ON THE STATE OF CONVERGENCE OF THE FLINT HILL SERIES

T. AGAMA

ABSTRACT. In this paper we study the convergence of the flint hill series of the form

$$\sum_{n=1}^{\infty} \frac{1}{(\sin^2 n)n^3}$$

via a certain method. The method works essentially by erecting certain pillars sufficiently close to the terms in the series and evaluating the series at those spots. This allows us to relate the convergence and the divergence of the series to other series that are somewhat tractable. In particular we show that the convergence of the flint hill series relies very heavily on the condition that for any small $\epsilon > 0$

$$\left| \sum_{i=0}^{\frac{n+1}{2}} \sum_{j=0}^{i} (-1)^{i-j} \binom{n}{2i+1} \binom{i}{j} \right|^{2s} \leq |(\sin^2 n)|n^{2s+2-\epsilon}$$

for some $s \in \mathbb{N}$.

1. Introduction

The flint hill series is the elementary series of the form

$$\sum_{n=1}^{\infty} \frac{1}{(\sin^2 n)n^3}.$$

The series seems to have gained much popularity in part due to the work of Pickover (See [1]), who studied and questioned its convergence. It is unknown whether the flint hill series converges or diverges and it is an open problem to determine the state of convergence of such series. As with most open problems finding a solution almost reduces to finding possible connections of the existing problem to other problems that seems quite amenable to current tools. For a fixed positive real number x, we call the smallest value of m such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^m}$$

the irrationality measure of x if the inequality holds only for a finite number of co-prime positive integers p, q. Alekseyev has shown that (See [2]) the convergence of the flint hill series is equivalent to asserting the irrationality measure of π as $\mu(\pi) \leq 2.5$. Nonetheless studying the convergence of the flint hill series via this approach can be quite hard, since establishing the irrationality measure $\mu(\pi) \leq 2.5$ is a terribly hard problem in its own right. Unfortunately there seems to be very

little progress along these lines and a resolution via this approach appears to be a seemingly far distant prospect.

In this paper we do not follow this orthodoxy; rather, we study the convergence of the flint hill series via a different approach. The method works basically by erecting pillars which are literally vertical lines in sufficiently small neighborhoods of the arguments of the terms in the series and leveraging a certain decomposition. This allows us to obtain equivalent forms of the flint hill series at the compromise of sufficiently higher powered polynomials and a certain local powered functions. By iterating the process at any given number of times we can then obtain a general equivalent form of the flint hill series. The convergence or the divergence of the flint hill series can then be ascertained if we can really say something substantial about its equivalent forms.

2. **Main result**

In this section we relate the convergence or the divergence of the flint hill series to other equivalent series via a certain method. This method pretty much would allow us to determine the state of convergence of the flint hill series provided we can say something concerning the convergence or the divergence of its equivalent forms. We launch the following basic and standard key inputs, which can be found in many elementary calculus textbooks. Nonetheless they feature prominently in our current studies.

Lemma 2.1. *The limit holds*

$$\lim_{n \longrightarrow a} \frac{\sin(n-a)}{n-a} = 1$$

equivalently

$$\lim_{m \longrightarrow 0} \frac{\sin m}{m} = 1.$$

Lemma 2.2. *The following identity holds*

$$\sin \delta = \sin(\frac{\delta}{n}) \sum_{i=0}^{\frac{n+1}{2}} \sum_{j=0}^{i} (-1)^{i-j} \binom{n}{2i+1} \binom{i}{j} \cos^{n-2(i-j)-1}(\frac{\delta}{n})$$

for any $\delta > 0$ and $n \in \mathbb{N}$ with $n > 1$.

Proof. This identity is easily obtained by writing

$$\sin \delta = \sin(\frac{\delta \cdot n}{n})$$

and applying the trigonometric identity

$$\sin(n\theta) = \sin \theta \sum_{i=0}^{\frac{n+1}{2}} \sum_{j=0}^{i} (-1)^{i-j} \binom{n}{2i+1} \binom{i}{j} \cos^{n-2(i-j)-1} \theta$$

which can be accessed on the Wikipedia page and due to Francois Viete.   □

Lemma 2.3. *The following asymptotic holds*

$$\sum_{n=1}^{k} \frac{1}{(\sin^2 n)n^3} \sim \sum_{n=1}^{k} \frac{(G(n))^2}{(\sin^2 n)n^5} \cdots \sim \sum_{n=1}^{k} \frac{(G(n))^{2s}}{(\sin^2 n)n^{2s+3}}$$

where

$$G(n) = \sum_{i=0}^{\frac{n+1}{2}} \sum_{j=0}^{i} (-1)^{i-j} \binom{n}{2i+1} \binom{i}{j}$$

for all $s \geq 1$ with $s \in \mathbb{N}$.

Proof. Appealing to lemma 2.1 and using the decomposition $\sin(n-a) = (\sin n)(\cos a - \blacksquare$
$\frac{(\sin a)\cos n}{\sin n})$, we can write the following asymptotic

$$\sum_{n=1}^{k} \frac{1}{(\sin^2 n)n^3} \sim \sum_{\substack{n=1 \\ a=n+\delta \\ \delta \longrightarrow 0^+}}^{k} \frac{(\cos a - \frac{(\sin a)\cos n}{\sin n})^2}{n^2(1-\frac{a}{n})^2 n^3}$$

$$= \sum_{\substack{n=1 \\ \delta \longrightarrow 0^+}}^{k} \frac{(\cos(n+\delta) - \frac{(\sin(n+\delta))\cos n}{\sin n})^2}{n^2(\frac{n+\delta}{n}-1)^2 n^3}$$

$$= \sum_{\substack{n=1 \\ \delta \longrightarrow 0^+}}^{k} \frac{(G(n))^2 \sin^2(\frac{\delta}{n}) \cos^{2n-4(i-j)-2}(\frac{\delta}{n})}{(\frac{\delta}{n})^2(\sin^2 n)n^5}$$

$$\sim \sum_{n=1}^{k} \frac{(G(n))^2}{(\sin^2 n)n^5}$$

by appealing to Lemma 2.1 and 2.2. where

$$G(n) = \sum_{i=0}^{\frac{n+1}{2}} \sum_{j=0}^{i} (-1)^{i-j} \binom{n}{2i+1} \binom{i}{j}.$$

By repeating the argument on $\sin n$ in the deduced finite sum we obtain

$$\sum_{n=1}^{k} \frac{(G(n))^2}{(\sin^2 n)n^5} \sim \sum_{n=1}^{k} \frac{(G(n))^4}{(\sin^2 n)n^7}$$

so that by iterating the argument in this manner we deduce the claimed chain of asymptotic. $\qquad\square$

Remark 2.4. Now we show that we can study the convergence of the flint hill series by examining its equivalent forms in the following result.

Theorem 2.5. *The flint hill series*

$$\sum_{n=1}^{\infty} \frac{1}{(\sin^2 n)n^3}$$

is convergent (resp. divergent) if and only if

$$\sum_{n=1}^{\infty} \frac{(G(n))^{2s}}{(\sin^2 n)n^{2s+3}}$$

is convergent (resp. divergent) for some $s \geq 1$ with $s \in \mathbb{N}$, where

$$G(n) = \sum_{i=0}^{\frac{n+1}{2}} \sum_{j=0}^{i} (-1)^{i-j} \binom{n}{2i+1} \binom{i}{j}.$$

Proof. Appealing to Lemma 2.3, we can write the asymptotic

$$\sum_{n=1}^{k} \frac{1}{(\sin^2 n)n^3} \sim \sum_{n=1}^{k} \frac{(G(n))^2}{(\sin^2 n)n^5} \cdots \sim \sum_{n=1}^{k} \frac{(G(n))^{2s}}{(\sin^2 n)n^{2s+3}}$$

for all $s \geq 1$ with $s \in \mathbb{N}$ where

$$G(n) = \sum_{i=0}^{\frac{n+1}{2}} \sum_{j=0}^{i} (-1)^{i-j} \binom{n}{2i+1} \binom{i}{j}$$

so that

$$\sum_{n=1}^{\infty} \frac{1}{(\sin^2 n)n^3}$$

is convergent (resp. divergent) if and only

$$\sum_{n=1}^{\infty} \frac{(G(n))^{2s}}{(\sin^2 n)n^{2s+3}}$$

is convergent (resp. divergent) for some $s \geq 1$ with $s \in \mathbb{N}$.   □

Remark 2.6. We now launch a criterion for the convergence of the flint hill series. The following could be considered as a test tool for deciding on the convergence or the divergence of the flint hill series, avoiding the studies of the irrationality measure of π which is generally a harder problem.

Theorem 2.7. *If for any small $\epsilon > 0$*

$$\left| \sum_{i=0}^{\frac{n+1}{2}} \sum_{j=0}^{i} (-1)^{i-j} \binom{n}{2i+1} \binom{i}{j} \right|^{2s} \leq |(\sin^2 n)|n^{2s+2-\epsilon}$$

for some $s \in \mathbb{N}$, then the flint hill series

$$\sum_{n=1}^{\infty} \frac{1}{(\sin^2 n)n^3}$$

converges.

Proof. For any small $\epsilon > 0$, we can write

$$\sum_{n=1}^{\infty} \frac{(G(n))^{2s}}{(\sin^2 n)n^{2s+3}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} < \infty$$

under the condition

$$\left| \sum_{i=0}^{\frac{n+1}{2}} \sum_{j=0}^{i} (-1)^{i-j} \binom{n}{2i+1} \binom{i}{j} \right|^{2s} \leq |(\sin^2 n)|n^{2s+2-\epsilon}.$$

Appealing to Theorem 2.5 then the flint hill series

$$\sum_{n=1}^{\infty} \frac{1}{(\sin^2 n)n^3} < \infty$$

and so it converges. □

References

1. Pickover, Clifford A *The mathematics of Oz: mental gymnastics from beyond the edge*, Cambridge University Press, 2002.
2. Alekseyev, Max A *On convergence of the Flint Hills series*, arXiv preprint arXiv:1104.5100, 2011.

DEPARTMENT OF MATHEMATICS, AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES, GHANA.
E-mail address: Theophilus@aims.edu.gh/emperordagama@yahoo.com