac{1}{2}\gamma_1^2 + \frac{1}{2}\gamma_2^2, \tag{183}$$

where V_i and γ_i $(i=1,2)$ are vector bundle and discrete twisting data in (30), respectively, in the visible $(i=1)$ and hidden $(i=2)$ sector. The number of 3-branes in F-theory is given by [31]

$$n_3 = \frac{\chi(X)}{24} - \sum_{i=1,2}\frac{1}{2}G_{Fi}^{(4)}\wedge G_{Fi}^{(4)}, \tag{184}$$

where $F^{(3)}\wedge H^{(3)}$ contribution from the three form fluxes of the Type IIB string theory are set to zero. The equality between the first terms in n_5 and n_3 was proved in [12, 25, 32]. The equality (182) was basically shown in [15]. When the $F^{(3)}\wedge H^{(3)}$ contribution is turned on, the n_3 in F-theory will be different from the original n_5 in a Heterotic compactification (that is no longer a dual). What we did so far in section 6 is basically to collect references (mainly [15, 51]) and tell a combined story.

The extra degree of freedom in the four-form flux $G_F^{(4)}$ (the 3-form flux $F^{(3)}$ and $H^{(3)}$ in the Type IIB language) brings about another issue. In the Heterotic theory description, γ on C_V has an alternative expression

$$\gamma_p = \int_{C^p} G_H^{(4)}, \tag{185}$$

where $C^p := l^p - x_8 = C^{ab;p} \bmod C_{A=1,2,3,4}$, because the difference from (163), $x_8\cdot G_H^{(4)}$, vanishes. In F-theory, however, the two natural guesses

$$\gamma_p = \int_{\pi^*(l^p)} G_F^{(4)}, \tag{186}$$

$$\gamma_p = \int_{\pi^*(C^p)} G_F^{(4)} \tag{187}$$

are not necessarily the same. The meaning of (186) is not even well-defined, because $\pi^*(l_p)$'s are well-defined two-cycles in dP_9, but their meaning has not been specified in $K3$. Depending on how the $\pi^*(l_p)$'s are defined in $K3$, (186) may or may not depend on the three-form fluxes $F^{(3)}$ (and $H^{(3)}$). $\pi^*(C_p)$ on the other hand, are naturally identified with one of two sets of two-cycles of $K3$ whose intersection form is $(-1)\times$ Cartan matrix of E_8. Since those two-cycles have vanishing intersection numbers with the two-cycles to which the three-form fluxes are associated with (see [51]), (187) does not depend on the choice of the extra discrete degrees of freedom in

F-theory, and is the same as (163, 185). Thus, we adopt (187) in translating γ in Heterotic theory into F-theory language. Note that γ_p's defined by (187) (and those by (186)) do not depend on the F_4 part of $G_F^{(4)}$.

6.1 Chirality in $\rho(V) = \wedge^2 V$

We are now ready to study the sheaf $\mathcal{F}_{\wedge^2 V}$ on the matter curve $\bar{c}_{\wedge^2 V}$, or $\widetilde{\mathcal{F}}_{\wedge^2 V}$ on the covering curves $\tilde{\bar{c}}_{\wedge^2 V}$. As we have learnt, divisors of the line bundles $\widetilde{\mathcal{F}}_{\wedge^2 V}$ always contain $\tilde{\pi}_{D*}\gamma$. Let us study what this contribution means in the F-theory language.

$\tilde{\pi}_D : D \rightarrow \tilde{\bar{c}}_{\wedge^2 V}$ is a degree-2 cover, allocating two points $\{p_i, p_j\}$ to a point in $\tilde{\bar{c}}_{\wedge^2 V}$ so that $p_i \boxplus p_j = e_0$. Let us denote the lines in I_8 for those two points (in the fundamental representation of $\mathrm{SU}(5)_{\mathrm{bdl}}$) as l^p and l^q $(p, q \in \{6^\flat, 6, 7, 8, 8^\sharp\})$; here, we consider those lines modulo C_A $(A = 1, 2, 3, 4)$ for the unbroken $\mathrm{SU}(5)_{\mathrm{GUT}}$ symmetry. Now

$$\tilde{\pi}_{D*}\gamma = \int_{l^p} G_H^{(4)} + \int_{l^q} G_H^{(4)} = \int_{C^p + C^q} G_H^{(4)} = \int_{C^{pq}} G_H^{(4)}. \tag{188}$$

Here, a topological relation $C^p + C^q \equiv C^{pq}$ (mod C_A $(A = 1, 2, 3, 4)$) between the 2-cycles in (**??**) and (**??**) was used in the last equality.

There is a uniqueness problem in translating the Heterotic theory result of $\tilde{\pi}_{D*}\gamma$ into F-theory language, as we encountered in translating $j^*\gamma$ into (186) or (187). We adopt

$$\tilde{\pi}_{D*}\gamma = \int_{\pi^*(C^{pq})} G_F^{(4)}, \tag{189}$$

in the same spirit as we chose (187) for $j^*\gamma$. This is the field strength of a gauge field obtained by integrating the 3-form field $C^{(3)}$ over the two-cycle $C_{a;}{}^{pq}$—a gauge field an $M2$-brane wrapped on the collapsed two-cycle $C_{a;}{}^{pq}$ is coupled to. As long as we adopt this rule of translation, the flux quanta associated with the non-E_8 part of the two-cycles in $K3$-fiber do not have an influence on the net chirality, or even on γ that describes a line bundle in F-theory.

It is interesting to note that the notion of the covering curve $\tilde{\bar{c}}_{\wedge^2 V}$ we introduced in sections 5 and **??** is not only for mathematical convenience. An $M2$-brane wrapped on a cycle $\pi^*(C^{pq})$ propagates on the covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$, not on the matter curve $\bar{c}_{\wedge^2 V}$, because the each point of the covering mater curve is in one to one correspondence with the collapsed two-cycle.

The chirality formula in this (pair of) irreducible representation(s) follows immediately:

$$\chi(\wedge^2 V) = \int_{\tilde{\bar{c}}_{\wedge^2 V}} \tilde{\pi}_{D*}\gamma = \int_{C^{pq} \times \tilde{\bar{c}}_{\wedge^2 V}} G_F^{(4)}. \tag{190}$$

This is quite a natural result, once again. But all the hard work that has led to this conclusion tells us that we do not need to add an extra contributions associated with codimension-3 singularities of F-theory; it was the part hardly accessible with limited intuition in F-theory, yet our study using the Heterotic–F theory duality shows that (190) is indeed fine.

This expression is an F-theory generalization of the Type IIB chirality formula in a corresponding system. Here, we imagine a Type IIB set up where five D7-branes are wrapped on a holomorphic four-cycle $\Sigma_{\mathbf{5}}$ of a Calabi–Yau 3-fold, and another D7-brane on another four-cycle $\Sigma_{\mathbf{1}}$. Topological U(1) gauge field configuration F_5 and F_1 is assumed on the both four-cycles, $\Sigma_{\mathbf{5}}$ and $\Sigma_{\mathbf{1}}$, respectively. Then, the net chirality in the $\mathrm{SU}(5)_{\mathrm{GUT}}$-$\bar{\mathbf{5}}$ representation is given by [39]:

$$\#(\bar{\mathbf{5}},\mathbf{1}^+) - \#(\mathbf{5},\mathbf{1}^-) = \int_{\Sigma_{\mathbf{5}}\cdot\Sigma_{\mathbf{1}}} i^*\left(\frac{F_1}{2\pi}\right) - i^*\left(\frac{F_5}{2\pi}\right). \tag{191}$$

This expression, written only in terms of local geometry around the D7–D7 intersection curve, is equivalent to the one in [42] given by pairing of D-brane charge vectors in K-theory [40, 43, 44]. The F-theory formula (190) is the most natural generalization of the local formula of the Type IIB string theory (191).

6.2 Chirality in $\rho(V) = \wedge^3 V$

It is now straightforward to provide an F-theory interpretation for the $\tilde{\pi}_{T\pm *}\gamma|_{T_\pm}$ contribution to the sheaves $\widetilde{\mathcal{F}}_{\wedge^3 V\pm}$ in (159). In the Heterotic theory description,

$$\tilde{\pi}_{T_\pm *}\gamma = \int_{l^p+l^q+l^r} G_H^{(4)} = \int_{C^p+C^q+C^r} G_H^{(4)} = \int_{C^{pqr}} G_H^{(4)}, \tag{192}$$

where C^{pqr}'s are now two-cycles that correspond to the roots in the $(\wedge^3 V, \mathbf{1}, \mathbf{2})$ of the group $\mathrm{SU}(6)\times\mathrm{SU}(3)\times\mathrm{SU}(2)\subset E_8$. In F-theory, this is replaced by $\int_{\pi^*(C^{pqr})} G_F^{(4)}$.

In the SU(6)-bundle compactification of the Heterotic string theory, there are two types of massless chiral multiplets in the $(\mathbf{1},\mathbf{2})$ representation of the unbroken symmetry group $\mathrm{SU}(3)\times\mathrm{SU}(2)$. One group of multiplets is $H^0(\bar{c}_{\wedge^3 V}; \nu_*\widetilde{\mathcal{F}}_{\wedge^3 V+})$, and the other $H^0(\bar{c}_{\wedge^3 V}; \nu_*\widetilde{\mathcal{F}}_{\wedge^3 V-})$ $\simeq [H^1(\bar{c}_{\wedge^3 V}; \widetilde{\mathcal{F}}_{\wedge^3 V+})]^\times$. Thus, a net chirality can be defined in the SU(2)-doublet sector as the difference between the degrees of freedom of the two groups. It is

$$\chi(\wedge^3 V)_+ := \chi(\bar{c}_{\wedge^3 V}; \nu_*\widetilde{\mathcal{F}}_{\wedge^3 V+}) = T_+\cdot\gamma = -\chi(\bar{c}_{\wedge^3 V}; \nu_*\widetilde{\mathcal{F}}_{\wedge^3 V-}). \tag{193}$$

In F-theory, this chirality is given by

$$\chi(\wedge^3 V)_+ = \int_{C^{pqr}\times\bar{c}_{\wedge^3 V}} G_F^{(4)}. \tag{194}$$

Here we want to study the effect of the (-1)-curves in the rational zero section in the spectrum of the effective theory. We will fix notation as,

$$\begin{aligned} \mathcal{F}^\bullet &:= Rp_* q^*\mathcal{E}, \\ \mathcal{L}^\bullet &:= \Phi^{\mathcal{P}}_{X'\to X'}(\mathcal{F}^\bullet). \end{aligned} \tag{195}$$

The goal then is to compute the zero-mode spectrum (i.e. bundle-valued cohomology groups) of $\mathcal{E}$ in X. Suppose the support of $\mathcal{L}^\bullet$ takes the most general form, this task reduces to computation of $R^1\pi_*\mathcal{E}$ by using Leray spectral sequence. To find this, first notice that inverse functor of Rq_*Lp^* is given by

$$\mathcal{E} = Rp_*(Lq^*\mathcal{F}^\bullet\otimes\mathcal{O}_{\tilde{X}}(e)). \tag{196}$$

Therefore we get,

$$\begin{aligned} R\pi_*\mathcal{E} &= R\pi'_*(\mathcal{F}^\bullet \otimes Rq_*\mathcal{O}_{\tilde{X}}(e)) \\ &= R\pi'_*(\mathcal{F}^\bullet), \end{aligned} \quad (197)$$

where we used $Rq_*\mathcal{O}_{\tilde{X}}(e) = \mathcal{O}_{X'}$. Next, one can use the same techniques as before to compute the $R\pi'_*\mathcal{F}^\bullet$ in terms of the "spectral data" in X',

$$R\pi_*\mathcal{E} = R\pi'_*\mathcal{F}^\bullet = R\pi'_*(\mathcal{L}^\bullet \otimes \mathcal{O}_{\sigma'}). \quad (198)$$

Naively the above result is the same as in the standard cases. But notice that $\mathcal{L}^\bullet$ is the Fourier-Mukai transform of a (may be non-WIT or singular) object $\mathcal{F}^\bullet$ in $D^b(X')$, and it may receive new contributions from the original (-1)-curve in X. In the example computed before, the component $[C'_2]$ doesn't intersect with the zero section, so the only contribution to the spectrum of the effective theory is through the line bundle over the component S.

7 Examples of Explicit Fourier-Mukai Transforms

The power of a Fourier-Mukai transform (and its inverse) is that in principle we can move freely between descriptions of stable vector bundles on elliptically fibered manifolds and the spectral data that we have been studying. In this section we now utilize this potential to explicitly compute FM transforms of stable bundles defined by the monad construction or by extension. Several explicit realizations of this type have been accomplished before in the literature and we will provide some generalizations. In particular, we will develop general tools that are applicable away from Weierstrass 3-folds. In these examples, we shall also observe that although we have derived general formulas for bundles defined via smooth spectral covers, this proves to be too limited to describe the explicit bundles we consider in the majority of cases. We will return to this point – namely that there remain important gaps in our description of general points in the moduli space of bundles. Beginning with the simplest possible elliptic CY 3-fold geometry – i.e. Weierstrass form, we will illustrate the ideas that can be generalized to compute the Fourier-Mukai transform of sheaves which are defined by extension sequences or monads.

7.1 Bundles Defined by Extension on Weierstrass CY Threefolds

To illustrate the techniques of taking explicit FM transforms, we begin with the simplest possible extension bundle – a rank two vector bundle defined by extension of two line bundles:

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow V_2 \longrightarrow \mathcal{L}_1^\vee \longrightarrow 0. \quad (199)$$

We require V_2 to be stable, and $c_1(V_2) = 0$. Note that a necessary (though not sufficient) constraint on the line bundles appearing in this sequence is that $\mathcal{L}_1$ must not be effective (i.e. have global sections). For such a stable bundle the restriction of V_2 over $E_t = \pi^{-1}(t)$ for a generic $t \in B$ is one of the following cases,

$$
\begin{aligned}
V_2|_{E_t} &= \mathcal{O}_{E_t} \oplus \mathcal{O}_{E_t}, \\
V_2|_{E_t} &= \mathcal{E}_2 \otimes \mathcal{F}, \quad deg(\mathcal{F}) = 0, \\
V_2|_{E_t} &= \mathcal{O}_{E_t}(-p - p_0) \oplus \mathcal{O}_{E_t}(p - p_0).
\end{aligned} \tag{200}
$$

In the first case, the support of the Fourier-Mukai sheaf (i.e. spectral cover), will be a non-reduced scheme (supported over the the section σ). In the second case $\mathcal{E}_2$ is the unique non trivial extension of trivial line bundles, and $\mathcal{F} = \mathcal{O}_{E_t}(p - p_0)$ for some p (here p_0 is the point on E_t chosen by the section), but for Weierstrass fibration, $p = p_0$ for generic fibers, and $V_2|_{E_t} = \mathcal{E}_2$. So again the spectral cover will be non-reduced and supported over the zero section. In the final case, the spectral cover can be non-singular. So it is clear that in the majority of cases, we *do not* expect the FM transform of V_2 to be in the same component of moduli space as a *smooth* spectral cover of the form described in Section 2. We will illustrate this effect with two choices of $\mathcal{L}_1$ below.

Applying the Fourier-Mukai functor to (199) produces a long exact sequence involving the FM transform of the line bundles defining V_2. Thus, we can compute $\Phi(V_2)$ if we can compute $\Phi(\mathcal{L}_1)$. To begin, the definition of the Poincare sheaf, (2.2) and (2.2), allows us to write the following short exact sequence:

$$
\begin{aligned}
0 \longrightarrow \pi_1^*\mathcal{L}_1 \otimes \mathcal{P} &\longrightarrow \pi_1^*(\mathcal{L}_1 \otimes \mathcal{O}_X(\sigma)) \otimes \pi_2^*(\mathcal{O}_X(\sigma) \otimes \pi^* K_B^*) \\
&\longrightarrow \delta_*(\mathcal{L}_1 \otimes \mathcal{O}_X(2\sigma) \otimes \pi^* K_B^*) \longrightarrow 0.
\end{aligned} \tag{201}
$$

Now, by applying, $R\pi_{2*}$ to the above sequence, we can compute $\Phi(\mathcal{L}_1)$,

$$
\begin{aligned}
0 &\longrightarrow \Phi^0(\mathcal{L}_1) \longrightarrow R^0\pi_{2*}\pi_1^*(\mathcal{L}_1 \otimes \mathcal{O}_X(\sigma)) \otimes (\mathcal{O}_X(\sigma) \otimes \pi^* K_B^*) \longrightarrow (\mathcal{L}_1 \otimes \mathcal{O}_X(2\sigma) \otimes \pi^* K_B^*) \to \\
&\longrightarrow \Phi^1(\mathcal{L}_1) \longrightarrow R^1\pi_{2*}\pi_1^*(\mathcal{L}_1 \otimes \mathcal{O}_X(\sigma)) \otimes (\mathcal{O}_X(\sigma) \otimes \pi^* K_B^*) \longrightarrow 0.
\end{aligned} \tag{202}
$$

With these general observations in hand, we will first consider the case where $\mathcal{L}_1 = \mathcal{O}_X(D_b)$ with D_b a divisor pulled back from the base, B_2. To use (202), in this case, $R\pi_{2*}\pi_1^*(\mathcal{L}_1 \otimes \mathcal{O}_X(\sigma))$ must be computed. To accomplish this, we can use the base change formula (see Appendix 9), which relates the following push-forwards,

$$
\begin{array}{ccc}
X \times_B X & \xrightarrow{\pi_1} & X \\
\downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi} \\
X & \xrightarrow{\pi} & B
\end{array}
$$

$$
R\pi_{2*}\pi_1^* \simeq \pi^* R\pi_* \tag{203}
$$

therefore $R\pi_{2*}\pi_1^*(\mathcal{L}_1 \otimes \mathcal{O}_X(\sigma)) = (\pi^* R\pi_* \mathcal{O}_X(\sigma)) \otimes \mathcal{O}_X(D_b)$. On the other hand, by Koszul sequence for the section (σ) we have,

$$
0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(\sigma) \longrightarrow \mathcal{O}_\sigma(K_B) \longrightarrow 0. \tag{204}
$$

It is well-known for Weierstrass CY elliptic fibration $\pi : X \longrightarrow B$, $R^0\pi_*\mathcal{O}_X = \mathcal{O}_B, R^1\pi_*\mathcal{O}_X = K_B$. So the above sequence implies $R\pi_*\mathcal{O}_X(\sigma) = \mathcal{O}_B$ and hence $R\pi_{2*}\pi_1^*(\mathcal{L}_1 \otimes \mathcal{O}_X(\sigma)) = \mathcal{O}_X$. Plugging this into (202), we see that this sequence is just Koszul sequence again which is twisted $\mathcal{O}_X(\sigma) \otimes \pi^* K_B^*$,

$$\Phi(\mathcal{L}_1) = \mathcal{O}_\sigma(D_b)[-1]. \tag{205}$$

We can apply this result then to obtain the FM transform of V_2 for this chosen line bundle to find

$$0 \longrightarrow \mathcal{O}_\sigma(D_b) \longrightarrow \Phi^1(V_2) \longrightarrow \mathcal{O}_\sigma(-D_b) \longrightarrow 0 \ . \tag{206}$$

In this case by the arguments given above, $\Phi^1(V_2)$ is supported over the section and its rank (when restricted over the support) is two (the rank is one when restricted to the modified support). As a result, from the arguments above, we do not expect the topology of this bundle to match the formulas given and indeed they do not though we will not yet make this comparison explicitly.

Let us not contrast this with another non-generic choice of line bundle,

$$\mathcal{L}_1 = \mathcal{O}_X(-\sigma + D_b). \tag{207}$$

In this case

$$\Phi(\mathcal{O}_X(\sigma + D_b)) = \mathcal{O}_X(-\sigma + K_B + D_b), \tag{208}$$

$$\Phi(\mathcal{O}_X(-\sigma + D_b)) = \mathcal{O}_X(\sigma + D_b)[-1]. \tag{209}$$

For the choice of line bundle in (207), the extension bundle V_2 is defined by a non-trivial element of the following space of extensions:

$$Ext^1(\mathcal{L}_1^\vee, \mathcal{L}_1) = H^1(X, \mathcal{L}_1^2) = H^0(B, \mathcal{O}_B(2D_b + c_1(B)) \oplus \mathcal{O}_B(2D_b - c_1(B))), \tag{210}$$

(note that the last equality follows from a Leray spectral sequence on the elliptic threefold (see (255)), and $R\pi_*\mathcal{O}_X(-2\sigma) = K_b \oplus K_b^{-1}$. As a brief aside, we remark here that the form of this space of extensions gives us some information about the form of the possible FM dual spectral cover.

It is clear from the expression above that if $2D_b + c_1(B)$ is not effective, then there exists no non-trivial extension, and the vector bundle is simply a direct sum $\mathcal{L}_1 \oplus \mathcal{L}_1^\vee$ (and therefore not strictly stable). If $2D_b + c_1(B) = 0$ there is only one non-zero extension. On the other hand, if the degree of D_b is large enough to make $2D_b - c_1(B)$ effective then for any generic choice of extension there are $(2d_b + c_1(B)) \cdot (2D_b - c_1(B))$ isolated curves which the spectral cover must wrap.

Returning to our primary goal of computing the FM transform of V_2, it can be observed that there is enough information in (208) and (209) to compute $\Phi(V_2)$ explicitly.

$$0 \longrightarrow \Phi^0(V_2) \longrightarrow \mathcal{O}_X(-\sigma + K_B - D_b) \xrightarrow{\mathrm{F}} \mathcal{O}_X(\sigma + D_b) \longrightarrow \Phi^1(V_2) \longrightarrow 0. \tag{211}$$

By fully faithfulness of Fourier-Mukai functor, one can show $F \in Ext^0(\mathcal{O}_X(-\sigma+K_B-D_b), \mathcal{O}_X(\sigma+D_b)) \simeq Ext^1(\mathcal{L}_1^\vee, \mathcal{L}_1)$. Therefore it is necessary $2D_b - c_1(B)$ be effective to have a non zero F, and $\Phi^0(V_2) = 0$ (and hence stability of V_2). Assuming that this is satisfied, we can find the Fourier-Mukai transform of V_2 as

$$\Phi(V_2) = \mathcal{O}_{2\sigma+2D_b-K_B}(\sigma + D_b). \tag{212}$$

At last we are in a position to compute the topological data, and directly compare the bundle constructed here with what would be expected from the formulas derived.

7.2 FM Transforms of Monad Bundles over Weierstrass 3-folds

In the following section we will provide an explicit construction of the spectral data a bundle defined via a monad. This construction is somewhat lengthy, but is useful to present in detail to demonstrate that FM transforms can be explicitly constructed for bundles that appear frequently in the heterotic literature.

Over a Weierstrass CY 3-fold of the form studied consider a bundle defined as a so-called "monad" (i.e. as the kernel of a morphism between two sums of line bundles over X_3):

$$0 \longrightarrow V \longrightarrow \oplus_{i=1}^{l}\mathcal{O}_X(n_i\sigma + D_i) \xrightarrow{\text{F}} \oplus_{j=1}^{k}\mathcal{O}_X(m_j\sigma + D_j) \longrightarrow 0, \tag{213}$$

where $Rank(V) = N = l - k$, and the divisors D_i are pulled back from the base, B_2. To compute the Fourier-Mukai transform V we will see that it is necessary to begin with the transform of line bundles of the form $\mathcal{O}_X(n_i\sigma + D_i)$, as well as the morphism $\Phi(F)$. With that information, we can compute $\Phi(V)$. We should point out that for the geometry in question, none of the n_i's nor m_j's are allowed to be negative. This is necessary for stability of the bundle. Upon applying the FM functor to (213), we get a sequence of the following form,

$$\begin{array}{l} 0 \longrightarrow \Phi^0(V) \longrightarrow \oplus'^{l}_{i=1}\Phi^0(\mathcal{O}_X(n_i\sigma + D_i)) \xrightarrow{\Phi(F_0)} \oplus'^{k}_{j=1}\Phi^0(\mathcal{O}_X(m_j\sigma + D_j)) \\ \hookrightarrow \Phi^1(V) \longrightarrow \oplus''^{l}_{i=1}\Phi^1(\mathcal{O}_X(n_i\sigma + D_i)) \longrightarrow \oplus''^{k}_{j=1}\Phi^1(\mathcal{O}_X(m_j\sigma + D_j)) \longrightarrow 0. \end{array} \tag{214}$$

In the diagram above we employ the sign $\oplus'$ to refer to the direct sum over the line bundles with positive definite relative degree, and use $\oplus''$ to mean the direct sum over the line bundles with with relative degree zero (i.e. pull back of line bundles in the base). So to compute the Fourier-Mukai transform of V we need to compute the Fourier-Mukai transform of the line bundles in (213). To do this, one can simply use the defining sequence of the diagonal divisor. Combining this with the sequence above, give the following diagram,

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & \ldots & \longrightarrow & \oplus'^{l}_{i=1}\Phi^0(\mathcal{O}_X(n_i\sigma + D_i)) & \xrightarrow{\Phi(F_0)} & \oplus'^{k}_{j=1}\Phi^0(\mathcal{O}_X(m_j\sigma + D_j)) & \longrightarrow & \ldots & & \\
 & & & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K_1 & \longrightarrow & \mathcal{A} \otimes \mathcal{O}_X(\sigma + c_1(B)) & \xrightarrow{F_0} & \mathcal{N} \otimes \mathcal{O}_X(\sigma + c_1(B)) & \longrightarrow & Q_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle ev} & & \downarrow{\scriptstyle ev} & & \downarrow & & \\
0 & \longrightarrow & K_2 & \longrightarrow & \oplus'^{l}_{i=1}\mathcal{O}_X((n_i+1)\sigma + D_i) \otimes \mathcal{O}_X(\sigma + c_1(B)) & \xrightarrow{F_0} & \oplus'^{k}_{j=1}\mathcal{O}_X((m_j+1)\sigma + D_j) \otimes \mathcal{O}_X(\sigma + c_1(B)) & \longrightarrow & Q_2 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}$$

Each column in the diagram is defines the Fourier-Mukai transform of the (direct sum of) line bundles by means of the resolution of the Poincare sheaf. Therefore in the second row $\mathcal{A}$ and $\mathcal{N}$ are the sheaves generated by the "fiberwise" global sections of the sheaves $\oplus' \mathcal{O}_X((n_j+1)\sigma + D_j)$ and $\oplus' \mathcal{O}_X((m_j+1)\sigma + D_j)$, respectively. The evaluation maps simply takes the global section, and evaluates the sheaf at each point. Finally, the map F_0 is simply the map induced by the monad map F itself (from (213)) on the line bundles with positive definite relative degree (which also acts on the "fiberwise" global sections too).

The most important parts of this diagram are the induced maps between the kernels and co-kernels, K_1, Q_1 and K_2, Q_2, respectively. The kernel and co-kernel of these maps give a rather explicit presentation of the spectral data, so we will give them specific names,

$$0 \longrightarrow \bar{\mathcal{L}} \longrightarrow K_1 \longrightarrow K_2 \longrightarrow \mathcal{L} \longrightarrow 0, \tag{215}$$

$$0 \longrightarrow \mathcal{M} \longrightarrow Q_1 \longrightarrow Q_2 \longrightarrow 0, \tag{216}$$

(note that the final map in the second line above must be surjective, otherwise it will be in contradiction with the commutativity of the middle two columns in the diagram.

Now, by careful diagram chasing, one can prove that the Fourier-Mukai transform of V can be given by the following (more consise) diagram,

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ & & \mathcal{L} & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & \mathcal{J} & \longrightarrow & \Phi^1(V) & \longrightarrow & \oplus''^{l}_{i=1}\Phi^1(\mathcal{O}_X(n_i\sigma + D_i)) & \longrightarrow & \oplus''^{k}_{j=1}\Phi^1(\mathcal{O}_X(m_j\sigma + D_j)) \longrightarrow 0 \\ & & \downarrow & & & & & & \\ & & \mathcal{M} & & & & & & \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

This construction is similar in spirit to the spectral data derived for monads and we will return to this.

To make this abstract formalism more concrete, it is helpful to consider an explicit example. Let us take X_3 to be a Weierstrass elliptically fibered threefold over $\mathbb{P}^2$, realized as a hypersurface in a toric variety, given by the charge data. Here the holomorphic zero section is determined by the divisor $z = 0$. As an explicit monad bundle over this manifold, consider the following short exact sequence:

$$0 \longrightarrow V \longrightarrow \mathcal{O}_X(2,3) \oplus \mathcal{O}_X(1,6) \oplus \mathcal{O}_X(0,1)^{\oplus 3} \xrightarrow{F} \mathcal{O}_X(3,12) \longrightarrow 0. \tag{217}$$

We first need to find the Fourier-Mukai of the line bundles. This can be done using the tools outlined in before and we simply summarize the results here:

$$\Phi(\mathcal{O}_X(D)) = \mathcal{O}_\sigma(K_B + D)[-1], \tag{218}$$

$$0 \longrightarrow \Phi^0(\mathcal{O}_X(2\sigma - K_B)) \longrightarrow \mathcal{O}_X(\sigma - 2K_B) \oplus \mathcal{O}_X(\sigma) \oplus \mathcal{O}_X(\sigma + K_B) \xrightarrow{ev} \mathcal{O}_X(4\sigma - 2K_B) \longrightarrow 0, \tag{219}$$

$$0 \longleftarrow \Phi^0(\mathcal{O}_X(\sigma - 2K_B)) \longrightarrow \mathcal{O}_X(\sigma - 3K_B) \oplus \mathcal{O}_X(\sigma - K_B) \longrightarrow \mathcal{O}_X(3\sigma - 3K_B) \longrightarrow 0, \tag{220}$$

$$0 \longrightarrow \Phi^0(\mathcal{O}_X(3\sigma - 4K_B)) \longrightarrow \mathcal{O}_X(\sigma - 5K_B) \oplus \cdots \oplus \mathcal{O}_X(\sigma - K_B) \xrightarrow{ev} \mathcal{O}_X(5\sigma - 5K_B) \longrightarrow 0, \tag{221}$$

where the middle bundles in the each of the short exact sequences above are the "fiberwise" global section of the line bundles in (213) denoted as $\mathcal{A}$ and $\mathcal{N}$ (twisted with $\mathcal{O}(\sigma + c_1(B))$). With this we have determined the columns of (215). We come now to our central claim in this section: If restriction of $\mathcal{L}$ on Σ is a trivial line bundle, then it is always possible to deform the "singular" spectral data to a "smooth" spectral data, such that it satisfies the generic formulae expected Otherwise it is impossible (generically). In particular if the restriction is a non-trivial degree zero line bundle, the deformation is obstructed.

First note that if $\mathcal{L}$ is defined as

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_2 \longrightarrow 0, \tag{222}$$

the restriction of $\mathcal{L}$ on S_1 and S_2 are

$$\begin{aligned} &\mathcal{L}_1 \otimes K_{S_2}|_{S_1}, \\ &\mathcal{L}_2, \end{aligned} \tag{223}$$

respectively. Therefore the line bundle induced over Σ lives in

$$Hom_\Sigma(\mathcal{L}_2, \mathcal{L}_1 \otimes K_{S_2}|_{S_1}) \simeq Ext^1_X(i_{S_2*}\mathcal{L}_2, i_{S_1*}\mathcal{L}_1), \tag{224}$$

corresponding to extensions. Conversely, if we define $\mathcal{L}$ as,

$$0 \longrightarrow \mathcal{L}_2 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_1 \longrightarrow 0, \tag{225}$$

the restriction of $\mathcal{L}$ on S_1 and S_2 are

$$\begin{aligned} &\mathcal{L}_2 \otimes K_{S_1}|_{S_2}, \\ &\mathcal{L}_1, \end{aligned} \tag{226}$$

respectively. Therefore the line bundle induced over Σ lives in

$$Hom_\Sigma(\mathcal{L}_1, \mathcal{L}_2 \otimes K_{S_1}|_{S_2}) \simeq Ext^1_X(i_{S_1*}\mathcal{L}_1, i_{S_2*}\mathcal{L}_2), \tag{227}$$

corresponding to the opposite extensions. If we rewrite the left hand side of (224) as,

$$\begin{aligned} & H^0(\Sigma, \mathcal{F}), \\ & \mathcal{F} := \mathcal{L}_1 \otimes \mathcal{L}_2^* \otimes K_{S_2}|_{S_1}, \end{aligned} \tag{228}$$

then 227 can be written as,

$$H^0(\Sigma, \mathcal{F}^* \otimes K_\Sigma). \tag{229}$$

Therefore we see if $\mathcal{F} \simeq \mathcal{O}_\Sigma$,then both extensions are possible, and we can deform the spectral data to generic "smooth" one described in FMW.

8 Basics about Derived Category

Since the Fourier-Mukai functor, which we use a lot in this paper, is a special integral transform, we devote this appendix on reviewing some key points about them. First of all note that any functor between two categories $F : \mathcal{A} \to \mathcal{B}$ induces a map between the space of morphisms,

$$Hom_{\mathcal{A}}(A, B) \to Hom_{\mathcal{B}}(F(A), F(B)), \tag{230}$$

where A, B are arbitrary objects of the category $\mathcal{A}$ (i.e. the map is "functorial"). In case the categories are additive the set of morphisms form an abelian group, and in the cases we are concerned in this paper they are actually $\mathbb{C}-$vector spaces. Abelian categories are particular additive categories that for any functor one can define kernel and cokernel. The specific category we need in this paper is $\mathbf{Coh}(X)$, i.e. the category of coherent sheaves over a variety X, and the categories derived from that.

A functor $F : \mathcal{A} \to \mathcal{B}$ is called full if the map ((230)) is surjective and it is called faithful if it is injective. So a fully faithful functor induces an isomorphism in ((230)).

A functor $G : \mathcal{B} \to \mathcal{A}$ is a right adjoint of $F : \mathcal{A} \to \mathcal{B}$, written as $F \dashv G$ if

$$Hom_{\mathcal{B}}(F(A), B) \sim Hom_{\mathcal{A}}(A, G(B)), \tag{231}$$

where $A \in \mathcal{A}$ and $B \in \mathcal{B}$ are any arbitrary object. In particular one can see

$$Hom_{\mathcal{B}}(F(A), F(B)) \sim Hom_{\mathcal{A}}(A, GoF(B)),$$

A functor $F : \mathcal{A} \to \mathcal{B}$ is called equivalence if there are functors $G, H : \mathcal{B} \to \mathcal{A}$ such that they satisfy the functor isomorphisms $GoF \sim id_{\mathcal{A}}$ and $FoH \sim id_{\mathcal{B}}$.
It is now easy to see [?] that if a functor is fully faithful and have both left and right adjoint then it is an equivalence.

Suppose $\mathcal{A}$ is an abelian category. Then one defines the category of complex $C(\mathcal{A})$, which it's objects are complexes of objects in
$mathcalA$,

$$A^\bullet := \ldots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \longrightarrow \ldots \tag{232}$$

such that $d^i \circ d^{i-1} = 0$. The morphisms in $C(\mathcal{A})$ between two objects $h : A^\bullet \to B^\bullet$ are defined by a collection of morphisms $\{h^i\}$ in $\mathcal{A}$ as,

$$\begin{array}{ccccccc} \ldots & \longrightarrow & A^{i-1} & \xrightarrow{d_A^{i-1}} & A^i & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle h^{i-1}} & & \downarrow{\scriptstyle h^i} & & \\ \ldots & \longrightarrow & B^{i-1} & \xrightarrow{d_B^{i-1}} & B^i & \longrightarrow & \ldots \end{array} \tag{233}$$

which must be commutative. There are several remarks that must be mentioned,

i) One can define the shift functor ,$T : C(\mathcal{A}) \to C(\mathcal{A})$, naturally in this category as,

$$\begin{aligned} A^\bullet[1] &:= T(A^\bullet), \\ (A^\bullet[1])^i = A^{i+1}, &\qquad d^i_{A^\bullet[1]} = -d^{i+1}_{A^\bullet}. \end{aligned} \tag{234}$$

ii) As usual one can define cohomology for complexes,

$$\mathcal{H}^i(A^\bullet) = \frac{Ker(d^i)}{Im(d^{i-1})}. \tag{235}$$

Two complexes $A^\bullet$, $B^\bullet$ are said to be Quasi Isomorphic if all of their cohomologies are isomorphic.

Roughly speaking, derived category is “derived” from the homotopy category by localizing with the ”ideal of quasi isomorphisms”. In other words $Ob(D(\mathcal{A})) := Ob(C(\mathcal{A}))$, and morphisms in $D(\mathcal{A})$ between two objects $A^\bullet$, $B^\bullet$ are like,

$$\begin{array}{ccccc} & & C^\bullet & & \\ & \overset{qis}{\swarrow} & & \overset{f}{\searrow} & \\ A^\bullet & & & & B^\bullet \end{array} \tag{236}$$

In general f is a general morphism in homotopy category. As a result if f is also a quasi isomorphism, then the corresponding morphism in the derived category is isomorphism. So in $\mathcal{A}$, if cohomology of two complex is isomorphic, then the complexes themselves are isomorphic. From now on we restrict ourselves to bounded derived categories, $D^b(\mathcal{A})$, which it's objects are isomorphic to complexes with bounded cohomology complexes.

If a functor $F : K(\mathcal{A}) \to K(\mathcal{B})$ between homotopy categories is compatible with quasi isomorphisms, i.e. it sends quasi isomorphisms to quasi isomorphisms (pr equivalently it sends acyclic complexes to acyclic complexes), then it naturally induces a functor on derived categories. But generally it may not happen, so one need to ‘derive’ a functor from F such that it is compatible with ‘localization’ of morphisms with quasi isomorphisms. This functor is called derived functor RF. Here we briefly describe the derived functors that we are going to use them in this paper.

From now on, we restrict ourselves with categories of coherent sheaves $Coh(X)$ and quasi coherent sheaves $Qcoh(X)$ over a variety X. In particular it is possible to show

$$D^b_{Coh(X)}(Qcoh(X)) \sim D^b(Coh(X)), \tag{237}$$

where the left hand side corresponds to derived category of complexes of quasi coherent sheaves which their cohomologies are coherent sheaves. One define the bounded derived category of X as $D^b(X) := D^b(Coh(X))$.

Here the goal is to find the derived functor of $f_* : Coh(X) \longrightarrow Coh(Y)$ induced from a projective (or at least proper) morphism of varieties $f : X \longrightarrow Y$.

If we have proper morphism of varieties $f : X \longrightarrow Y$, then the (right) direct image $Rf_* : D^b(X) \longrightarrow D^b(Y)$ is defoned in the following way,

1) For any complex of coherent sheaves $A^\bullet$ with bounded cohomology, we have an injective resolution $A^\bullet \longrightarrow I(A^\bullet)$.

2) Define

$$\begin{aligned} Rf_*(A^\bullet) &:= f_*(I(A^\bullet)), \\ R^i f_*(A^\bullet) &:= \mathcal{H}^i(f_*(I(A^\bullet))). \end{aligned} \tag{238}$$

Lets start by the following definition,

A complex in $\mathcal{I}^\bullet \in C(\mathcal{M}od(X))$ is called injective COMPLEX if the right exact functor $Hom^\bullet_{C(\mathcal{M}od(X))}(\ldots, \mathcal{I}^\bullet) : C(\mathcal{M}od(X)) \longrightarrow \mathcal{A}b$ maps any acyclic complex to another acyclic complex (or equivalently map any quasi isomorphism to another quasi isomorphism).

Now it can be proved a bounded bellow complex of injective sheaves is actually an injective complex. So as before for a complex $A^\bullet$ one can define a resolution by injective objects $B^\bullet \to \mathcal{I}^\bullet$, and define

$$RHom^\bullet_{C(\mathcal{M}od(X))}(A^\bullet, \ldots) : D^b(X) \longrightarrow D^b(\mathcal{A}b), \tag{239}$$

$$RHom^i_{C(\mathcal{M}od(X))}(A^\bullet, B^\bullet) := \mathcal{H}^i(Hom_{C(\mathcal{M}od(X))}(A^\bullet, \mathcal{I}^\bullet)). \tag{240}$$

Without getting into more details, we state that relative to the first "variable" (i.e. $A^\bullet$), the functor defined above is consistent with the quasi isomorphisms. So if we consider $RHom$ as a functor on the first variable, it naturally induces a well defied functor in the derived category. Therefore,

$$RHom : D^0(X) \times D^b(X) \longrightarrow D(\mathcal{A}b), \tag{241}$$

where $D^0(X)$ is the opposite category of $D(X)$. $Ext^i_{D(X)}(A^\bullet, B^\bullet) := R^i Hom(A^\bullet, B^\bullet)$.

So far we only considered the global Hom functor, but in the case of sheaves one can define a local version [27]$\mathcal{H}om$,

$$R\mathcal{H}om_{\mathcal{O}_X} : D^0(X) \times D^b(X) \longrightarrow D^b(X), \tag{242}$$

and similar to the global version one has local "ext" sheaves,

$$\mathcal{E}xt^i_{\mathcal{O}_X}(A^\bullet, B^\bullet) := R^i\mathcal{H}om_{\mathcal{O}_X}(A^\bullet, B^\bullet). \tag{243}$$

Lets start by reviewing some standard facts,

i) For any sheaf A, the functor $A \otimes \dots$ is right exact, and A is flat if $A \otimes \dots$ is exact.

ii) For any coherent sheaf A, there is a flat resolution of finite length

$$\dots \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_0 \longrightarrow A \longrightarrow 0, \tag{244}$$

where $\mathcal{F}_i$'s are flat sheaves.

iii) One can define the tensor product of two complexes $A^\bullet \otimes B^\bullet$ as a double complex.

iv) A flat complex is defined as complex $\mathcal{P}^\bullet$, which the functor $\mathcal{P}^\bullet \otimes \dots$, maps acyclic complexes to acyclic complexes (or equivalently quasi isomorphism to quasi isomorphism).

v) A bounded above (in particular bounded) complex of flat sheaves is a flat complex. For a bounded complex of coherent sheaves, $B^\bullet$, then (using point (ii)) one can find a quasi isomorphism $\mathcal{P}^\bullet \longrightarrow B^\bullet$. If $\mathcal{P}^\bullet$ is both flat and acyclic, then $B^\bullet \otimes \mathcal{P}^\bullet$ is again acyclic for any complex $B^\bullet$.

As before one can define the derived tensor product as,

$$RF_{A^\bullet} := A^\bullet \otimes^L \dots : D^b(X) \longrightarrow D^b(X), \tag{245}$$

$$RF^i_{A^\bullet}(B^\bullet) = \mathcal{H}^i(A^\bullet \otimes \mathcal{P}^\bullet). \tag{246}$$

Note that the process of defining derived tensor product is symmetric, and one could define it using the first variable. Also if there is a quasi isomorphism $A^\bullet \xrightarrow{qis} B^\bullet$, then we have a functor isomorphism $F_{A^\bullet} \sim F_{B^\bullet}$. So naturally the derived tensor product descends to a well defined functor in derived category relative to the first variable,

$$\dots \otimes^L \dots : D^b(X) \times D^b(X) \longrightarrow D^b(X). \tag{247}$$

$$\mathcal{T}or_i(A^\bullet, B^\bullet) := \mathcal{H}^{-i}(A^\bullet \otimes^L B^\bullet). \tag{248}$$

Finally we are at the position to review the definition the left derived functor for the pullback of a morphism $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$. As before, we recall some basic facts and then compare with the general definition.

i) Recall that the pull back of a sheaf under f is defined as,

$$f^*(\mathcal{F}) := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}. \tag{249}$$

ii) There is a projective resolution for every coherent sheaf,

$$\dots \longrightarrow \mathcal{P}^1 \longrightarrow \mathcal{P}^0 \longrightarrow \mathcal{F} \longrightarrow 0, \tag{250}$$

This induces a quasi isomorphism for any bounded complex of coherent sheaves (at least bounded above) one gets a quasi isomorphism $\mathcal{P}^\bullet \xrightarrow{qis} \mathcal{F}^\bullet$.

So by combining these facts and what we learned for derived tensor product we can write,

$$
\begin{aligned}
Lf^*(\mathcal{F}^\bullet) &:= \mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}^\bullet, \\
L_i f^*(\mathcal{F}^\bullet) &:= \mathcal{H}^{-i}(f^*(\mathcal{P}^\bullet)).
\end{aligned} \tag{251}
$$

Here we collect the identities that are going to be useful in the calculations throughout this paper.

Lets start with following general theorem, Suppose $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{C}$ be functors between abelian categories such that $G(K_F) \subset K_G$ (look at the definition of derived functors). Then one gets the following identity,

$$R(G \circ F) = RG \circ RF. \tag{252}$$

This theorem looks pretty simple, but it allows us to combine derived functors. Basically it says there is a spectral sequence,

$$E_2^{p,q} := R^pG(R^q(F)) \Longrightarrow E_\infty^{p+q} := R^{p+q}G \circ F. \tag{253}$$

Here we review some of the applications. First lets consider the direct image of a bounded complex,

$$R^i f_*(\mathcal{H}^j(\mathcal{F}^\bullet)) \Rightarrow R^{i+j} f_*\mathcal{F}^\bullet. \tag{254}$$

Obviously one can write a similar spectral sequence formula to compute the derived functor of complexes. Another example is the global section functor over a variety X, $\Gamma : Coh(X) \longrightarrow \mathcal{A}b$. The direct images of this functor are just the cohomology of sheaves [27], i.e. $R^i\Gamma(\mathcal{F}) = H^i(X, \mathcal{F})$. Now let combine this with the direct image functor induced by a proper morphism $f : X \longrightarrow Y$,

$$
\begin{aligned}
\Gamma_Y : Coh(Y) :\longrightarrow point, \quad \Gamma_X = \Gamma_Y \circ f_* : Coh(X) \longrightarrow point, \\
R\Gamma_X(\mathcal{F}) = R\Gamma_Y \circ Rf_*(\mathbb{F}), \\
E_2^{p,q} = H^p(Y, R^q f_*\mathcal{F}) \Rightarrow E_\infty^{p+q} = H^{p+q}(X, \mathcal{F}).
\end{aligned} \tag{255}
$$

Last line is nothing but Leray spectral sequence. As the final example consider the relation between local extension $\mathcal{E}xt$, and the global extension Ext,

$$R\Gamma \circ R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = RHom_{D^b(X)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet). \tag{256}$$

In particular if we apply this to concentrated complexes at zero position (i.e. a single coherent sheaf), we get the following famous result,

$$H^i(X, \mathcal{E}xt^j_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) \Rightarrow Ext^{i+j}_X(\mathcal{F}, \mathcal{G}) \tag{257}$$

[Base change formula] Consider the following commutative diagram of proper morphisms,

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{g'} & Y' \end{array}$$

Then, in general, there is a morphism of functors ,

$$Lf'^{*}Rg'_{*} \longrightarrow Rf_{*}Lg^{*}. \tag{258}$$

In particular if f (g) is flat, then f' (g') is flat, and the above morphism is actually isomorphism of functor. One of the main properties of Fourier Mukai functor is its compatibility with the base change, and therefore the theorem above will be very useful.

[Dualizing Complex] Consider a proper morphism $f_X \longrightarrow Y$, it's dualizing complex is defined as,

$$Hom_{D^b(Y)}(Rf_*\mathcal{F}^\bullet, \mathcal{G}^\bullet) = Hom_{D^b(X)}(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet). \tag{259}$$

In particular it satisfies the identities,

$$f^!\mathcal{G}^\bullet = Lf^*\mathcal{G} \otimes^L f^!\mathcal{O}_Y, \tag{260}$$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle g} \\ & & Z \end{array} \quad s.t. \quad h = g \circ f \Longrightarrow h^! = f^! \circ g^!. \tag{261}$$

So by the first identity one only needs to know the dualizing complex of morphism relative to the structure sheaf. A morphism is called *Gorenstein* if the dualizing complex is a concentrated complex, i.e. $f^!\mathcal{O}_Y = \Omega[k]$ for some $k \in \mathbb{Z}$. There two specific cases that will be useful for us in this paper,

Flat Fibration In this case $f^!\mathcal{O}_Y = \omega_{X/Y}[n]$, where n is the relative dimension (i.e. the dimension of the fibers), and $\omega_{X/Y} = \omega_X \otimes f^*\omega_Y$.

Complete intersection This is an inclusion morphism $f : X \hookrightarrow Y$ where X is a complete intersection of varieties in Y. In this case $f^!\mathcal{O}_Y = det(\mathcal{N})[-d]$, where $\mathcal{N}$ is the normal bundle, and d is the codimension of X is Y.

The definition above is called Grothendieck-Verdier duality, and it is a general form of Serre duality. There is also a local version of this duality,

$$R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*\mathcal{F}^\bullet, \mathcal{G}^\bullet) = Rf_*R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^\bullet, Lf^*\mathcal{G}^\bullet \otimes^L f^!\mathcal{O}_Y). \tag{262}$$

One can define derived dual of a complex $\mathcal{F}^\bullet \in D^b(X)$ as,

$$\mathcal{F}^{\bullet\vee} := R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{O}_X). \tag{263}$$

$$R\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq R\mathcal{H}om(\mathcal{O}_X, \mathcal{F}^{\bullet\vee} \otimes^L \mathcal{G}^\bullet) \simeq \mathcal{F}^{\bullet\vee} \otimes^L \mathcal{G}^\bullet \tag{264}$$

$Rf_* \dashv Lf^*$,

$$RHom_{D^b(Y)}(\mathcal{F}^\bullet, Rf_*\mathcal{G}^\bullet) \simeq RHom_{D^b(X)}(Lf^*\mathcal{F}^\bullet, \mathcal{G}^\bullet), \tag{265}$$
$$R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}^\bullet, Rf_*\mathcal{G}^\bullet) \simeq Rf_*R\mathcal{H}om_{\mathcal{O}_X}(Lf^*\mathcal{F}^\bullet, \mathcal{G}^\bullet). \tag{266}$$

[The Projection Formula is]

$$Rf_*(Lf^*\mathcal{F}^\bullet \otimes^L \mathcal{G}^\bullet) = \mathcal{F}^\bullet \otimes^L Rf_*\mathcal{G}^\bullet. \tag{267}$$

The commutative diagram bellow for a projective morphism f,

$$\begin{array}{ccc} f^{-1}(p) & \xhookrightarrow[i_f]{} & X \\ {\scriptstyle f_p}\downarrow & & \downarrow{\scriptstyle f} \\ p & \xhookrightarrow[i]{} & Y \end{array} \tag{268}$$

we get the following results when $\mathcal{F} \in Coh(X)$. They will be very useful in many cases, and also give a rather clear intuitive picture about the direct images,

$$Li^*Rf_*\mathcal{F} \longrightarrow Rf_{p*}(Li_f^*\mathcal{F}),$$
$$\phi^j : (Li^*Rf_*\mathcal{F})^j = Tor_{-j}^{i^{-1}\mathcal{O}_Y}(Rf_*\mathcal{F}, \mathcal{O}_p) = R^jf_*\mathcal{F} \otimes \mathcal{O}_p \longrightarrow H^j(f_p^{-1}(p), i_f^*\mathcal{F}). \tag{269}$$

It is proved in [27] III.12.10 that ϕ^j is isomorphism if and only of it is surjective, and $R^jf_*\mathcal{F}$ is locally free if and only if ϕ^{j-1} is surjective.

9 Integral Functors

In this section we briefly review the main features of integral functors, specially the Fourier Mukai functors which are the important special cases. Let $D^b(X)$ and $D^b(Y)$ be the derived category of varieties X and Y. Consider the following morphisms,

$$\begin{array}{ccccc} & & X \times Y & & \\ & {\scriptstyle \pi_X}\swarrow & & \searrow{\scriptstyle \pi_Y} & \\ X & & & & Y \end{array} \tag{270}$$

Then the integral functor $\Phi_{X\to Y}^{\mathcal{P}^\bullet}$ is defined in the following way,

$$\Phi_{X\to Y}^{\mathcal{P}^\bullet} : D^b(X) \longrightarrow D^b(Y),$$

$$\Phi_{X\to Y}^{\mathcal{P}^\bullet}(\dots) := R\pi_{Y*}(\pi_X^*(\dots) \otimes^L \mathcal{P}^\bullet), \tag{271}$$

where π_X and π_Y are projections to the corresponding factors, and $\mathcal{P}^\bullet$ is the kernel of the transform. Note that π_X is a flat morphism, so $L\pi_X^* = \pi_X^*$. In particular if the integral transform of a sheaf $\mathcal{E}$ (consider it as complex which is only non-zero at the zero entry, i.e. concentrated on the zero position) is concentrated the ith position, it is called a WIT_i sheaf.

Note that any integral functor is a composition of three exact functors in derived categories, derived inverse image, derived tensor product and derived direct image. So $\Phi^{\mathcal{P}^\bullet}_{X\to Y}$ is also an exact functor. In particular, to any short exact sequence there is an associated long exact sequence induced by that integral functor.

We are particularly interested in "relative" integral transforms. Suppose $\Phi^{\mathcal{K}}_{X\to Y}: D^b(X) \longrightarrow D^b(Y)$ be an integral transform, for any variety T, the corresponding relative integral functor (relative to T) $\Phi^{\mathcal{K}^\bullet_T}_{X\times T\to Y\times T}$ is defined as

$$\begin{array}{ccccc} & & X\times Y\times T & & \\ & \swarrow^{\pi_{X\times T}} & \downarrow{\scriptstyle \pi_{X\times Y}} & \searrow^{\pi_{Y\times T}} & \\ X\times T & & X\times Y & & Y\times T \end{array}$$

$$\Phi^{\mathcal{K}^\bullet_T}_T(\dots) := R\pi_{Y\times T_*}(\pi^*_{X\times T}(\dots) \otimes^L \mathcal{K}^\bullet_T),$$

$$\mathcal{K}^\bullet_T := \pi^*_{X\times Y}\mathcal{K}^\bullet. \tag{272}$$

Now consider a morphism of varieties $f: S \longrightarrow T$, and the induced relative morphisms: $f_X: S\times X \longrightarrow T\times X$ and $f_Y: S\times Y \longrightarrow T\times Y$, then one can prove the following functorial isomorphism,

$$Lf^*_Y\Phi_T(\mathcal{E}^\bullet) \simeq \Phi_S(Lf^*_X\mathcal{E}^\bullet), \tag{273}$$

with $\mathcal{E}^\bullet \in D^b(X\times T)$. In particular if $j_t: t \longrightarrow T$ be the inclusion of a point t, then the identity above gives,

$$Lj^*_t\Phi_T(\mathcal{E}^\bullet) = \Phi_t(Lj^*_t\mathcal{E}^\bullet), \tag{274}$$

This has important consequences: first of all if $\mathcal{E}$ is a sheaf, one can prove (by checking the spectral sequences of the combined functors),

$$\Phi^{n_m}_t(j^*_t\mathcal{E}) \simeq j^*_t\Phi^{n_m}_T(\mathcal{E}), \tag{275}$$

where n_m is the maximal integer that either $\Phi^{n_m}_t$ or $\Phi^{n_m}_T$ is non zero. Moreover, if both $\mathcal{E}$ and $\Phi^i_T(\mathcal{E})$ are flat over T, then $\mathcal{E}_t$ is WIT_i relative to Φ_t if and only if $\mathcal{E}$ is WIT_i relative to Φ_T. This is an important point, and when we want to describe the Fourier-Mukai transform of vector bundles which are unstable over some non generic elliptic fibers, or when we need to deal with general coherent sheaves, it is going to help us.

Finally we mention that there are similar result for non trivial fibration, which we discuss briefly later. For now, let's move on to review Fourier-Mukai functors briefly.

A Fourier Mukai functor is an integral functor which is also an exact equivalence.

Probably the first important point about Fourier-Mukai functors is that any equivalence can be written as Fourier-Mukai,

[Orlov's representability theorem] Let X and Y be two smooth projective varieties, and let

$$F : D^b(X) \longrightarrow D^b(Y)$$

be a fully faithful exact functor. If F admits right and left adjoint functors, then there exists an object $\mathcal{P}^\bullet \in D^b(X \times Y)$ unique up to isomorphism such that F is isomorphic to a Fourier Mukai functor $\Phi^{\mathcal{P}}_{X\to Y}$.

There is a partial inverse to this theorem, due to Bondal and Orlov, which states when an integral functor is indeed fully faithful, i.e. it puts constraints over the kernel of the transform,

Let X and Y be smooth projective varieties. Consider $\Phi^{\mathcal{P}^\bullet}_{X\to Y} : D^b(X) \longrightarrow D^b(Y)$ with $\mathcal{P}^\bullet$ in $D^b(X \times Y)$. Then $\Phi^{\mathcal{P}^\bullet}_{X\to Y}$ is a fully faithful functor if and only if $\mathcal{P}^\bullet$ is a strongly simple object over X, i.e.

$$Hom^i_{D^b(Y)}(Lj^*_{x_1}\mathcal{P}^\bullet, Lj^*_{x_2}\mathcal{P}^\bullet) = 0 \quad unless \quad x_1 = x_2 \quad and \quad 0 \leq i \leq dimX; \tag{276}$$
$$Hom^0_{D^b(Y)}(Lj^*_{x}\mathcal{P}^\bullet, Lj^*_{x}\mathcal{P}^\bullet) = \mathbb{C}. \tag{277}$$

In addition, if $Lj^*_x\mathcal{P}^\bullet$ is a special object of $D^b(Y)$, i.e. $Lj^*_x\mathcal{P}^\bullet \otimes K_Y \simeq Lj^*_x\mathcal{P}^\bullet$, then $\Phi^{\mathcal{P}^\bullet}_{X\to Y}$ is an equivalence. In particular if both X and Y are both smooth Calabi-Yau varieties, and the kernel is a strongly simple object, then the corresponding integral functor is a Fourier-Mukai functor.

It is worth to mention another very important property of Fourier-Mukai functors, and that is these kind of integral functors are sensitive to smoothness and " Calabi-Yau ness", and dimension. In other words, if tow varieties X and Y are Fourier-Mukai partners (their derived category are equivalent), then X is smooth if and only if Y is smooth (this proved by Serre's criterion on regular local rings of finite homological dimension), and X is Calabi-Yau if and only if Y is Calabi-Yau (this is proved by using Grothendieck-Verdier duality), and both of them must have the same dimension. There are also other geometrical constraints which are induced by the equivalence condition, but we ignore them here.

We finish this section by quickly deriving the inverse transform of a Fourier-Mukai functor $\Phi^{\mathcal{P}^\bullet}_{X\to Y}$. Since for an equivalence of categories, the adjoint functor is actually the inverse functor, one can find it easily for the Fourier Mukai functor as follows,

$$\begin{aligned} RHom_{D^b(Y)}(\Phi^{\mathcal{P}^\bullet}_{X\to Y}(\mathcal{F}^\bullet), \mathcal{G}^\bullet) &= RHom_{D^b(X\times Y)}(\pi^*_X\mathcal{F}^\bullet, \pi^*_Y\mathcal{G}^\bullet \otimes^L \mathcal{P}^{\bullet\vee} \otimes \pi^*_X\omega_X[n]) \\ &= RHom_{D^b(X)}(\mathcal{F}^\bullet, R\pi_{X*}(\pi^*_Y\mathcal{F}^\bullet \otimes^L \mathcal{P}^{\bullet\vee} \otimes \pi^*_X\omega_X[n])) \\ &= RHom_{D^b(X)}(\mathcal{F}^\bullet, \Phi^{\mathcal{P}^{\bullet\vee}\otimes\pi^*_X\omega_X[n]}_{Y\to X}(\mathcal{G}^\bullet)), \end{aligned} \tag{278}$$

where $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are generic objects of derived category of varieties X and Y, n is the dimension of both X and Y, and ω_X is the canonical sheaf of X. Therefor the "inverse transform" is itself a Fourier Mukai functor,

$$\Phi^{\mathcal{P}^{\bullet\vee}\otimes\pi^*_X\omega_X[n]}_{Y\to X}. \tag{279}$$

10 Heterotic Hyperflux

The special plan in the following part will be to elucidate how heterotic hyperflux works in the presence of localized gauge fields. To this end, we shall first review some features of bulk axion couplings to gauge theory degrees of freedom. Then, we review the hyperflux mechanism for F-theory compactification, and then translate this to our heterotic construction. In this section we exhibit a heterotic dual to the hyperflux mechanism. The main idea is to show that an abelian flux $U(1)_Y \subset SU(5)_{GUT}$ can be activated, but which also decouples from all bulk axions.

To frame our discussion, let us briefly review some aspects of the hyperflux mechanism in F-theory. We begin with F-theory compactified on a threefold base B, and study the worldvolume theory of a seven-brane with gauge group G wrapping $\mathbb{R}^{3,1} \times S$ for some Kähler surface S. In the eight-dimensional gauge theory, we have the terms,

$$S_{\rm 10D} \supset -M_*^4 \int\limits_{\mathbb{R}^{3,1}\times S} \mathrm{Tr}(F_{8D} \wedge *_8 F_{8D}) + \int\limits_{\mathbb{R}^{3,1}\times S} i^*(C_4) \wedge \mathrm{Tr}(F_{8D} \wedge F_{8D}) + M_*^6 \int\limits_{\mathbb{R}^{3,1}\times B} dC_4 \wedge *_{10} dC_4, \quad (280)$$

where $i^*(C_4)$ is the pullback of the bulk four-form potential C_4 onto $\mathbb{R}^{3,1} \times S$, F_{8D} is the 8D field strength, and M_* is a characteristic UV scale where the large volume approximation breaks down.

Suppose now we expand this theory around a non-trivial internal gauge field flux valued in some abelian subgroup $U(1) \subset G$. For ease of exposition, we treat all gauge fields as abelian. We decompose the form content of the eight-dimensional field strength as,

$$F_{8D} = F_{4D} + F_S, \quad (281)$$

for some non-zero background value of F_S. We also decompose the four-form C_4 into a basis of internal harmonic two-forms on B,

$$C_4 = r^\alpha \wedge b_\alpha, \quad (282)$$

where b_α is a two-form on B, and r^α is a two-form on $\mathbb{R}^{3,1}$ dual to an axion. Expanding around this background, we get the four-dimensional terms,

$$S_{\rm 4D} \supset -\frac{1}{4g_{U(1)}^2} \int\limits_{\mathbb{R}^{3,1}} F_{4D} \wedge *_4 F_{4D} + \int\limits_{\mathbb{R}^{3,1}} r^\alpha \wedge F_{4D} \int\limits_{S} i^*(b_\alpha) \wedge F_S + M_*^2 \int\limits_{\mathbb{R}^{3,1}} dr^\alpha \wedge *_4 dr_\alpha. \quad (283)$$

The middle term is a coupling between an axion and a gauge field. When it is non-zero, the abelian gauge field picks up a large mass of order M_*.

In F-theory GUTs, such couplings can be eliminated provided,

$$\int\limits_S i^*(b_\alpha) \wedge F_S = 0, \quad (284)$$

for all harmonic two-forms b_α on B. This can be arranged by a trivialization condition of the divisor dual to F_S inside of B. The embedding $i : S \to B$ induces the pullback map for cohomology,

$$i^* : H^2(B) \to H^2(S). \quad (285)$$

So, a nontrivial relative cohomology allows us to generate a hyperflux which decouples from all bulk axions.

Now, in heterotic strings, this GUT breaking mechanism would at first appear to be absent. As explained, for heterotic strings compactified on a Calabi-Yau threefold, the hyperflux mechanism is unavailable. This is because of the interaction terms in the ten-dimensional action,

$$S_{\rm 10D} \supset -M_*^6 \int\limits_{\mathbb{R}^{3,1}\times M} \frac{1}{g^2}{\rm Tr}(F_{10D} \wedge *_{10} F_{10D}) + \int\limits_{\mathbb{R}^{3,1}\times M} |d\Lambda + A \wedge F|^2 , \tag{286}$$

where Λ is the two-form potential of the heterotic theory. Let us now expand around a background value of the internal field strength F_M. Decompose Λ into a basis of harmonic two-forms λ_α on M,

$$\Lambda = c^\alpha \wedge \lambda_\alpha, \tag{287}$$

with c^α an axion of the four-dimensional theory. Then, upon expanding with respect to an internal flux,

$$F_{10D} = F_{4D} + F_M, \tag{288}$$

the four-dimensional effective action contains the terms,

$$S_{\rm 4D} \supset -\frac{1}{4g^2_{U(1)}} \int\limits_{\mathbb{R}^{3,1}} F_{4D} \wedge *_4 F_{4D} + \int\limits_{\mathbb{R}^{3,1}} r^\alpha \wedge F_{4D} \int\limits_{M} *_6\, \lambda_\alpha \wedge F_M + M_*^2 \int\limits_{\mathbb{R}^{3,1}} dc^\alpha \wedge *_4 dc_\alpha, \tag{289}$$

where r^α is the two-form dual to the axion c^α in four-dimensions. Again, the middle term is responsible for the Stückelberg mechanism of the four-dimensional effective theory. In the standard heterotic compactification on a Calabi-Yau threefold, the harmonic two-forms λ_α and F_M are both representatives of elements in $H^2(M)$, so the hyperflux mechanism is unavailable.

With a position dependent dilaton, however, we can localize the profile of the heterotic gauge fields. It is therefore worth revisiting whether the hyperflux mechanism holds in heterotic models. In fact, localization is by itself not enough to ensure that a given heterotic gauge bundle configuration will decouple from the axions. The main idea will be to formally construct a non-trivial vector bundle on the "standard" middle region $M_{\rm mid}$, and then show that in the full geometry $M_{\rm het}$, it trivializes. In other words, we consider the embedding

$$i : M_{\rm mid} \to M_{\rm het}, \tag{290}$$

and seek a non-trivial kernel to the pushforward

$$i_* : H_4(M_{\rm mid}, \mathbb{Z}) \to H_4(M_{\rm het}, \mathbb{Z}). \tag{291}$$

The localization of the ten-dimensional gauge fields near the gluing regions D_L and D_R means that effectively, the GUT breaking flux is localized on this lower-dimensional component of the geometry. To construct examples of gauge field configurations which trivialize in the full geometry, we can first construct a line bundle over $M_{\rm mid}$ which, upon gluing, trivializes in the full geometry $M_{\rm het}$. Along these lines, recall that $M_{\rm mid}$ is given by an elliptic fibration with section over a base D. We shall assume that there are at least two effective divisors σ_1, σ_2 with homology classes $[\sigma_i] \in H_2(D, \mathbb{Z})$ such that $\sigma_1 - \sigma_2$ is trivial inside of M_L, but is non-trivial inside of M_{mid}. This can happen because in M_{mid} there is a section to the fibration, so σ_1 and σ_2 lift to two non-trivial divisors S_1, S_2 with homology classes $[S_i] \in H_4(M_{\text{mid}}, \mathbb{Z})$. So, let us consider the line bundle $\mathcal{L}_{\text{mid}} = \mathcal{O}_{M_{\text{mid}}}(S_1 - S_2)$. Under the embedding map, we can pushforward $\mathcal{L}_{\text{mid}}$ to a rank one sheaf on M_{het}. Observe, however, that since $[S_1] = [S_2]$ in $H_4(M_{\text{het}}, \mathbb{Z})$, that the topology of the line bundle is globally trivial, even though there is a non-trivial flux localized along D_L and D_R. Indeed, upon restriction of $\mathcal{L}_{\text{mid}}$ to D, we get the line bundle $\mathcal{O}_D(\sigma_1 - \sigma_2)$. As consequence of this topological mechanism, all couplings to bulk axions automatically vanish. This includes model-dependent axions coming from harmonic two-forms of M_{het}, as well as the contribution from the universal axion of a heterotic compactification. In our analysis, we have used the gluing to the middle region as a means to track this possibility. Following up on the discussion it would be quite interesting to understand this purely from the perspective of vector bundles on M_L. Finally, note that any holomorphic vector bundle on M_{mid} which trivializes in the full geometry will automatically define a consistent solution to the Hermitian Yang-Mills equations. The reason is that the Hermitian $(1,1)$ form J_{mn} is a bulk mode defined over the entire geometry M_{het}. So, there is automatically a representative flux which satisfies the condition $F_{mn}J^{mn} = 0$. One check of the duality we can already perform involves the construction of a heterotic hyperflux. In the F-theory model, suppose we have a seven-brane wrapping a divisor $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$. There is a single generator H for the homology ring of $\mathbb{P}^3$ whereas there are two generators σ_1 and σ_2 for $\mathbb{P}^1 \times \mathbb{P}^1$. Indeed, the two-cycle $\sigma_1 - \sigma_2$ trivializes in $\mathbb{P}^3$. The seven-brane two-form flux Poincaré dual to this class gives a configuration which decouples from the bulk axions.

We can now see how a similar mechanism operates in the heterotic dual configuration. We showed how to build up a heterotic gauge field configuration which breaks $SU(5)_{\text{GUT}}$ to the Standard Model gauge group by activating a flux in the $U(1)_Y \subset SU(5)_{\text{GUT}}$ subgroup. First, we construct a line bundle on M_{mid} given by

$$\mathcal{L}_{\text{mid}} = \mathcal{O}_{M_{\text{mid}}}(S_1 - S_2), \tag{292}$$

where S_i are the divisor classes coming from the two $K3$ fibers of the elliptic fibration $T^2 \to M_{\text{mid}} \to \mathbb{P}^1_{(1)} \times \mathbb{P}^1_{(2)}$. Upon restriction to the base $D = \mathbb{P}^1_{(1)} \times \mathbb{P}^1_{(2)}$, the line bundle becomes

$$\mathcal{L}_{\text{mid}}|_D = \mathcal{O}_D(\sigma_1 - \sigma_2), \tag{293}$$

where σ_i is the divisor class for one of the $\mathbb{P}^1_{(i)}$ factors. The important feature is that this class $\sigma_1 - \sigma_2$ trivializes in M_{het}.

Finally, the heterotic dual for X_R is M_R, that is, another copy of M_L, and is constructed in the same way as M_L. In the heterotic theory, these geometric building blocks are then glued together to construct the full compact six-manifold,

$$M_{\text{het}} = M_L \cup_{D_L} M_{\text{mid}} \cup_{D_R} M_R. \tag{294}$$

To complete the analysis, we also need to specify the profile of the heterotic fields on the other side of the duality. Here, we must again take our guidance from the F-theory geometry. First of all, deep in the middle region M_{mid}, we have a standard compactification of heterotic strings on a Calabi-Yau threefold. This means the heterotic dilaton can be taken to be a constant, and there is no three-form flux switched on. A particularly interesting feature of this specific heterotic dual is that the presence of more than one $K3$ fibration in the F-theory geometry means we have various string/string dualities in the heterotic theory. Now, as we move closer to the gluing regions, the curvature of the metric becomes more pronounced. Additionally, we can see that the profile of the dilaton as well as the three-form flux also changes. Near the gluing locus D_L, we see in particular that the profile of the string coupling becomes weakly coupled, while it can be bigger deep in the M_L and M_{mid} regions. This enforces the localization of the heterotic gauge fields near the gluing region, which is simply the heterotic dual of the familiar localization of gauge theory degrees of freedom in the F-theory geometry. Finally, deep in the regions M_L and M_R, we can see that fluxes must be switched on. The simplest way to see this is to observe that even after deleting D_L to reach M_L, we still have a non-compact positive curvature six-manifold. Indeed, to reach a non-compact Calabi-Yau threefold, we would have needed to delete a $K3$ surface. It is beyond the scope of the present work to find an explicit solution to the metric and background fluxes in this region, though we can see that the duality with F-theory clearly predicts the existence of such a solution.

11 Conclusion

In this note we have proposed a generalization of heterotic/F-theory duality. Within heterotic/F-theory duality, the constrained geometric arena – Weierstrass from for both the heterotic and F-theory Calabi-Yau backgrounds – has long been a frustrating obstacle to studying new phenomena. Within heterotic effective theories for example, there are a number of interesting effects that are believed to have interesting F-theory duals, including perhaps novel mechanisms for moduli stabilization such as the linking of bundle and complex structure moduli in the heterotic theory through the condition of holomorphy and potentially new 4-dimensional $\mathcal{N} = 1$ dualities including heterotic threefolds admitting multiple elliptic fibrations and hence leading to multiple, related dual F-theory fourfolds, the F-theory duals of heterotic target space duality or F-theory duals of known “standard model like” heterotic compactifications. However in all cases, these theories have crucially involved decidedly non-Weierstrass geometry on the heterotic side. These questions have formed the motivation for the present work. We believe that here we have taken important first steps towards extending the geometries for which explicit heterotic/F-theory duals can be constructed. There remain however, important open questions. First, as mentioned above, we require new and more robust tools to address the general case of a higher rank Mordell-Weil group with rational generators. In addition, as illustrated in the explicit examples constructed all the formulas we have derived in this work have been limited by the restriction of smoothness of the spectral cover. In general many examples in the literature have demonstrated that smooth vector bundles do not necessarily correspond to smooth spectral covers. Indeed, this observation has been a powerful tool in determining the effective physics of T-brane solutions in F-theory. By placing the constraint of smoothness on the spectral data, we are clearly loosing information about general components of the bundle moduli space. Finally, there remain interesting open questions about how to determine the full Picard groups of spectral covers since these are surfaces of general type, this is a notoriously hard problem in algebraic geometry, and a number of interesting possibilities remaining to be explored related to higher co-dimensional behavior in moduli superspaces i.e. so-called "jumping" phenomena or Noether-Lefschetz problems.

On the F-theory side, the building blocks of the duality are non-compact elliptically fibered Calabi-Yau fourfolds which also admit a $K3$ fibration. These are glued together to form a compact elliptic Calabi-Yau fourfold which need not have a global $K3$ fibration. On the heterotic side, the $K3$ fiber of each F-theory building block is replaced by a T^2 fiber. In the heterotic description, the gluing also involves a non-trivial three-form flux and position dependent dilaton. Using our proposal, we reach new compact examples of heterotic/F-theory duality pairs. This leads to a localization of heterotic gauge field degrees of freedom in various regions of the geometry, and also provides a heterotic version of the hyperflux mechanism for breaking GUT groups. In other words, we have used F-theory to argue for the existence of a new class of heterotic flux vacua. In the remainder of this section we discuss some additional avenues of investigation. In this work we have mainly focussed on the general contours of our proposal, emphasizing in particular the simple form of the geometric F-theory building blocks. It would clearly be useful to confirm in purely heterotic terms the exact form of the background fields necessary to solve the equations of motion. Along these lines, it would be important to verify that the resulting low energy effective action defined by the heterotic compactification indeed matches to the one defined by the F-theory model. In the case of heterotic compactification on a model with a large radius limit, there is a simple topological check which can be performed. It would be interesting to extend this analysis to the class of flux vacua considered here. On the other hand, one might instead take the F-theory geometry as a definition of what a generalized heterotic vacuum ought to be. From this perspective, the relevant issue is to demonstrate existence of a solution and its topology rather than a direct construction of all background fields. Along these lines, one ingredient which would be very interesting to work out in more detail concerns the construction of heterotic vector bundles on branched covers of twistor space. Roughly speaking, our proposal points to a generalization of the standard spectral cover construction which should hold even when the elliptic fibration of the heterotic model does not possess a holomorphic section. Another generalization concerns giving a heterotic dual description of T-branes for such flux vacua. Expanding on these details further would be most interesting. Finally, though we used the F-theory dual to motivate the existence of a heterotic hyperflux mechanism, it should be possible to realize examples of heterotic hyperflux even if there is no F-theory dual. Compared with standard Calabi-Yau compactification, the main ingredient we have identified is a position dependent dilaton profile to trap the 10D gauge fields on regions of the geometry, and the existence of vector bundles which are non-trivial on components of a gluing construction, but which are globally trivial. This points to a potentially vast generalization of heterotic model building. All of these problems deserve further attention and are necessary for a general study of heterotic/F-theory duality. We hope to continue to explore them in future research work.

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