Discrete Noether's Theorem

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June 12, 2021

Abstract

Conservation law for discrete symmetry

I try to write a generalization of Noether's theorem to the discrete case.

If a Lagrangian is invariant for a discrete simmetry S then there is a conservation law for each point (accessible) in the space:

$$\begin{aligned}
&\mathcal{L}[R(q)] = \mathcal{L}(q) \\
&0 = \frac{d\mathcal{L}[S(q)]}{dt} - \frac{d\mathcal{L}(q)}{dt} = \frac{d\mathcal{L}(Q)}{dt} - \frac{d\mathcal{L}(q)}{dt} = \\
&= \frac{\partial \mathcal{L}(Q)}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial \mathcal{L}(Q)}{\partial \dot{Q}}\frac{\partial \dot{Q}}{\partial q} - \frac{\partial \mathcal{L}(q)}{\partial q} - \frac{\partial \mathcal{L}(q)}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q} = \\
&= \left[\frac{d}{dt}\frac{\partial \mathcal{L}(Q)}{\partial \dot{Q}}\right]\frac{\partial Q}{\partial q} + \frac{\partial \mathcal{L}(Q)}{\partial \dot{Q}}\frac{\partial \dot{Q}}{\partial q} - \left[\frac{d}{dt}\frac{\partial \mathcal{L}(q)}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}(q)}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q} = \\
&= \frac{d}{dt}\left[\frac{\partial \mathcal{L}(Q)}{\partial \dot{Q}}\frac{\partial Q}{\partial q} - \frac{\partial \mathcal{L}(q)}{\partial \dot{q}}\right]
\end{aligned}$$

then along the trajectory (where the Eulero-Lagrange equation $\frac{d}{dt}\frac{\partial \mathcal{L}(q)}{\partial \dot{q}} = \frac{\partial \mathcal{L}(q)}{\partial q}$ is true) there is a conservation law.

This conservation law is true for each point of the trajectory, and for each discrete symmetry $Q(t) = S[q(t)]$.

The symmetry can be a linear transformation $Q_n = \pm Aq \mp nB$ that are reflection, discrete spatial translation, discrete spatial rotation, etc.

I write some example of discrete Noether's theorem:

- **discrete translation** $Q_n = S^n(q) = q + n\Delta$: the derivative is $\partial_q Q = \partial_q S(q) = 1$, then the generalized momentum is constant $p(Q_n) = p(q + n\Delta)$ and $q \in [0, \Delta]$: the generalized momentum is an arbitrary function in an interval $[0, \Delta]$, and it is replicated indefinitely in $[n\Delta, (n+1)\Delta]$. If $\Delta \to 0$ then there is an invariant momentum in each point of the space, the usual Noether's theorem.

- **discrete rotation** $\theta_n = S^n(\theta) = \theta + n2\pi/M$: the derivative is $\partial_\theta S(\theta) = 1$, the angular momentum is an arbitrary function in an interval $[0, 2\pi/M]$ and $p(\Theta_n) = p(\theta + n2\pi/M)$

- **reflection** $Q_n = (-1)^n q$: the derivative is $\partial_q S(q) = -1$, then the generalized momentun reverses with every reflection, but this happen only if the reflection point is on the trajectory (to satisfy the Eulero-Lagrange equation). It seem that there is an invariant in quantum mechanics (for example parity), then there is an invariant in classical mechanics. For example the classical Hamiltonian $E = \frac{m}{2}\left[\dot{r}^2 + r^2\dot{\theta})^2 + r^2 \sin^2\theta\dot{\phi}^2\right] + V(r)$ where $\mathcal{L}(r) = \mathcal{L}(-r)$ has the invariant $m\dot{r}(\vec{r}) = \pm m\dot{r}(-\vec{r})$, that is the invariant obtained from the infinitesimal transformation $R = r - \epsilon r$ transformed in the discrete with $\epsilon = 2$. The discrete Noether's theorem give the correct sign.