Seeking String Theory in Free Algebras

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Abstract. After decades of study, string theory still lacks a formulation which is non-perturbative, background independent and duality invariant. The free Lie algebra can be resolved into cyclic structures which can be mapped onto a closed string-like state in a given background. From this observation the idea that string theories may be reformulated in background free algebraic terms is explored.

Introduction

String theory is the culmination of unification progression in foundational physics, bringing together general, relativity, quantum theory and gauge theories into a single consistent framework. Further progress has now stalled on three fronts. On the Experimental side the problem is that direct observation of stringy effects almost certainly requires probing the Plank scale. On the theory side the fundamental issue is that string theory is only known via its perturbative formulation and some duality principles. For example there is no theory to attack the question of what happens in string theory above its Hagedorn temperature, or to fully understand the quantum mechanics of black holes. In the middle ground, phenomenologists looking for the low-energy effective theory from strings are also stuck because string theory has a vast landscape of vacuum solutions, with insufficient information from theory or experiment to determine the correct one for our universe.

Contrary to the opinion of some, these problems do not mean that string theory has failed and should be abandoned. Nothing can be ruled out with purely epistemological arguments that assume physics has to be tractable to human technology and understanding. Some commentators wrongly claim that string theory's popularity among theorists is merely due to its mathematical beauty. This is not the case. The correct description of our world most likely has a consistent perturbation theory around a fixed space-time background manifold describing gravitons and other particles. We know enough about the mathematics of renormalisation and other consistency requirements in this regime to narrow down the possible range of theories, within reasonable assumptions, leaving string theories as the only viable known options [Arkani-Hamed]. It remains an outside possibility that some alternative non-perturbative theory exists, but the fact that anything remains on the perturbative side given the strong combination of constraints is a strong indication that theorists are on the right track. Interest in string theory is thus driven by mathematical consistency requirements and the need to fit within the combined parameters of observation (i.e quantum mechanics and general relativity.)

If no new empirical data is found, then the best hope of unlocking further progress in fundamental physics is to find the mathematical key that provides a full formulation of string theory from the theory side. With this in hand it may be possible to explore the phenomenological landscape or discover non-perturbative string processes that lead to new understanding amenable to observation. Work in this direction has largely focussed on M-theory [Schreiber, Duff], a hypothetical model of membranes in 11-dimentional super-space from which all lower dimensional string theories may be derived. The best known non-perturbative formulation is a matrix theory [Susskind], but even this is incomplete and too constrained to allow the full range of compactification to be explored. Matrix theories also exist directly for string theories [Motl]

Whatever string theory is, it is likely to be something of importance to pure mathematics as well as fundamental physics. Even in its current incomplete form it has helped solve mathematical problems with applications to mirror-manifold symmetry [MS], monstrous moonshine [Borcherds] and geometric Langlands [Witten]. There is a sense that if string theory had not been found by physicists it would eventually be necessary for mathematicians to discover it in some other form. It is that pure mathematical form that is now needed to advance the physics.

Algebraic approach

String theory includes dualities between sectors in different spacetimes and even dynamic topology change. This suggests that a pregeometric approach is required for a fundamental formulation. The partial success of Matrix theory indicates that an algebraic synthesis might be possible. We therefore turn away from geometry to algebra.

How fundamental are the strings in string theory? Are they just an emergent feature of the perturbation theory, or should they be present in the underlying non-perturbative formulation? Loop Quantum Gravity is the main approach to background independent quantum gravity but the relationship between its developers and string theorists can best be described as tribal, so that LQG and ST are commonly seen as competing alternatives. A more open view is that there are more similarities than differences between the two theories [baez]. If this is correct then we can expect loops and worldsheets to appear in some form in non-perturbative string theory.

It is a common view of string theorists that symmetry in string theory is emergent rather than fundamental. This is because different gauge symmetries appear in different string theories which are linked through dualities. Also, the diffeomorphism groups on topologically different manifolds are distinct. Nevertheless, it is possible that these symmetries are residuals of a larger fundamental symmetry that is partially hidden in each sector. The holographic principle suggests that bulk degrees of freedom are redundant. This would be the case if string theory was formulated in terms of a complete symmetry, meaning that there is one continuous degree of symmetry for each bulk field variable.

In the algebraic approach followed here, the strings come from cyclic structures in free Lie algebras. Symmetry is expected to play an important part, ultimately in the form of supersymmetries, quantum groups and also higher dimensional n-groups.

Free algebras

For present purposes, an algebra is a vector space over the complex numbers $\mathbb{C}$ with a bilinear product operation that is associative and unital, but not necessarily commutative.

The free algebra F_n over n independent elements e_i is the algebra generated by linear combinations of free products of those elements. A general element $\hat{a} \in F_n$ can be written

$$\hat{a} = a + \sum_i a^i e_i + \sum_{ij} a^{ij} e_i e_j + \sum_{ijk} a^{ijk} e_i e_j e_k + \cdots$$

Only a finite number of the superscripted complex valued components $a^{i\dots k}$ will be non-zero because only finite applications of sums and products are used. F_n can thus be regarded as an algebra of polynomials over n non-commuting variables.

If the basis $\{e_i\}$ of the vector space V is regarded as an alphabet of letters then the basis of F_n is the set of words of any length (including zero) using this alphabet. Since the number of words is unlimited, the algebra is infinite dimensional as a vector space. These words form a monoid (semigroup with unit) under the word concatenation operation, corresponding to multiplication in the algebra. The free algebra is graded over non-negative integers given by the length of these words.

Free algebras are closely related to tensor algebras $T(V)$ of a vector space V over $\mathbb{C}$ which can be written using vector space direct sums and tensor products as

$$T(V) = \mathbb{C} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$$

When V is the n dimensional vector space spanned by the set of basis elements $\{e_i\}$, a general element of $T(V)$ has the same structure as an element of F_n but without the finite restriction on the number of non-zero components. It also extends the concept of free algebras by including infinite dimensional vector spaces V, such as the space of continuous functions on a manifold.

Dual algebras

A vector space V over $\mathbb{C}$ has a dual space V^* defined as the space of linear functions from V to $\mathbb{C}$.

An algebra A is a vector space, so it has dual A^* which is a vector space, but it is a coalgebra rather than an algebra. The product on A is a bilinear function $\nabla_A: A \otimes A \longrightarrow A, \nabla_A(u \otimes v) = uv$, but instead of a product, A^* has a coproduct $\Delta_{A^*}: A^* \longrightarrow A^* \otimes A^*$. If $v \in A^*$and $u, w \in A$, then Δ_{A^*} is defined by $\Delta_{A^*}(v)(u \otimes w) = v(uw)$

Likwise, a coalgebra A with a coproduct $\Delta_A: A \longrightarrow A \otimes A$ has a dual A^* with a product $\nabla_{A^*}: A^* \otimes A^* \longrightarrow A^*$ given for $u, w \in A^*$ and $v \in A$ by $\nabla_{A^*}(u \otimes w)(v) = (u \otimes w)\big(\Delta_A(v)\big)$

A structure A with both an associative product ∇_A and coproduct Δ_A (and unit and counit) is both an algebra and a coalgebra. If the coproduct is an algebra homomorphism from $A \otimes A$ to A then it is called a bialgebra. The dual of a bialgebra is another bialgebra switching the roles of product and coproduct. Bialgebras can be extended to Hopf algebras by adding an antipode operator. For simplicity we ignore the Hopf algebra structure here.

Bialgebra homomorphism

A homomorphism between bialgebras A and B is a mapping $\theta: A \to B$ that commutes with its operations, i.e. θ is linear and $\theta\big(\nabla_A(u \otimes v)\big) = \nabla_B\big(\theta(u)\otimes\theta(v)\big)$, $(\theta \otimes \theta)\big(\Delta_A(u)\big)$=$\Delta_B(\theta(u))$ etc.

Suppose in particular that there is a homomorphism $\theta: A \to C^*$ between an algebra A and the dual of another algebra C. Then θ is a linear mapping from A to a linear mappings on C to $\mathbb{C}$. This is equivalent to a linear mapping (using the same symbol) $\theta(A \otimes C) \to \mathbb{C}$, with the homomorphism conditions becoming

$$\theta(\nabla_A(a \otimes b) \otimes c) = (\theta \otimes \theta)\big(a \otimes b \otimes \Delta_B(c)\big)$$

$$\theta\big(a \otimes \nabla_B(c \otimes d)\big) = (\theta \otimes \theta)(\Delta_A(a) \otimes c \otimes d)$$

The symmetry between these equations tells us that a homomorphism from A to the dual of C is equivalent to a homomorphism from C to the dual of A

Shuffle Product

The free algebra, or tensor algebra $T(V)$ can be enhanced to form a bialgebra by adding the commutative shuffle product to its dual $T^*(V)$. The shuffle product denoted by the symbol ⧢ combines basis elements of the dual algebra in a way that befits the name e.g.

$$\nabla\big(e^i e^j \otimes e^k\big) = e^i e^j ⧢ e^k = e^i e^j e^k + e^i e^k e^j + e^k e^i e^j$$

In general the shuffle product of two words is the sum over all words combining the letters of the two words, in permutations that preserve the ordering of the letters in the original words. It name is due to the similarity with a rifle shuffle in a card game.

The coproduct for $T^*(V)$ is then the dual of the tensor product, sometimes called the deconcatenation operator because it is the sum over all ways of splitting a word into two parts that concatenate to give the original word, e.g.

$$\Delta\big(e^i e^j e^k\big) = 1 \otimes e^i e^j e^k + e^i \otimes e^j e^k + e^i e^j \otimes e^k + e^i e^j e^k \otimes 1$$

String state algebra

An open string in a vector space V is a parameterised piece-wise continuous curve function $c(t), 0 < t \le T$ into V. The limit T is a non-negative real number that can be different for each curve and is considered to be a feature of the curve. Let $c(V)$ be the set of all such curves.

$c(V)$ is a semi-group under concatenation of curves. i.e.

$$c_1(t) \cdot c_2(t) = \begin{cases} c_1(t), 0 < t \le T_1 \\ c_2(t - T_1), T_1 < t \le T_1 + T_2 \end{cases}$$

The empty curve with $T = 0$ is an identity element.

A semi-group algebra $\mathcal{C}(V)$ spans $c(V)$ over $\mathbb{C}$ extending this product by linearity to form the algebra product. A co-product that extends this to a bialgebra is given by $\Delta(c) = c \otimes c$ for curves in $c(V)$, extended by linearity to $\mathcal{C}(V)$.

The dual $\mathcal{C}^*(V)$ is a bialgebra of functions from $c(V)$ to $\mathbb{C}$, i.e. it is a quantum state for strings. The commutative product is simply the component-wise product of these functions.

Iterated integration

The key to an algebraic formulation of string theory has to be a means to project algebraic structures onto geometric ones. A starting point would be a bialgebra homomorphism from $T^*(V)$ to $C^*(V)$. Both of these bialgebras have a commutative product. The product of $T(V)$ is concatenation of words made from strings of discrete letters, while the product of $C(V)$ is concatenation of piecewise continuous strings. Surprisingly there is an exact homomorphism mapping the discrete to the continuous using iterated integration.

Let $w = e^{i_1} \dots e^{i_r}$ be a word of length r in $T^*(V)$ and $c = c(t), 0 \le t < T$ a curve in $\otimes$. Define

$$\theta(w \otimes c) = \int_0^T dt_1\, c^{i_1}(t_1) \int_0^{t_1} dt_2\, c^{i_2}(t_2) \dots \int_0^{t_{r-1}} dt_r\ c^{i_r}(t_r)$$

θ can be extended to a complex valued function on $T^*(V) \otimes C(V)$ by linearity.

It can then be verified using sum rules for iterated integrals that

$$\theta\big((u \sqcup\!\sqcup v) \otimes c\big) = \theta(u \otimes c)\theta(v \otimes c)$$

$$\theta\big(v \otimes (a \cdot b)\big) = (\theta \otimes \theta)\big(\Delta(v) \otimes (a \otimes b)\big)$$

This confirms that θ provides a bialgebra homomorphism from $T^*(V)$ to $C^*(V)$.

Spacetime is a manifold rather than a vector space so is there another way to do the mapping with manifolds? Take $V_{\mathfrak{M}}$ to be the vector space of complex valued scalar functions on a manifold $\mathfrak{M}$. An element of $T^*(V_{\mathfrak{M}})$ is determined by a set of scalar functions $v(\boldsymbol{x}_1, \dots, \boldsymbol{x}_r)$ of r points on $\mathfrak{M}$ for $= 0, 1, 2, \dots$. Shuffle products are formed using shuffles to the arguments.

An bialgebra $S(\mathfrak{M})$ can be defined on functions from continuous curves $\boldsymbol{x}(t)$ in $\mathfrak{M}$. The bialgebra homomorphism φ from $T^*(V_{\mathfrak{M}})$ to $S(\mathfrak{M})$ is determined by

$$\varphi\big(\boldsymbol{x}(t)\big) = \sum_r \int_0^T dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{r-1}} dt_r\, v(\boldsymbol{x}(t_1), \dots, \boldsymbol{x}(t_r))$$

Free Lie Algebra

Open strings by themselves are not sufficient to describe string theory with gravity. Loops are essential for graviton states. To find these we must consider the structure of free lie algebras.

A vector space V can be used to generate a free lie algebra $L(V)$ by successively generating commutators modulo anti-symmetry and the Jacobi identities.

$L(V)$ is graded over the positive integers with the grade one part being V and the grade two part being antisymmetric tensor product of V with itself. It is known that in general the grade r part $V^r = \otimes^r V$ has a basis over lyndon words of necklaces of length r. This is true over any field but the proof in general is messy. For the special case of the free lie algebra over the complex field an easier construction is possible.

An index rotation operator R with $R^r = 1$ can be defined to act on V^r such that

$$R(v_1 \otimes v_2 \otimes \ldots \otimes v_r) = v_2 \otimes \ldots \otimes v_r \otimes v_1, \qquad v_i \in V$$

A cyclic subspace is defined $C^r \subset V^r = \{t \in V^r | R(t) = e^{\frac{2\pi i}{r}} \times t\}$

The indexes for base elements of this space C^r are identified with aperiodic necklaces which are in one-to-one correspondence with Lyndon words.

An operator P from V^r to C^r is given by

$$P = \sum_{k=1}^{r} R^k e^{-\frac{2\pi i k}{r}}$$

Define an infinite dimensional graded space $L(V) = C^1 \oplus C^2 \oplus \ldots$

A bilinear bracket operator can be defined on $L(V)$ by

$$[A, B] = P_{r+s}(A \otimes B - B \otimes A), \qquad A \in C^r, B \in C^s$$

This is extended to all elements of $L(V)$ using the bilinear property of the bracket.

With this definition the bracket operator is a Lie product for the free lie algebra generated freely over V. This can be verified by checking anti-symmetry and the Jocabi identity.

The universal enveloping algebra for $L(V)$ is isomorphic to the tensor algebra $T(V)$. Basis elements of the algebra are single necklaces but the universal enveloping algebra consists also of products of multiple necklaces. It's bialgebra dual has a commutative shuffle product which has an induced form of elements of the lie algebra as cyclic shuffles of necklaces. There is then a homomorphism mapping from this algebra to quantum states of multiple loops on a manifold.

In the early days of string theory some theorists such as Kaku sought to form a lie algebra or super-lie algebra on string loops so that a Langangian could be defined that respected the symmetry [Kaku]. This failed because the proposed Lie algebra did not quite close correctly, and string field theory was developed in other ways, but without providing much insight into non-perturbative formulations. The approach described here provides a homomorphism from the dual of the universal enveloping algebra of the free lie algebra to string states. By principles of duality this should also give a mapping from the dual of string states back to the free lie algebra. A field theory on the free lie algebra then becomes a string field theory.

To widen the scope it may be possible to extend the principles of these mappings from algebras to string theory by treating Feynman diagrams as algebraic constructs and extending the ideas to higher dimensional algebras such as n-categories.

There are relationships between free lie algebras, iterated integrals, conformal field theory and polylogarithms. This gives a strong sense that the approach advocated here embodies the right mathematics to find the origins of string theory. I urge string theorists and mathematicians to investigate further this possibility.

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