Why Did Einstein Avoid The Definition of Covariant Tensor?

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ABSTRACT

The aim of this article is to draw the attention of readers in basic tensor algebra and its applications to a strange method used by Einstein to introduce one of the basic identities of tensor algebra in one of his most important written work. A general probable reason is given.

In his book *The Foundation of General Theory of Relativity* written in 1915, before introducing the field equations which summarizes the theory of general relativity, Einstein went over differential geometric background related to tensor analysis, tensor calculus and Riemann Geometry as expected, but what didn't go as expected is the way he followed to introduce the mathematical identity that the product of multiplication of a covariant and a contra-variant tensors of the same order and the same indices is invariant.

For example two vectors; covariant and contra-variant (tensors of rank 1) A^{v}, B_{v} :

$$A^{v} B_{v} = \text{invariant}$$

The sensible and reasonable sequence to introduce this mathematical fact is to:

1) **Define the Covariant Tensor by the nature of its components and the law of their transformation**
2) **Define the Contra-variant Tensor by the nature of its components and the law of their transformation**
3) **Use mathematical reasoning to prove that these two definitions lead to this identity**

However, this was not the way followed by Einstein, and he instead:

1) **Defined the components and transformation of Contra-variant Tensor**
2) **Used the Identity as a starting point and defined Covariant Tensor as the components with transformation laws that make them satisfy the identity**
3) **Try to derive the formula of transformation law of the Covariant Tensor from this definition**

This strange method looks like introducing and proving the content of Pythagorean Theorem to your students like this:

Let us first define what we call right angled triangle (the reason for this name will be introduced latter) as the triangle in which the square of one side is equal to the sum of squares of the other two sides. I will also prove that the triangle which satisfies these requirements is right angled!

Here is the method used by Einstein:

Contravariant Four-vector. The line-element is defined by the four components dx_ν whose transformation law is expressed by the equation

$$dx'_\sigma = \sum_\nu \frac{\partial x'_\sigma}{\partial x_\nu} dx_\nu \tag{5}$$

The dx'_σ are expressed as linear and homogeneous function of dx_ν; we can look upon the differentials of the co-ordinates dx_ν as the components of a tensor, which we designate specially as a contravariant Four-vector. Everything which is defined by Four quantities A^ν, with reference to a co-ordinate system, and transforms according to the same law,

$$A'^\sigma = \sum_\nu \frac{\partial x'_\sigma}{\partial x_\nu} A^\nu \tag{5a}$$

we may call a contravariant Four-vector. From (5a), it follows at once that the sums $(A^\sigma \pm B^\sigma)$ are also components of a four-vector, when A^σ and B^σ are so; corresponding relations hold also for all systems afterwards introduced as "tensors" (Rule of addition and subtraction of Tensors).

Covariant Four-vector. We call four quantities A_ν as the components of a covariant four-vector, when for any choice of the contravariant four-vector B^ν

$$\sum_\nu A_\nu B^\nu = \textbf{invariant} \tag{6}$$

From this definition follows the law of transformation of the covariant four-vectors.

If we substitute in the right band side of the equation

$$\sum_\sigma A'_\sigma B'^\sigma = \sum_\nu A_\nu B^\nu$$

the expressions
$$\sum_\sigma \frac{\partial x_\nu}{\partial x'_\sigma} B'^\sigma$$

for B^ν following from the inversion of the equation (5a) we get

$$\sum_\sigma B'^\sigma \sum_\nu \frac{\partial x_\nu}{\partial x'_\sigma} A_\nu = \sum_\sigma B'^\sigma A'_\sigma$$

As in the above equation B'^σ are independent of one another and perfectly arbitrary,

it follows that the transformation law is: —

$$A'_\sigma = \sum \frac{\partial x_\nu}{\partial x'_\sigma} A_\nu \tag{7}$$

Now, the question which imposes itself before anyone who place confidence in Einstein's intelligence is: Why did he take such a roundabout route to introduce this identity? It seems that he recognized something wrong in the steps of proving the identity in the ordinary way in the textbooks of tensor algebra in front of him and then tried to skip the problem rather than giving it much of his time and effort to solve as he used to do with such occasional problems which he thinks that their solution will not affect his route to the theory which he want to create or the form and validity of a theory that he created.