Deriving the Pythagorean Theorem Using Infinitesimal Area

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Abstract

The Pythagorean Theorem is derived by performing an infinitesimal rotation of a right triangle and using the equation for arc length and the equation for the area of a triangle.

A derivation of the Pythagorean Theorem is developed that uses infinitesimal area and simple geometric concepts. An arbitrary right triangle is rotated by an infinitesimal angle, Θ, about the vertex connecting sides A and C, as illustrated in Fig. 1. Side A sweeps out an infinitesimal triangular region that has height A and base equal to A times Θ, as illustrated in Fig. 2. The size of Θ is large in the figures for better visualization.

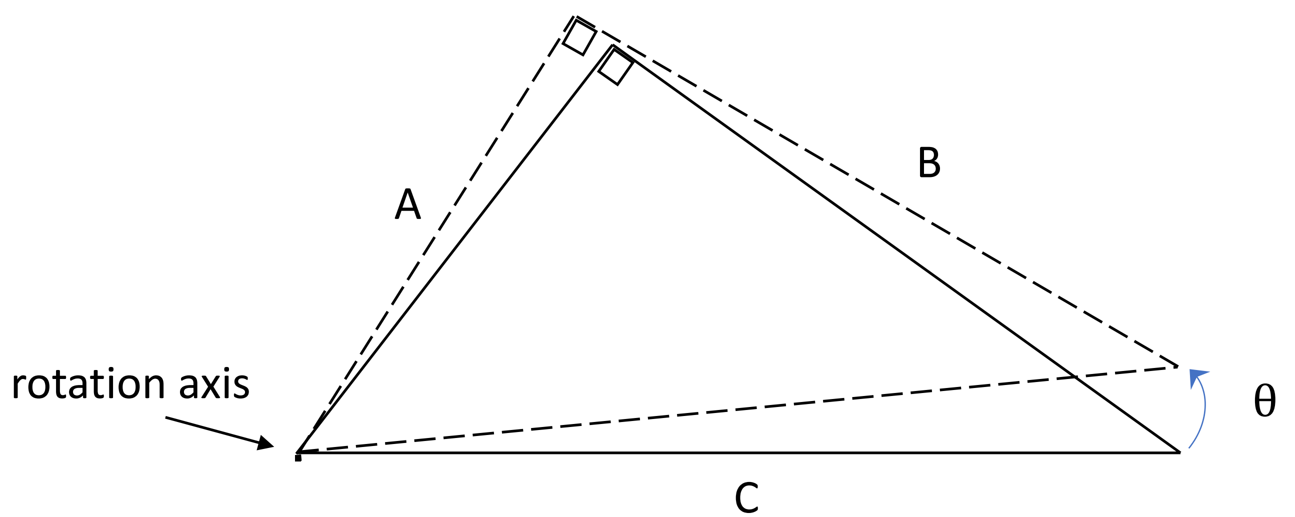

Fig. 1. A right triangle is shown before and after it is rotated by an infinitesimal angle.

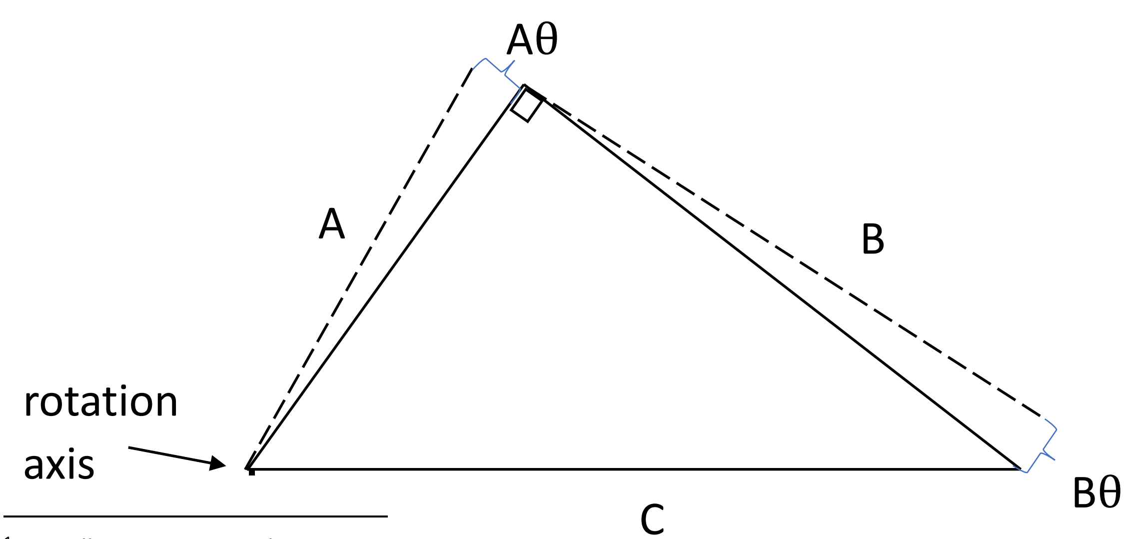

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Fig. 2. The base and heights of the infinitesimal triangles with sides A and B are shown.

The area, σ_A, of the triangular region swept out by side A is given by the base, AΘ, times the height, A, divided by 2, as shown in eq. (1).

$$\sigma_A = \frac{A^2\theta}{2} \quad (1)$$

In a similar fashion, the area swept out by side C in Fig. 3 is shown in eq. (2).

$$\sigma_C = \frac{C^2\theta}{2} \quad (2)$$

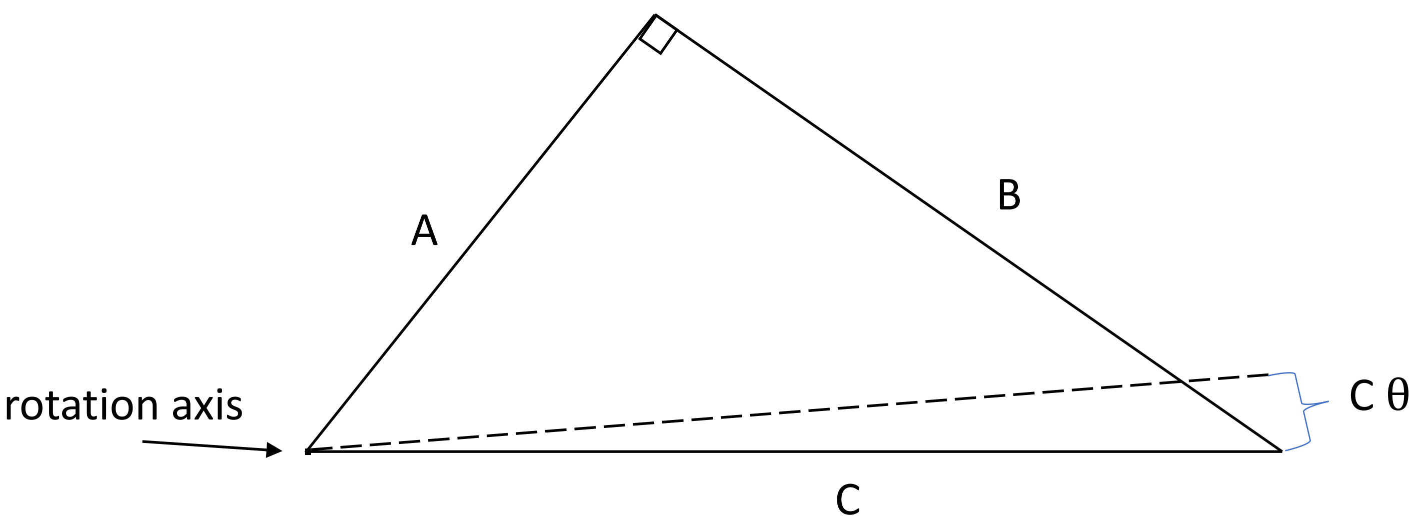

Fig. 3. The base and height of the infinitesimal triangle with side C are shown.

Note that both sides A and C rotate about the same axis. Since side B is connected rigidly to side A, as shown in Fig. 2, it also rotates by an amount Θ and sweeps out area given in eq. (3).

$$\sigma_B = \frac{B^2\theta}{2} \quad (3)$$

Side B is also displaced by an amount A times Θ in a direction parallel to side B, as shown in Fig. 1. The 90 degree angle ensures that the displacement is parallel to B. This parallel displacement does not contribute to the area swept out by side B, because only motion perpendicular to side B can sweep out area.

Since the areas σ_A and σ_B add to the area of the initial unrotated triangle and σ_C subtracts from the area of the initial unrotated triangle, the areas must be equal, as shown in eq. (4). This is because the area of the rotated triangle is equal to the area of the initial unrotated triangle.

$$\sigma_A + \sigma_B = \sigma_C = \frac{A^2\theta}{2} + \frac{B^2\theta}{2} = \frac{C^2\theta}{2} \tag{4}$$

When eq. (4) is divided by Θ/2, we obtain the Pythagorean Theorem shown in eq. (5).

$$A^2 + B^2 = C^2 \tag{5}$$