formulas related with the integral : $\int_0^\infty \frac{\cosh x}{\cosh 5x}\, dx$

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abstract

In this note we give some formulas related with the integral:

$$\int_0^\infty \frac{\cosh x}{\cosh 5x}\, dx = \frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}$$

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1. Introduction

Recall that (Gradshteyn and Ryzhik , 7 th. ed. , 2007 , pag. 371 , 3.511.4) :

$$\int_0^\infty \frac{\cosh x}{\cosh 5x}\, dx = \frac{\pi}{10}\sec\left(\frac{\pi}{10}\right) = \frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}} \tag{1}$$

where

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} \tag{2}$$

In this note we give some formulas related with (1) .

2. Formulas

Entry 1.

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}} = \int_0^1 \cosh^{-1}\left(\sqrt{\frac{\sqrt{5x^2+4x}+5x}{8x}}\right) dx \tag{3}$$

Entry 2.

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}} = \int_0^1 \ln\left(\sqrt{\frac{\sqrt{5x^2+4x}+5x}{8x}} + \sqrt{\frac{\sqrt{5x^2+4x}-3x}{8x}}\right) dx \tag{4}$$

Entry 3.

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}=\frac{1}{2\sqrt{5}}\coth^{-1}\left(\frac{3}{\sqrt{5}}\right)-\frac{\ln 2}{5}+\int_0^1 \ln\left(1+\sqrt{\frac{\sqrt{5x^2+4x}-3x}{\sqrt{5x^2+4x}+5x}}\right)dx \tag{5}$$

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}=-\frac{3}{2\sqrt{5}}\coth^{-1}\left(\frac{3}{\sqrt{5}}\right)-\ln 2+\int_0^1 \ln\left(1+\sqrt{\frac{\sqrt{5x^2+4x}+5x}{\sqrt{5x^2+4x}-3x}}\right)dx \tag{6}$$

Entry 4.

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}=\int_0^\infty \frac{e^{-4x}}{1-e^{-2x}+e^{-4x}-e^{-6x}+e^{-8x}}\,dx \tag{7}$$

Entry 5.

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}=\int_0^1 \frac{1-x^2}{1+10x^2+5x^4}\,dx \tag{8}$$

Entry 6.

$$\frac{\pi}{5}\sqrt{\frac{1}{5+\sqrt{5}}}=\int_0^1 \sqrt{\frac{1-x}{1+10x+\sqrt{1+40x+80x^2}}}\,dx \tag{9}$$

Entry 7.

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}=\int_0^{1/2}\frac{\cosh x}{\cosh 5x}\,dx+\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{e^{-(5n+2)}}{5n+2}+\frac{e^{-(5n+3)}}{5n+3}\right) \tag{10}$$

Entry 8. for $a>\ln\left(\frac{1}{4}\left(\sqrt{2\left(3+\sqrt{13}\right)}+\sqrt{2\left(11+\sqrt{13}\right)}\right)\right)$ we have

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}=\int_0^a\frac{\cosh x}{\cosh 5x}\,dx+\frac{\tanh a}{32}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+2}\left(\frac{3}{4}\right)^n\left(\frac{1}{\sinh a}\right)^{2n+4}F\left(\frac{1}{2},\,1,\,n+3,\,\frac{1}{\cosh^2 a}\right)\sum_{k=0}^{[n/2]}(-1)^k 3^{-2k}\binom{n-k}{k} \tag{11}$$

Entry 9. for $a>0$ we have

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}=\int_0^a\frac{\cosh x}{\cosh 5x}\,dx+\frac{1}{32}\sum_{n=0}^{\infty}\frac{1}{n+2}\left(\frac{5}{4}\right)^n\left(\frac{1}{\cosh a}\right)^{2n+4}F\left(\frac{1}{2},\,n+2,\,n+3,\,\frac{1}{\cosh^2 a}\right)\sum_{k=0}^{[n/2]}(-1)^k 5^{-k}\binom{n-k}{k} \tag{12}$$

Entry 10.

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}=\sum_{n=0}^{\infty}2^{-n-2}\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{k}\binom{k}{m}\frac{(-1)^m}{n+2k+m+2}F\left(n+1,\,1,\,\frac{n+2k+m+4}{2},\,\frac{1}{2}\right) \tag{13}$$

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}=\sum_{n=0}^{\infty}2^{-n-2}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\sum_{m=0}^{k}\binom{k}{m}\frac{(-1)^m}{n+k+m+2}F\left(n+1,\,1,\,\frac{n+k+m+6}{4},\,\frac{1}{2}\right) \tag{14}$$

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k 2^{-k-2}}{n+3k+2}\binom{n}{k}F\left(k+1,\,1,\,\frac{n+3k+4}{2},\,\frac{1}{2}\right) \tag{15}$$

Entry 11.

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}\binom{n}{k}\binom{4n-2k+2}{n+1}^{-1} \tag{16}$$

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}} = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\binom{2n+2k+2}{n}^{-1}\frac{1}{n+2k+2} \tag{17}$$

Entry 12.

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}} = \sum_{n=0}^{\infty}3^{-n-1}\sum_{k=0}^{n}(-2)^k\binom{n}{k}\left(\frac{1}{5k+2}+\frac{1}{5k+3}\right) \tag{18}$$

$$\pi\sqrt{\frac{2}{5+\sqrt{5}}} = \sum_{n=0}^{\infty}\frac{5^{-2n}}{2n+1}\sum_{k=0}^{n}\frac{(-1)^k 5^{2k}E_k}{(2k)!\,(2n-2k)!} + \sum_{n=0}^{\infty}(-1)^n\left(\frac{e^{-\left(2n+\frac{4}{5}\right)}}{2n+\frac{4}{5}}+\frac{e^{-\left(2n+\frac{6}{5}\right)}}{2n+\frac{6}{5}}\right) \tag{19}$$

where E_k are the Euler numbers

$$\{E_k : k=0,\ 1,\ 2,\ ..\} = \{1,\ 1,\ 5,\ 61,\ 1385,\ ...\} \tag{20}$$

Entry 13. for $a > 0$ we have

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{e^{-(10n+4)a}}{10n+4}+\frac{e^{-(10n+6)a}}{10n+6}\right) + \sum_{n=0}^{\infty}\left(\frac{1+e^{-10a}}{3+e^{-10a}}\right)^{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2}{1+e^{-10a}}\right)^{k+1}\left(\frac{1-e^{-(10k+4)a}}{10k+4}+\frac{1-e^{-(10k+6)a}}{10k+6}\right) \tag{21}$$

Entry 14.

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{10n+4}+\frac{1}{10n+6}\right) \tag{22}$$

Entry 15.

$$4\pi\sqrt{\frac{2}{5+\sqrt{5}}} = \int_0^1\left(\sqrt{\frac{10-5x+2\sqrt{5(5-6x+x^2)}}{x}} - \sqrt{\frac{10-5x-2\sqrt{5(5-6x+x^2)}}{x}}\right)dx \tag{23}$$

Entry 16.

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}} = \frac{1}{11}\sum_{n=0}^{\infty}(-1)^n\left(\frac{5}{11}\right)^n\left(\frac{1}{4n+1}F\left(n+1,\ 1,\ 2n+\frac{3}{2},\ \frac{10}{11}\right) - \frac{1}{4n+3}F\left(n+1,\ 1,\ 2n+\frac{5}{2},\ \frac{10}{11}\right)\right) \tag{24}$$

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}} = \frac{1}{36}\sum_{n=0}^{\infty}\left(\frac{5}{9}\right)^n\left(\frac{1}{4n+1}F\left(2n+2,\ 1,\ 2n+\frac{3}{2},\ \frac{5}{6}\right) - \frac{1}{4n+3}F\left(2n+2,\ 1,\ 2n+\frac{5}{2},\ \frac{5}{6}\right)\right) \tag{25}$$

Entry 17.

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}} = \frac{1}{2}\int_0^1\left(\left(\frac{1+\sqrt{1-x^2}}{x}\right)^{1/5} - \left(\frac{x}{1+\sqrt{1-x^2}}\right)^{1/5}\right)dx \tag{26}$$

Entry 18. for $0 < a < 1$ we have

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{a^{10n+4}}{10n+4}+\frac{a^{10n+6}}{10n+6}\right)+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{n}\sum_{r=0}^{2n+4k+3}\binom{n}{k}\binom{n}{m}\binom{2n+4k+3}{r}\frac{(-1)^{m+r}\,2^{n-m}\,(1-a)^{n+m+r+1}}{n+m+r+1} \tag{27}$$

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{a^{10n+4}}{10n+4}+\frac{a^{10n+6}}{10n+6}\right)+\sum_{n=0}^{\infty}\frac{(1-a)^{n+1}}{n+1}\sum_{k=0}^{n}\binom{n}{k}2^n F_1\left(n+1,\,-2n-4k-3,\,-n,\,n+2,\,1-a,\,\frac{1-a}{2}\right) \tag{28}$$

Remark. F_1 is the Appell hypergeometric function.

Entry 19. for $a \geq 0$ we have

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}} = \int_0^a \frac{\cosh x}{\cosh 5x}\,dx + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\sinh a + (10n+5)\cosh a}{(5n+2)(5n+3)}\right)e^{-(10n+5)a} \tag{29}$$

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}} = \frac{5}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(5n+2)(5n+3)} \tag{30}$$

Entry 20.

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}} = \frac{1}{8}F\left(1,\,1,\,\frac{7}{5},\,\frac{1}{2}\right)+\frac{1}{12}F\left(1,\,1,\,\frac{8}{5},\,\frac{1}{2}\right) \tag{31}$$

Entry 21.

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}} = \int_0^{\infty}\frac{\sinh^3 x}{16+12\sinh^2 x+\sinh^4 x}\,dx \tag{32}$$

Entry 22.

$$\frac{\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}} = \frac{1}{8}-\int_0^{1/16}\sqrt{\sqrt{20+\frac{1}{x}}-5}\,dx+\int_{1/16}^{1}\sqrt{\sqrt{\frac{4}{5}+\frac{1}{5x}}-1}\,dx \tag{33}$$

Remark. In this note $F(a,\,b,\,c,\,x)$ is the Gauss hypergeometric function.

References

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