Three Identities, Number Pi, Appell Function

Edgar Valdebenito

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abstract

We give three identities that involving number Pi and Appell function:

$$F_1\left(n+\frac{3}{2}, -n-\frac{1}{2}, 2, n+\frac{5}{2}; \frac{x}{2}, x\right)$$

keywords: Number Pi , Appell function , Identities

I. Introduction

Recall that:

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} \tag{1}$$

$$F_1(a, b, c, d; x, y) = \sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(c)_n}{(d)_{m+n}\, m!\, n!}x^m y^n \quad , \quad \max(|x|, |y|) < 1 \tag{2}$$

where $F_1(a, b, c, d; x, y)$ is the Appell hypergeometric function.

In this note we give three formulas that involving the number Pi and the Appell function.

II. Three identities

Entry 1. For

$$\alpha = \frac{\pi}{6} \quad , \quad \beta = \frac{\pi}{3} \quad , \quad x = \sqrt{3} - 1 \tag{3}$$

$$u = 2\sqrt{\frac{3\sqrt{3}-4}{11}} \quad , \quad v = \frac{2}{11}\sqrt{\frac{164+97\sqrt{3}}{11}} \tag{4}$$

$$p = 1-\sqrt{1-\frac{3\sqrt{3}-4}{11}} \quad , \quad q = 1-\sqrt{1-\frac{3\left(164+97\sqrt{3}\right)}{1331}} \tag{5}$$

we have

$$\frac{(1+2x)\,\alpha}{\sqrt{1-u^2\sin^2\alpha}}-\frac{\beta}{\sqrt{1-v^2\sin^2\beta}}=\frac{(1+2x)\,2p\sqrt{2p}}{u}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\left(p/\left(2u^2\right)\right)^n}{(2n+1)(2n+3)}F_1\left(n+\frac{3}{2},-n-\frac{1}{2},2,n+\frac{5}{2};\frac{p}{2},p\right)-\frac{2q\sqrt{2q}}{v}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\left(q/\left(2v^2\right)\right)^n}{(2n+1)(2n+3)}F_1\left(n+\frac{3}{2},-n-\frac{1}{2},2,n+\frac{5}{2};\frac{q}{2},q\right)\tag{6}$$

Entry 2. For

$$\alpha=\frac{\pi}{4}\;,\quad\beta=\frac{\pi}{3}\;,\quad x=\left(\sqrt{2}-1\right)\left(\sqrt{3}-1\right)\tag{7}$$

$$u=\left(\left(\sqrt{2}-1\right)\left(\sqrt{3}-1\right)\right)^{3/2}\sqrt{\frac{2+\left(\sqrt{2}-1\right)\left(\sqrt{3}-1\right)}{1+2\left(\sqrt{2}-1\right)\left(\sqrt{3}-1\right)}}\tag{8}$$

$$v=\sqrt{\left(\sqrt{2}-1\right)\left(\sqrt{3}-1\right)}\left(\frac{2+\left(\sqrt{2}-1\right)\left(\sqrt{3}-1\right)}{1+2\left(\sqrt{2}-1\right)\left(\sqrt{3}-1\right)}\right)^{3/2}\tag{9}$$

$$p=1-\sqrt{1-\frac{\left(\sqrt{2}-1\right)^3\left(\sqrt{3}-1\right)^3\left(2+\left(\sqrt{2}-1\right)\left(\sqrt{3}-1\right)\right)}{2\left(1+2\left(\sqrt{2}-1\right)\left(\sqrt{3}-1\right)\right)}}\tag{10}$$

$$q=1-\sqrt{1-\frac{3\left(\sqrt{2}-1\right)\left(\sqrt{3}-1\right)\left(2+\left(\sqrt{2}-1\right)\left(\sqrt{3}-1\right)\right)^3}{4\left(1+2\left(\sqrt{2}-1\right)\left(\sqrt{3}-1\right)\right)^3}}\tag{11}$$

equation (6) holds.

Entry 3. For

$$\alpha=\frac{\pi}{6}\;,\quad\beta=\frac{\pi}{4}\;,\quad x=\frac{4\left(1+\sqrt{2}\right)}{1+\sqrt{2}+\sqrt{3}}-2\tag{12}$$

$$u=4\left(\frac{1+\sqrt{2}-\sqrt{3}}{1+\sqrt{2}+\sqrt{3}}\right)^{3/2}\sqrt{\frac{2\left(1+\sqrt{2}\right)}{5+5\sqrt{2}-3\sqrt{3}}}\tag{13}$$

$$v=8\sqrt{\frac{2\left(1+\sqrt{2}-\sqrt{3}\right)}{1+\sqrt{2}+\sqrt{3}}}\left(\frac{1+\sqrt{2}}{5+5\sqrt{2}-3\sqrt{3}}\right)^{3/2}\tag{14}$$

$$p=1-\sqrt{1-\frac{8\left(1+\sqrt{2}\right)\left(1+\sqrt{2}-\sqrt{3}\right)^3}{\left(5+5\sqrt{2}-3\sqrt{3}\right)\left(1+\sqrt{2}+\sqrt{3}\right)^3}}\tag{15}$$

$$q=1-\sqrt{1-\frac{64\left(1+\sqrt{2}\right)^3\left(1+\sqrt{2}-\sqrt{3}\right)}{\left(5+5\sqrt{2}-3\sqrt{3}\right)^3\left(1+\sqrt{2}+\sqrt{3}\right)}}\tag{16}$$

equation (6) holds.

Entry 4. entry 1 is equivalent to

$$-\frac{11\pi}{468}\sqrt{750-378\sqrt{3}} = \sqrt{50+208\sqrt{3}-6\sqrt{3693+567\sqrt{3}}}$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{(2n+1)(2n+3)}\left(\frac{4+3\sqrt{3}-\sqrt{39+21\sqrt{3}}}{8}\right)^{n}F_1\left(n+\frac{3}{2},-n-\frac{1}{2},2,n+\frac{5}{2};\frac{p}{2},p\right)-$$

$$\frac{1}{11}\sqrt{\frac{1+\sqrt{3}}{22}}\left(77-33\sqrt{3}-\sqrt{9130-5214\sqrt{3}}\right)^{3/2} \tag{17}$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{(2n+1)(2n+3)}\left(\frac{-164+97\sqrt{3}-\sqrt{55615-32107\sqrt{3}}}{8}\right)^{n}F_1\left(n+\frac{3}{2},-n-\frac{1}{2},2,n+\frac{5}{2};\frac{q}{2},q\right)$$

where

$$p = 1-\sqrt{\frac{3\left(5-\sqrt{3}\right)}{11}}\quad,\quad q = 1-\frac{1}{11}\sqrt{\frac{839-291\sqrt{3}}{11}} \tag{18}$$

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