QUANTUM PARTIAL AUTOMORPHISMS OF FINITE GRAPHS

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Abstract. The partial automorphisms of a graph X having N vertices are the bijections $\sigma : I \to J$ with $I, J \subset \{1, \ldots, N\}$ which leave invariant the edges. These bijections form a semigroup $\widetilde{G}(X)$, which contains the automorphism group $G(X)$. We discuss here the quantum analogue of this construction, with a definition and basic theory for the quantum semigroup of quantum partial automorphisms $\widetilde{G}^+(X)$, which contains both $G(X)$, and the quantum automorphism group $G^+(X)$. We comment as well on the case $N = \infty$, which is of particular interest, due to the fact that $\widetilde{G}^+(X)$ is well-defined, while its subgroup $G^+(X)$, not necessarily, at least with the currently known methods.

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Introduction

Associated to any finite graph X, having N vertices, is its quantum automorphism group $G^+(X) \subset S_N^+$, obtained as the subgroup of Wang's quantum permutation group [26] which leaves invariant the edges. This quantum group contains the usual automorphism group, $G(X) \subset G^+(X)$, but this inclusion is in general not an isomorphism. When this latter inclusion is proper, the graph X is said to have quantum symmetries, and the study of $G^+(X)$ is an interesting question. The basic theory here goes back to [2], [3], [11], [12],

2010 *Mathematics Subject Classification.* 46L65.

Key words and phrases. Graph automorphism, Partial permutation.

and then to [6], [7], [8], papers from the mid 00s. More recent work on the subject, mostly from the mid and late 10s, includes the papers [13], [14], [17], [18], [19], [20], [21], [22], [23], [24], solving some old questions, and making the link with advanced graph theory, and with nonlocal games and quantum information theory.

We will be interested here in the semigroup of partial automorphisms $\widetilde{G}(X)\subset\widetilde{S}_N$, and in its quantum analogue $\widetilde{G}^+(X)\subset\widetilde{S}_N^+$. These are objects which are far more specialized than $G(X)$ and $G^+(X)$, and explaining our motivations will be our first goal.

In what regards the classical automorphisms, the main object to be studied is definitely the group $G(X)$, with the bigger semigroup $\widetilde{G}(X)$ being something rather abstract, having little to no interest. However, in what regards the quantum automorphisms, while the quantum group $G^+(X)$ remains of course the main object to be studied, talking about the bigger quantum semigroup $\widetilde{G}^+(X)$ makes sense, and can be potentially helpful, due to a number of subtle algebraic and analytic reasons, as follows:

(1) As a first piece of motivation, the semigroup $\widetilde{S}_N^+$ was introduced in [10], the work there being motivated by the fact that the semigroups $G\subset\widetilde{S}_N^+$ encode the combinatorics of the partial Hadamard matrices, with the main interest in these latter matrices coming from the work in [15], which shows that the Hadamard Conjecture problematics is far more tractable in the rectangular matrix setting. All this is quite heavy, with a considerable amount of algebraic and analytic work to be done, and from this perspective, the study of the quantum semigroups of type $\widetilde{G}^+(X)\subset\widetilde{S}_N^+$ can only be useful.

(2) As a second piece of motivation, the semigroup $\widetilde{S}_N^+$ is part of a whole family of semigroups, including as well objects of type $\widetilde{O}_N^+$, $\widetilde{U}_N^+$, discussed in [4], [5] and subsequent papers, and with all these semigroups being the key ingredients for constructing objects like free Grassmannians, flag manifolds, Stiefel manifolds, and so on. In short, all this is related to the development of "free geometry". As before with the partial Hadamard matrix program, there is a lot of work here to be done, and from this perspective, the study of the quantum semigroups of type $\widetilde{G}^+(X)\subset\widetilde{S}_N^+$ can only be useful.

(3) As a third piece of motivation, one big question is that of understanding what the quantum automorphism group $G^+(X)$ of an infinite graph X is. This is a notoriously difficult question, the problem coming from the fact that we can have both compact quantum group actions $G\curvearrowright X$ and discrete quantum group actions $\Gamma\curvearrowright X$, and also from the fact that for the simplest infinite graph, having no edges at all, the quantum symmetry group S_∞^+ is not well-defined [16]. From this perspective, looking at subgroups of the semigroups $\widetilde{G}^+(X)$, which are well-defined, is a good way to be followed.

Summarizing, we have a number of good motivations for studying $\widetilde{G}^+(X)$. Observe that all these motivations are of genuine "quantum" nature, having no classical counterpart. Observe also that all this puts $\widetilde{G}(X)$ into a more favorable light. Indeed, this apparently anecdotical semigroup gains in this way some motivations to be studied, for the simple reason that this semigroup appears as the classical version of $\widetilde{G}^+(X)$.

We will develop in this paper the basic theory of the construction $X \to \widetilde{G}^+(X)$. Besides the precise definition of $\widetilde{G}^+(X)$, which is not exactly obvious, and which will be one of our main results here, we will work out semigroup analogues of some of the basic results from [2], [6] regarding the construction $X \to G^+(X)$. As we will see, with a good definition for $\widetilde{G}^+(X)$ in hand, some of the basic results will extend in a straightforward way, some other part of the basic results will extend in a tricky way, and some other part of the basic results will not extend at all. For the precise statements of our results, which require some discussion and new definitions, we refer to the body of the paper.

In what concerns the possible continuations of the present work, there are several of them, as part of the long-term programs (1,2,3) mentioned above, and we intend to come back to this in due time. Among these programs (3) is the most recent, and the most graph-theoretical one as well, and we will comment on this at the end of the paper.

The paper is organized as follows: 1 is a preliminary section, in 2-3 we construct the semigroup of quantum partial automorphisms $\widetilde{G}^+(X)$ and we discuss its basic properties, in 4-5-6 we discuss a number of more advanced properties of $\widetilde{G}^+(X)$, notably with results about color independence and cycles, and in 7 we discuss the general question of locating $G^+(X)$ inside $\widetilde{G}^+(X)$, in the case where the graph is infinite, $|X| = \infty$.

1. Quantum permutations and partial permutations

Let us start with the following definition, which is standard:

Definition 1.1. *A partial permutation of $\{1 \ldots, N\}$ is a bijection $\sigma : I \simeq J$, with $I, J \subset \{1, \ldots, N\}$. We denote by $\widetilde{S}_N$ the semigroup formed by such partial permutations.*

Observe that we have $S_N \subset \widetilde{S}_N$. The embedding $u : S_N \subset M_N(0,1)$ given by permutation matrices can be extended to an embedding $u : \widetilde{S}_N \subset M_N(0,1)$, as follows:

$$u_{ij}(\sigma) = \begin{cases} 1 & \text{if } \sigma(j) = i \\ 0 & \text{otherwise} \end{cases}$$

By looking at the image of this embedding, we see that $\widetilde{S}_N$ is in bijection with the matrices $M \in M_N(0,1)$ having at most one 1 entry on each row and column.

In what follows we will be interested in certain algebraic aspects of the partial permutations, and their quantum analogues. Before starting, let us record however the following result, which shows that the bare partial permutations are very interesting objects:

Proposition 1.2. *The number of partial permutations is given by*

$$|\widetilde{S}_N|=\sum_{k=0}^N k!\binom{N}{k}^2$$

that is, $1,2,7,34,209,\ldots$, *and with* $N\to\infty$ *we have* $|\widetilde{S}_N|\simeq N!\sqrt{\frac{\exp(4\sqrt{N}-1)}{4\pi\sqrt{N}}}$.

Proof. Indeed, in terms of partial bijections $\sigma:I\simeq J$ as in Definition 1.1, we can set $k=|I|=|J|$, and this leads to the formula in the statement. Equivalently, in the $M_N(0,1)$ picture, k is the number of 1 entries, $\binom{N}{k}^2$ corresponds to the choice of the k rows and k columns for these 1 entries, and $k!$ comes from positioning the 1 entries. Finally, the asymptotic formula if well-known, see OEIS, sequence A002720. □

Getting back now to our present purposes, which are mostly algebraic, we have so far an inclusion of semigroups $S_N\subset\widetilde{S}_N$, along with a useful linear algebra interpretation of it. In functional analysis terms, following [10], [26], the result is as follows:

Proposition 1.3. *The algebras* $C(\widetilde{S}_N)\to C(S_N)$ *have presentations as follows,*

$$C(S_N)=C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\text{magic}\right)$$

$$C(\widetilde{S}_N)=C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\text{submagic}\right)$$

with "submagic" meaning formed of projections, pairwise orthogonal on rows and columns, and with "magic" assuming in addition that the row and column sums are 1.

Proof. This is standard, by using the Gelfand theorem. Indeed, this theorem shows that the algebras on the right must be of the form $C(X_N)$, $C(\widetilde{X}_N)$, for certain compact spaces $X_N\subset\widetilde{X}_N$, and by using the coordinate functions u_{ij} we obtain $X_N\subset\widetilde{X}_N\subset M_N(0,1)$, with the equations for $X_N\subset\widetilde{X}_N$ being those for $S_N\subset\widetilde{S}_N$. See [10], [26]. □

Still following [10], [26], the above presentation result, along with the fact that S_N is a finite group and $\widetilde{S}_N$ is a finite semigroup, has the following quantum analogue:

Theorem 1.4. *We have universal* C^**-algebras* $C(\widetilde{S}_N^+)\to C(S_N^+)$ *as follows,*

$$C(S_N^+)=C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\text{magic}\right)$$

$$C(\widetilde{S}_N^+)=C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\text{submagic}\right)$$

and the underlying compact quantum spaces $S_N^+\subset\widetilde{S}_N^+$ *are respectively a compact quantum group, and a compact quantum semigroup, which are infinite at* $N\geq 4,2$.

Proof. Since the entries u_{ij} are projections, we have $||u_{ij}|| \leq 1$, and so the universal C^*-algebras in the statement are indeed well-defined. Next, by using the universality property of these algebras, we can define in both cases morphisms of algebras, as follows:

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$$

$$\varepsilon(u_{ij}) = \delta_{ij}$$

$$S(u_{ij}) = u_{ji}$$

In the case of $C(S_N^+)$ the matrix $u = (u_{ij})$ is unitary, so the Woronowicz axioms in [27], [28] are satisfied, and S_N^+ is a compact quantum group. In the case of $C(\widetilde{S}_N^+)$ the matrix $u = (u_{ij})$ is no longer unitary, and we can only say that $\widetilde{S}_N^+$ is a compact quantum semigroup with subantipode, with the subantipode condition meaning that we have:

$$m_3(S \otimes id \otimes S)\Delta_2 = S$$

Finally, the simplest example of a magic matrix having noncommuting entries appears at $N = 4$, as follows, with p, q being suitable projections on $H = l^2(\mathbb{N})$:

$$u = \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

As for the simplest submagic matrix having noncommuting entries, this appears at $N = 2$, as follows, with p, q, r, s being suitable projections on $H = l^2(\mathbb{N})$:

$$u = \begin{pmatrix} p \oplus 0 & 0 \oplus q \\ 0 \oplus r & s \oplus 0 \end{pmatrix}$$

But this gives the last assertion. For details on all this, see [10], [26]. □

As a conclusion to all this, we have a diagram as follows, with all maps being inclusions, and with the vertical inclusions being liberation operations:

$$\begin{tikzcd} S_N^+ \arrow[r] & \widetilde{S}_N^+ \\ S_N \arrow[u] \arrow[r] & \widetilde{S}_N \arrow[u] \end{tikzcd}$$

Finally, let us mention that, in addition to what has been said above, there are many known things about S_N^+, with virtually all important properties of S_N having their quantum counterpart. As for $\widetilde{S}_N^+$, which is more of a technical object, a bit like $\widetilde{S}_N$, there is a growing body of literature here. See [1], [4], [5], [9], [10], [26] and related papers.

2. Graphs and their quantum partial automorphisms

Let us get now into the graph problematics. Things are quite tricky here, and this already in what regards the automorphism groups, where two possible definitions were proposed in the quantum case [2], [11]. The statement from [2], that we will use here, along with the classical definition, suitably formulated, are as follows:

Proposition 2.1. *Given a graph X with N vertices, and adjacency matrix $d \in M_N(0,1)$, the algebra of functions on its automorphism group $G(X) \subset S_N$ is given by:*

$$C(G(X)) = C(S_N) \Big/ \Big\langle du = ud \Big\rangle$$

The quantum automorphism group of X is defined by the following formula,

$$C(G^+(X)) = C(S_N^+) \Big/ \Big\langle du = ud \Big\rangle$$

which defines indeed a closed subgroup $G^+(X) \subset S_N^+$, whose classical version is $G(X)$.

Proof. In order to prove the first assertion, consider the standard coordinates on $S_N \subset O_N$, which are the following characteristic functions:

$$u_{ij} = \chi\left(\sigma \in S_N \Big| \sigma(j) = i\right)$$

With this formula in hand, $du = ud$ is equivalent to $d_{ij} = d_{\sigma(i)\sigma(j)}$, and this gives the first assertion. As for the second assertion, this is standard, coming from the fact that $du = ud$ can be rewritten as $d \in End(u)$, which is a relation of "Hopf type". See [2]. $\square$

As already mentioned, the above might look quite straightforward, but it is not. The point indeed is that an equally natural idea is that of saying that the quantum automorphism group should act on both the vertices and edges of X, and this leads to a certain intermediate quantum group $G(X) \subset G^\times(X) \subset G^+(X)$, constructed in [11], and having a number of interesting technical uses. For a recent discussion here, see [23].

Let us discuss now the partial automorphism groups, and their quantum analogues. Things are quite tricky here as well, and skipping a discussion of the various possible wrong ways that can be taken, the idea will be that of using the following relation, with R, C being the diagonal matrices formed by the row and column sums of u:

$$R(du - ud)C = 0$$

In order to explain this, let us begin with the classical case. Here the definition of $\widetilde{G}(X)$, along with the corresponding presentation of $C(\widetilde{G}(X))$, and with a result as well about what happens when imposing the condition $du = ud$, is as follows:

Proposition 2.2. *Given a graph X with N vertices, and adjacency matrix $d \in M_N(0,1)$, consider its partial automorphism semigroup, given by:*

$$\widetilde{G}(X) = \left\{ \sigma \in \widetilde{S}_N \middle| d_{ij} = d_{\sigma(i)\sigma(j)},\ \forall i,j \in Dom(\sigma) \right\}$$

We have then the following formula, with $R = diag(R_i)$, $C = diag(C_j)$, with R_i, C_j being the row and column sums of the associated submagic matrix u:

$$C(\widetilde{G}(X)) = C(\widetilde{S}_N) \Big/ \Big\langle R(du - ud)C = 0 \Big\rangle$$

Moreover, when using the relation $du = ud$ instead of the above one, we obtain a certain semigroup $\bar{G}(X) \subset \widetilde{G}(X)$, which can be strictly smaller.

Proof. We have two assertions here, the idea being as follows:

(1) With the convention $i \sim j$ when i,j are connected by an edge of X, the definition of $\widetilde{G}(X)$ from the statement reformulates as follows:

$$\widetilde{G}(X) = \left\{ \sigma \in \widetilde{S}_N \middle| i \sim j, \exists\, \sigma(i), \exists\, \sigma(j) \implies \sigma(i) \sim \sigma(j) \right\}$$

We have the following computations:

$$(du)_{ij}(\sigma) = \sum_k d_{ik} u_{kj}(\sigma) = \sum_{k \sim i} u_{kj}(\sigma) = \begin{cases} 1 & \text{if } \sigma(j) \sim i \\ 0 & \text{otherwise} \end{cases}$$

$$(ud)_{ij}(\sigma) = \sum_k u_{ik} d_{kj}(\sigma) = \sum_{k \sim j} u_{ik}(\sigma) = \begin{cases} 1 & \text{if } \sigma^{-1}(i) \sim j \\ 0 & \text{otherwise} \end{cases}$$

Here the "otherwise" cases include by definition the cases where $\sigma(j)$, respectively $\sigma^{-1}(i)$, is undefined. We have as well the following formulae:

$$R_i(\sigma) = \sum_j u_{ij}(\sigma) = \begin{cases} 1 & \text{if } \exists\, \sigma^{-1}(i) \\ 0 & \text{otherwise} \end{cases}$$

$$C_j(\sigma) = \sum_i u_{ij}(\sigma) = \begin{cases} 1 & \text{if } \exists\, \sigma(j) \\ 0 & \text{otherwise} \end{cases}$$

Now by multiplying the above formulae, we obtain the following formulae:

$$(R_i (du)_{ij} C_j)(\sigma) = \begin{cases} 1 & \text{if } \sigma(j) \sim i \text{ and } \exists\, \sigma^{-1}(i) \text{ and } \exists\, \sigma(j) \\ 0 & \text{otherwise} \end{cases}$$

$$(R_i (ud)_{ij} C_j)(\sigma) = \begin{cases} 1 & \text{if } \sigma^{-1}(i) \sim j \text{ and } \exists\, \sigma^{-1}(i) \text{ and } \exists\, \sigma(j) \\ 0 & \text{otherwise} \end{cases}$$

We conclude that the relations in the statement, which read $R_i(du)_{ij}C_j = R_i(ud)_{ij}C_j$, when applied to a given $\sigma \in \widetilde{S}_N$, correspond to the following condition:

$$\exists\,\sigma^{-1}(i),\ \exists\,\sigma(j) \implies [\sigma(j) \sim i \iff \sigma^{-1}(i) \sim j]$$

But with $i = \sigma(k)$, this latter condition reformulates as follows:

$$\exists\,\sigma(k),\ \exists\,\sigma(j) \implies [\sigma(j) \sim \sigma(k) \iff k \sim j]$$

Thus we must have $\sigma \in \widetilde{G}(X)$, and we obtain the presentation result for $\widetilde{G}(X)$.

(2) Regarding now the second assertion, the simplest counterexample here is simplex X_N, having N vertices and edges everywhere. Indeed, the adjacency matrix of this simplex is $d = \mathbb{I}_N - 1_N$, with $\mathbb{I}_N$ being the all-1 matrix, and so the commutation of this matrix with u corresponds to the fact that u must be bistochastic. Thus, u must be in fact magic, and we obtain $\bar{G}(X_N) = S_N$, which is smaller than $\widetilde{G}(X_N) = \widetilde{S}_N$. $\square$

With the above result in hand, we are led to the following statement:

Theorem 2.3. *Given a graph X with N vertices, and adjacency matrix $d \in M_N(0,1)$, the following construction, with R, C being the diagonal matrices formed by the row and column sums of u, produces a quantum semigroup with subantipode $\widetilde{G}^+(X) \subset \widetilde{S}_N^+$,*

$$C(\widetilde{G}^+(X)) = C(\widetilde{S}_N^+)\Big/\Big\langle R(du - ud)C = 0\Big\rangle$$

called quantum semigroup of quantum partial automorphisms of X, whose classical version is $\widetilde{G}(X)$. Moreover, when using $du = ud$ instead of the above relation, we obtain a semigroup $\bar{G}^+(X) \subset \widetilde{G}^+(X)$, which can be strictly smaller.

Proof. We have to construct morphisms Δ, ε, S and then prove the last two assertions, and the proof goes as follows:

(1) In order to construct the comultiplication Δ, consider the following elements:

$$U_{ij} = \sum_k u_{ik} \otimes u_{kj}$$

We must prove that the relations in the statement are satisfied by $U=(U_{ij})$. For this purpose, observe that we have, by using the fact that u is submagic:

$$\begin{aligned}
R_i^U(dU)_{ij}C_j^U &= \sum_k U_{ik}\sum_l d_{il}U_{lj}\sum_m U_{mj}\\
&= \sum_{kn} u_{in}\otimes u_{nk}\sum_{lo} d_{il}u_{lo}\otimes u_{oj}\sum_{mp} u_{mp}\otimes u_{pj}\\
&= \sum_{klmnop} d_{il}\cdot u_{in}u_{lo}u_{mp}\otimes u_{nk}u_{oj}u_{pj}\\
&= \sum_{klmno} d_{il}\cdot u_{in}u_{lo}u_{mo}\otimes u_{nk}u_{oj}\\
&= \sum_{klno} d_{il}\cdot u_{in}u_{lo}\otimes u_{nk}u_{oj}
\end{aligned}$$

On the other hand, we have as well the following formula:

$$\begin{aligned}
R_i(du)_{ij}C_j &= \sum_k u_{ik}\sum_l d_{il}u_{lj}\sum_m u_{mj}\\
&= \sum_{klm} d_{il}\cdot u_{ik}u_{lj}u_{mj}\\
&= \sum_{kl} d_{il}\cdot u_{ik}u_{lj}
\end{aligned}$$

Now by getting back to our computation, we can finish it as follows:

$$\begin{aligned}
R_i^U(dU)_{ij}C_j^U &= \sum_{klno} d_{il}\cdot u_{in}u_{lo}\otimes u_{nk}u_{oj}\\
&= \sum_{kn} u_{in}\otimes u_{nk}\sum_{lo} d_{il}\cdot u_{lo}\otimes u_{oj}\\
&= \sum_{kl} d_{il}\cdot\Delta(u_{ik})\Delta(u_{lj})\\
&= \Delta\left(\sum_{kl} d_{il}\cdot u_{ik}u_{lj}\right)\\
&= \Delta(R_i(du)_{ij}C_j)
\end{aligned}$$

(2) The second computation that we need is similar. We first have:

$$\begin{aligned}
R_i^U(Ud)_{ij}C_j^U
&= \sum_kU_{ik}\sum_lU_{il}d_{lj}\sum_mU_{mj}\\
&= \sum_{kn}u_{in}\otimes u_{nk}\sum_{lo}d_{lj}u_{io}\otimes u_{ol}\sum_{mp}u_{mp}\otimes u_{pj}\\
&= \sum_{klmnop}d_{lj}\cdot u_{in}u_{io}u_{mp}\otimes u_{nk}u_{ol}u_{pj}\\
&= \sum_{klmnp}d_{lj}\cdot u_{in}u_{mp}\otimes u_{nk}u_{nl}u_{pj}\\
&= \sum_{kmnp}d_{lj}\cdot u_{in}u_{mp}\otimes u_{nk}u_{pj}
\end{aligned}$$

On the other hand, we have as well the following formula:

$$\begin{aligned}
R_i(ud)_{ij}C_j
&= \sum_ku_{ik}\sum_lu_{il}d_{lj}\sum_mu_{mj}\\
&= \sum_{klm}d_{lj}\cdot u_{ik}u_{il}u_{mj}\\
&= \sum_{km}d_{lj}\cdot u_{ik}u_{mj}
\end{aligned}$$

Now by getting back to our computation, we can finish it as follows:

$$\begin{aligned}
R_i^U(Ud)_{ij}C_j^U
&= \sum_{kmnp}d_{lj}\cdot u_{in}u_{mp}\otimes u_{nk}u_{pj}\\
&= \sum_{kn}u_{in}\otimes u_{nk}\sum_{mp}d_{lj}\cdot u_{mp}\otimes u_{pj}\\
&= \sum_{km}d_{lj}\cdot\Delta(u_{ik})\Delta(u_{mj})\\
&= \Delta\left(\sum_{km}d_{lj}\cdot u_{ik}u_{mj}\right)\\
&= \Delta(R_i(ud)_{ij}C_j)
\end{aligned}$$

(3) We can now construct Δ, based on the formulae found in (1,2), namely:

$$R_i^U(dU)_{ij}C_j^U=\Delta(R_i(du)_{ij}C_j)$$

$$R_i^U(Ud)_{ij}C_j^U=\Delta(R_i(ud)_{ij}C_j)$$

Indeed, we know that the quantities on the right are equal, and the quantities on the left follow to be equal as well. Thus we can define Δ by $u_{ij}\to U_{ij}$, as desired.

(4) Regarding now ε, the algebra in the statement has indeed a morphism ε defined by $u_{ij}\to\delta_{ij}$, because the following relations are trivially satisfied:

$$R_i(d1_N)_{ij}C_j=R_i(1_Nd)_{ij}C_j$$

(5) Regarding now S, we must prove that we have a morphism S given by $u_{ij}\to u_{ji}$. Here the best is to use the reformulation of the relations in the statement mentioned before the statement itself, which is as follows, with $R=diag(R_i)$ and $C=diag(C_j)$:

$$R(du-ud)C=0$$

Indeed, this formula is a certain equality of $N\times N$ matrices. Now when transposing this formula, we obtain:

$$C^t(u^td-du^t)R^t=0$$

Now since C^t,R^t are respectively the diagonal matrices formed by the row sums and column sums of u^t, we conclude that the relations $R(du-ud)C=0$ are satisfied by the transpose matrix u^t, and this gives the existence of the subantipode map S.

(6) The fact that we have $\widetilde{G}^+(X)_{class}=\widetilde{G}(X)$ follows from $(S_N^+)_{class}=S_N$.

(7) Finally, the last assertion follows from the last assertion in Proposition 2.2, by taking classical versions, the simplest counterexample being the simplex. □

Summarizing, we have a good liberation inclusion $\widetilde{G}(X)\subset\widetilde{G}^+(X)$, that we will study in what follows. We will sometimes use the "wrong" semigroups $\bar{G}(X)\subset\bar{G}^+(X)$ as well, for certain technical purposes, and the full picture includes of course the automorphism groups $G(X)\subset G^+(X)$ as well. The diagram formed by these objects is as follows:

$$\begin{array}{ccccc}
G^+(X) & \longrightarrow & \bar{G}(X) & \longrightarrow & \widetilde{G}^+(X)\\
\uparrow & & \uparrow & & \uparrow\\
G(X) & \longrightarrow & \bar{G}(X) & \longrightarrow & \widetilde{G}(X)
\end{array}$$

Here all the maps are inclusions, and the vertical maps are liberations. On the left we have automorphism groups, on the right we have partial automorphism semigroups, and in the middle we have the "wrong" semigroups, which can be technically useful objects.

3. General properties, simplices and complementation

In this section and in the next three ones we study the basic properties of the operation $X\to\widetilde{G}^+(X)$, by taking some inspiration from [2], where the basic properties of the operation $X\to G^+(X)$ were established. Let us start with a useful technical result, providing us with some alternative formulations of the relations that we use:

Proposition 3.1. *Given a $N\times N$ submagic matrix u, set $R=diag(R_i)$ and $C=diag(C_j)$, with R_i,C_j being the row and column sums. We have then $Ru=u=uC$, and given a matrix $d\in M_N(\mathbb C)$, the following conditions are equivalent:*

1. *$R(du-ud)C=0$.*
2. *$Rdu=udC$.*
3. *$R_i(du)_{ij}C_j=R_i(ud)_{ij}C_j$.*
4. *$R_i(du)_{ij}=(ud)_{ij}C_j$.*

Proof. In order to check the equalities $Ru=u=uC$, consider the standard basis $e_1,\ldots,e_N$ of the space $\mathbb C^N$. We have the following computation:

$$\begin{aligned}
Ru(e_i)&=R\left(\sum_ke_k\otimes u_{ki}\right)\\
&=\sum_ke_k\otimes R_ku_{ki}\\
&=\sum_{kl}e_k\otimes u_{kl}u_{ki}\\
&=\sum_ke_k\otimes u_{ki}\\
&=u(e_i)
\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}
uC(e_i)&=u(e_i\otimes C_i)\\
&=\sum_ke_k\otimes u_{ki}C_i\\
&=\sum_{kl}e_k\otimes u_{ki}u_{li}\\
&=\sum_ke_k\otimes u_{ki}\\
&=u(e_i)
\end{aligned}$$

Thus, the equalities $Ru=u=uC$ hold indeed, and with this in hand, (1) $\iff$ (2) is clear. As for (1) $\iff$ (3) and (2) $\iff$ (4), these are both clear from definitions. □

As a first result now regarding the correspondence $X\to\widetilde{G}^+(X)$, we have:

Proposition 3.2. *For any finite graph X we have*

$$\widetilde{G}^+(X)=\widetilde{G}^+(X^c)$$

where X^c is the complementary graph.

Proof. The adjacency matrices of a graph X and of its complement X^c are related by the following formula, where $\mathbb{I}_N$ is the all-1 matrix:

$$d_X + d_{X^c} = \mathbb{I}_N - 1_N$$

Thus, in order to establish the formula in the statement, we must prove that:

$$R_i(\mathbb{I}_N u)_{ij} C_j = R_i(u\mathbb{I}_N)_{ij} C_j$$

For this purpose, let us recall that, the matrix u being submagic, its row sums and column sums R_i, C_j are projections. By using this fact, we have:

$$R_i(\mathbb{I}_N u)_{ij} C_j = R_i C_j C_j = R_i C_j$$

$$R_i(u\mathbb{I}_N)_{ij} C_j = R_i R_i C_j = R_i C_j$$

Thus we have proved our equality, and the conclusion follows. □

Let us discuss now some basic product operations. Following [25], we first have the following standard construction:

Proposition 3.3. *Given two semigroups $G \subset \widetilde{S}_N^+$ and $H \subset \widetilde{S}_M^+$, with submagic matrices denoted u, v, we can construct a semigroup $G \,\hat{*}\, H \subset \widetilde{S}_{M+N}^+$ by setting*

$$C(G \,\hat{*}\, H) = C(G) * C(H)$$

with submagic matrix $w = diag(u, v)$.

Proof. Since u, v are submagic, so is $w = diag(u, v)$, and the construction of Δ, ε and of the subantipode map S is standard, as in the quantum group case. □

With the above notion in hand, we can formulate:

Proposition 3.4. *Assuming that we have semigroup actions $G_i \curvearrowright X_i$, we have a semigroup action as follows:*

$$G_1 \,\hat{*}\, \ldots \,\hat{*}\, G_n \curvearrowright X_1 \sqcup \ldots \sqcup X_n$$

In particular, we have an inclusion of semigroups, as follows:

$$\widetilde{G}^+(X_1) \,\hat{*}\, \ldots \,\hat{*}\, \widetilde{G}^+(X_n) \subset \widetilde{G}^+(X_1 \sqcup \ldots \sqcup X_n)$$

Proof. The submagic matrix of the semigroup $G_1 \,\hat{*}\, \ldots \,\hat{*}\, G_n$ and the adjacency matrix of the graph $X_1 \sqcup \ldots \sqcup X_n$ are by definition block diagonal, as follows:

$$u = \begin{pmatrix} u^{(1)} & & \\ & \ddots & \\ & & u^{(n)} \end{pmatrix} \quad , \quad d = \begin{pmatrix} d^{(1)} & & \\ & \ddots & \\ & & d^{(n)} \end{pmatrix}$$

Regading the row and column sum matrices of u, these are of a similar form:

$$R=\begin{pmatrix}R^{(1)}&&\\&\ddots&\\&&R^{(n)}\end{pmatrix}\quad,\quad C=\begin{pmatrix}C^{(1)}&&\\&\ddots&\\&&C^{(n)}\end{pmatrix}$$

Now since the relations $Rdu=udC$ are satisfied over each block, they are satisfied globally, and this gives the first assertion. The second assertion follows from it. □

There are many other types of product operations for graphs, discussed in [1], [2], [6], [7], [8], [14], [22], [23], which are more specialized, and whose semigroup extension is less straightforward. We will be back to this later, after developing some general theory.

4. Colored oriented graphs and color independence

Following [2], one basic thing to be done, which is of key importance, is that of examining the stability of the condition $R(du-ud)C=0$ under the joint spectral and color decomposition of d. We are therefore naturally led into an extension of our formalism, using matrices $d\in M_N(\mathbb C)$, so let us start here with the following definition:

Definition 4.1. *Associated to any complex-colored oriented graph X, with adjacency matrix $d\in M_N(\mathbb C)$, is its semigroup of partial automorphisms, given by*

$$\widetilde{G}(X)=\left\{\sigma\in\widetilde{S}_N\Big|d_{ij}=d_{\sigma(i)\sigma(j)},\ \forall i,j\in Dom(\sigma)\right\}$$

as well as its quantum semigroup of quantum partial automorphisms, given by

$$C(\widetilde{G}^+(X))=C(\widetilde{S}_N^+)\Big/\Big\langle R(du-ud)C=0\Big\rangle$$

where $R=diag(R_i)$, $C=diag(C_j)$, with R_i,C_j being the row and column sums of u.

Here the fact that $\widetilde{G}^+(X)$ is indeed a quantum semigroup follows from the proof of Theorem 2.3, which does not use $d\in M_N(0,1)$. Observe that the proof of Proposition 2.2 does not use either $d\in M_N(0,1)$, and so we have the following formula:

$$C(\widetilde{G}(X))=C(\widetilde{S}_N)\Big/\Big\langle R(du-ud)C=0\Big\rangle$$

Thus, the inclusion $\widetilde{G}(X)\subset\widetilde{G}^+(X)$ is a liberation, and as a conclusion, everything that has been said in sections 2-3 above extends to the present "colored" setting.

With this extended formalism in hand, let us discuss now the color independence. The point here is that the various automorphism groups $\Gamma=G,\widetilde{G},G^+,\widetilde{G}^+$ should be all invariant under "change of colors", meaning replacing complex numbers appearing all across $d \in M_N(\mathbb{C})$ by other complex numbers. In other words, the claim is that for $\Gamma = G, \widetilde{G}, G^+, \widetilde{G}^+$, the following implication should hold:

$$\left[d_{ij} = d_{kl} \iff d'_{ij} = d'_{kl}\right] \implies \Gamma(X) = \Gamma(X')$$

For $\Gamma = G, \widetilde{G}$ this is just obvious from definitions. For $\Gamma = G^+$, however, this is not exactly trivial, and the proof here, from [2], consists of an algebraic trick, combined with an analytic argument, making it for a half a page proof. In what follows we will discuss the remaining case, $\Gamma = \widetilde{G}^+$, by suitably adapting the proof in [2]. As we will see, there will be many new computations needed, and some tricks too. Let us start with:

Definition 4.2. *We let m, γ be the multiplication and comultiplication of $\mathbb{C}^N$,*

$$m(e_i \otimes e_j) = \delta_{ij} e_i$$

$$\gamma(e_i) = e_i \otimes e_i$$

and we denote by $m^{(p)}, \gamma^{(p)}$ their iterations, given by the formulae

$$m^{(p)}(e_{i_1} \otimes \ldots \otimes e_{i_1}) = \delta_{i_1 \ldots i_p} e_{i_1}$$

$$\gamma^{(p)}(e_i) = e_i \otimes \ldots \otimes e_i$$

with p components in the last formula, $e_1, \ldots, e_N$ being the standard basis of $\mathbb{C}^N$.

We will a number of technical results. Let us start with:

Proposition 4.3. *We have the following formulae,*

$$m^{(p)} u^{\otimes p} = u m^{(p)}$$

$$u^{\otimes p} \gamma^{(p)} = \gamma^{(p)} u$$

valid for any submagic matrix u.

Proof. (1) We have the following computation, valid for any indices $i_1, \ldots, i_p$:

$$\begin{aligned}
m^{(p)} u^{\otimes p}(e_{i_1} \otimes \ldots \otimes e_{i_p}) &= m^{(p)} \left(\sum_{j_1 \ldots j_p} e_{j_1} \otimes \ldots \otimes e_{j_p} \otimes u_{j_1 i_i} \ldots u_{j_p i_p} \right) \\
&= \sum_j e_j \otimes u_{j i_1} \ldots u_{j i_p} \\
&= \delta_{i_1 \ldots i_p} \sum_j e_j \otimes u_{j i_1}
\end{aligned}$$

We have as well the following computation, which proves the first formula:

$$\begin{aligned}
um^{(p)}(e_{i_1}\otimes\ldots\otimes e_{i_p}) &= u(\delta_{i_1\ldots i_p}e_{i_1})\\
&= \delta_{i_1\ldots i_p}u(e_{i_1})\\
&= \delta_{i_1\ldots i_p}\sum_j e_j\otimes u_{ji_1}
\end{aligned}$$

(2) We have the following computation, valid for any index i:

$$\begin{aligned}
u^{\otimes p}\gamma^{(p)}(e_i) &= u^{\otimes p}(e_i\otimes\ldots\otimes e_i)\\
&= \sum_{j_1\ldots j_p} e_{j_1}\otimes\ldots\otimes e_{j_p}\otimes u_{j_1i}\ldots u_{j_pi}\\
&= \sum_j e_j\otimes\ldots\otimes e_j\otimes u_{ji}
\end{aligned}$$

We have as well the following computation, which proves the second formula:

$$\begin{aligned}
\gamma^{(p)}u(e_i) &= \gamma\left(\sum_j e_j\otimes u_{ji}\right)\\
&= \sum_j e_j\otimes\ldots\otimes e_j\otimes u_{ji}
\end{aligned}$$

Summarizing, we have proved both formulae in the statement. $\square$

We will need as well a second technical result, as follows:

Proposition 4.4. *We have the following formulae, with u,m,γ being as before,*

$$m^{(p)}R^{\otimes p}d^{\otimes p}\gamma^{(p)} = Rd^{\times p}$$

$$m^{(p)}d^{\otimes p}C^{\otimes p}\gamma^{(p)} = d^{\times p}C$$

and with $\times$ being the componentwise, or Hadamard, product of matrices.

Proof. (1) We have the following computation, valid for any index i:

$$\begin{aligned}
m^{(p)}R^{\otimes p}d^{\otimes p}\gamma^{(p)}(e_i) &= m^{(p)}R^{\otimes p}d^{\otimes p}(e_i\otimes\ldots\otimes e_i)\\
&= m^{(p)}R^{\otimes p}\left(\sum_{j_1\ldots j_p}e_{j_1}\otimes\ldots\otimes e_{j_p}\otimes d_{j_1i}\ldots d_{j_pi}\right)\\
&= m^{(p)}\left(\sum_{j_1\ldots j_p}e_{j_1}\otimes\ldots e_{j_p}\otimes R_{j_1}\ldots R_{j_p}d_{j_1i}\ldots d_{j_pi}\right)\\
&= \sum_j e_j\otimes R_j^pd_{ji}^p\\
&= \sum_j e_j\otimes R_jd_{ji}^p
\end{aligned}$$

We have as well the following computation, which proves the first formula:

$$\begin{aligned}
Rd^{\times p}(e_i) &= R\left(\sum_j e_j\otimes d_{ji}^p\right)\\
&= \sum_j e_j\otimes R_jd_{ji}^p
\end{aligned}$$

(2) We have the following computation, valid for any index i:

$$\begin{aligned}
m^{(p)}d^{\otimes p}C^{\otimes p}\gamma^{(p)}(e_i) &= m^{(p)}d^{\otimes p}C^{\otimes p}(e_i\otimes\ldots\otimes e_i)\\
&= m^{(p)}d^{\otimes p}(e_i\otimes\ldots\otimes e_i\otimes C_i)\\
&= m\left(\sum_{j_1\ldots j_p}e_{j_1}\otimes\ldots\otimes e_{j_p}\otimes d_{j_1i}\ldots d_{j_pi}C_i\right)\\
&= \sum_j e_j\otimes d_{ji}^pC_i
\end{aligned}$$

We have as well the following computation, which proves the second formula:

$$\begin{aligned}
d^{\times p}C(e_i) &= d^{\times p}(e_i\otimes C_i)\\
&= \sum_j e_j\otimes d_{ji}^pC_i
\end{aligned}$$

Thus, we have proved both formulae in the statement. □

We can now prove a key result, as follows:

Proposition 4.5. *We have the following formulae, with u, m, γ being as before,*

$$m^{(p)}(Rdu)^{\otimes p}\gamma^{(p)} = Rd^{\times p}u$$

$$m^{(p)}(udC)^{\otimes p}\gamma^{(p)} = ud^{\times p}C$$

and with $\times$ being the componentwise product of matrices.

Proof. (1) By using the formulae in Proposition 4.3 and Proposition 4.4, we obtain:

$$\begin{aligned} m^{(p)}(Rdu)^{\otimes p}\gamma^{(p)} &= m^{(p)}R^{\otimes p}d^{\otimes p}u^{\otimes p}\gamma^{(p)} \\ &= m^{(p)}R^{\otimes p}d^{\otimes p}\gamma^{(p)}u \\ &= Rd^{\times p}u \end{aligned}$$

(2) Once again by using Proposition 4.3 and Proposition 4.4, we have:

$$\begin{aligned} m^{(p)}(udC)^{\otimes p}\gamma^{(p)} &= m^{(p)}u^{\otimes p}d^{\otimes p}C^{\otimes p}\gamma^{(p)} \\ &= um^{(p)}d^{\otimes p}C^{\otimes p}\gamma^{(p)} \\ &= ud^{\times p}C \end{aligned}$$

Thus, we have proved both formulae in the statement. □

We can now prove the color independence result, as follows:

Theorem 4.6. *The quantum semigroup of quantum partial isomorphisms of a graph is subject to the "independence on the colors" formula*

$$\left[d_{ij} = d_{kl} \iff d'_{ij} = d'_{kl}\right] \implies \widetilde{G}^+(X) = \widetilde{G}^+(X')$$

valid for any graphs X, X', having adjacency matrices d, d'.

Proof. Given a matrix $d \in M_N(\mathbb{C})$, consider its color decomposition, which is as follows, with the color components d_c being by definition 0-1 matrices:

$$d = \sum_{c \in \mathbb{C}} c \cdot d_c$$

We want to prove that a given quantum semigroup G acts on (X, d) if and only if it acts on (X, d_c), for any $c \in \mathbb{C}$. For this purpose, consider the following linear space:

$$E_u = \left\{ f \in M_N(\mathbb{C}) \middle| Rfu = ufC \right\}$$

In terms of this space, we want to prove that we have:

$$d \in E_u \implies d_c \in E_u, \forall c \in \mathbb{C}$$

For this purpose, observe that we have the following implication, as a consequence of the formulae established in Proposition 4.5 above:

$$Rdu = udC \implies Rd^{\times p}u = ud^{\times p}C$$

We conclude that we have the following implication:

$$d \in E_u \implies d^{\times p} \in E_u, \forall p \in \mathbb{N}$$

But this gives the result, exactly as in [2], via the standard linear algebra fact that the color components d_c can be obtained from the componentwise powers $d^{\times p}$. □

In contrast with what happens for the groups or quantum groups, in the semigroup setting we do not have a spectral decomposition result as well. To be more precise, consider as before the following linear space, associated to a submagic matrix u:

$$E_u = \left\{ d \in M_N(\mathbb{C}) \Big| Rdu = udC \right\}$$

It is clear that E_u is a linear space, containing 1, and the following computation, using $*$ and then S, shows that E_u is stable under the adjoint operation $*$ too:

$$\begin{aligned} Rdu = udC &\implies u^t d^* R = C d^* u^t \\ &\implies R d^* u = u d^* C \end{aligned}$$

We also know from Theorem 4.6 above that E_u is stable under color decomposition. However, E_u is not stable under taking products, and so is not an algebra, in general.

This phenomenon will prevent us, in what follows, to work out semigroup analogues of the various results in [1], [2], [6], [7], [8], [14], [22], [23]. These results are indeed of spectral nature, and do not have analogues in the present setting. In short, we are reaching to the conclusion, already formulated in the introduction, that when passing from quantum groups to quantum semigroups some results extend in a straightforward way, some other results extend in a tricky way, and some other results do not extend at all.

5. Basic examples, oriented and unoriented cycles

As explained in the previous section, in what regards the potential basic tools for the study of $\widetilde{G}^+(X)$, namely the color decomposition and the spectral decomposition, one is available, while the other one isn't. Thus, we have to take now our distances with the theory from the quantum group case [1], [2], [6], [7], [8], [14], [22], [23], and rather focus on the aspects which are purely semigroup-theoretical. As we will soon see, this will actually lead to some interesting conclusions, which are worth the passage to semigroups.

One basic finding from [2] states that the oriented and unoriented cycles have no quantum symmetry, with the exception of the square C_4, whose complement is disconnected, and which has quantum symmetry. We will see here that the situation changes in the semigroup setting, by becoming more interesting, with quantum symmetry present.

Let us start with a discussion regarding the usual partial permutation semigroups. In order to discuss the oriented and unoriented cycles, we make the following convention:

Definition 5.1. *In the context of the partial permutations $\sigma : I \to J$, with $I, J \subset \{1, \ldots, N\}$, we decompose the domain set I as a disjoint union*

$$I = I_1 \sqcup \ldots \sqcup I_p$$

with each I_r being an interval consisting of consecutive numbers, and being maximal with this property, and with everything being taken cyclically.

In other words, we represent the domain set $I \subset \{1, \ldots, N\}$ on a circle, with 1 following $1, \ldots, N$, and then we decompose it into intervals, in the obvious way. With this convention made, in the case of the oriented cycle, we have the following result:

Proposition 5.2. *For the oriented cycle $C_N^{\to}$ we have*

$$\widetilde{G}(C_N^{\to}) = \widetilde{\mathbb{Z}}_N$$

with the semigroup on the right consisting of the partial permutations

$$\sigma : I_1 \sqcup \ldots \sqcup I_p \to J$$

which are cyclic on any I_r, given there by $i \to i + k_r$, for a certain $k_r \in \{1, \ldots, N\}$.

Proof. According to the definition of $\widetilde{G}(X)$, we have the following formula:

$$\widetilde{G}(C_N^{\to}) = \left\{\sigma \in \widetilde{S}_N \Big| d_{ij} = d_{\sigma(i)\sigma(j)},\ \forall i, j \in Dom(\sigma)\right\}$$

On the other hand, the adjacency matrix of $C_N^{\to}$ is given by:

$$d_{ij} = \begin{cases} 1 & \text{if } j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus, the condition defining $\widetilde{G}(C_N^{\to})$ is as follows:

$$j = i + 1 \iff \sigma(j) = \sigma(i) + 1,\ \forall i, j \in Dom(\sigma)$$

But this leads to the conclusion in the statement. □

In the case of the unoriented cycle, the result is as follows:

Proposition 5.3. *For the unoriented cycle C_N we have*

$$\widetilde{G}(C_N) = \widetilde{D}_N$$

with the semigroup on the right consisting of the partial permutations

$$\sigma : I_1 \sqcup \ldots \sqcup I_p \to J$$

which are dihedral on any I_r, given there by $i \to \pm_r i + k_r$, for a certain $k_r \in \{1, \ldots, N\}$, and a certain choice of the sign $\pm_r \in \{-1, 1\}$.

Proof. The proof here is similar to the proof of Proposition 5.2. Indeed, the adjacency matrix of C_N is given by:

$$d_{ij} = \begin{cases} 1 & \text{if } j = i \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus, the condition defining $\widetilde{G}(C_N)$ is as follows:

$$j = i \pm 1 \iff \sigma(j) = \sigma(i) \pm 1, \ \forall i, j \in Dom(\sigma)$$

But this leads to the conclusion in the statement. □

An interesting question is whether the semigroups $\widetilde{\mathbb{Z}}_N, \widetilde{D}_N$ are related by a formula similar to $D_N = \mathbb{Z}_N \rtimes \mathbb{Z}_2$. This is not exactly the case, at least with the obvious definition for the $\rtimes$ operation, because at the level of cardinalities we have:

Theorem 5.4. *The cardinalities of $\widetilde{\mathbb{Z}}_N, \widetilde{D}_N$ are given by the formulae*

$$|\widetilde{\mathbb{Z}}_N| = 1 + NK_1(N) + \sum_{p=2}^{[N/2]} N^p K_p(N)$$

$$|\widetilde{D}_N| = 1 + NK_1(N) + \sum_{p=2}^{[N/2]} (2N)^p K_p(N)$$

where $K_p(N)$ counts the sets having p cyclic components, $I = I_1 \sqcup \ldots \sqcup I_p$.

Proof. The first formula is clear from the description of $\widetilde{\mathbb{Z}}_N$ from Proposition 5.2, because for any domain set $I = I_1 \sqcup \ldots \sqcup I_p$, we have N choices for each scalar k_r, producing a cyclic partial permutation $i \to i + k_r$ on the interval I_r. Thus we have, as claimed:

$$|\widetilde{\mathbb{Z}}_N| = \sum_{p=0}^{[N/2]} N^p K_p(N)$$

In the case of $\widetilde{D}_N$ the situation is similar, with Proposition 5.3 telling us that the N choices at the level of each interval I_r must be now replaced by $2N$ choices, as to have a dihedral permutation $i \to \pm_r i + k_r$ there. However, this is true only up to a subtlety, coming from the fact that at $p = 1$ the choice of the ± 1 sign is irrelevant. Thus, we are led to the formula in the statement, with $2N$ factors everywhere, except at $p = 1$. □

In relation with the above, the computation of the numbers $K_p(N)$ is an interesting problem. At $p = 1$ the formula, which is actually not entirely obvious, is:

$$K_1(N) = N^2 - N + 1$$

At higher p we do not know the exact formula. As mentioned above, having this would be interesting, in order to understand the relation between $\widetilde{\mathbb{Z}}_N, \widetilde{D}_N$.

Regarding now the quantum symmetries, both the oriented cycle and the unoriented cycle are known from [2] to not have quantum symmetry in the quantum group sense, unless we are in the special case of the square C_4, which has disconnected complement, and which has quantum group symmetries. However, the proof in [2] is of spectral nature, using the Fourier diagonalization of the corresponding adjacency matrices, and since we do not have spectral decomposition results in our present semigroup setting, as explained at the end of section 4 above, such methods do not apply.

In fact, both the oriented cycle and the unoriented cycle do have quantum symmetries in the present semigroup setting. We have here, as a first illustration:

Proposition 5.5. *For the oriented and unoriented* 2*-cycles, which coincide, we have*

$$\widetilde{G}^+(C_2)=\widetilde{G}^+(C_2^\to)=\widetilde{S}_2^+$$

and this quantum semigroup is infinite.

Proof. The adjacency matrix of the oriented or unoriented 2-cycle is as follows:

$$d=\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}1&1\\1&1\end{pmatrix}-\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Thus the commutation with d is automatic, and we obtain the formula in the statement. As for the last assertion, this is known since [10], and explained in section 1 above. □

As explained in [10], the semigroup $\widetilde{S}_2^+$ is elementary to compute, and basically appears from the matrices given in section 1 above, namely:

$$u=\begin{pmatrix}p\oplus 0&0\oplus q\\0\oplus r&s\oplus 0\end{pmatrix}$$

The next computation appears at $N=3$. In what regards the unoriented cycle C_3, this is the triangle, which by Proposition 3.2 has quantum symmetries as follows:

$$\widetilde{G}^+(C_3)=\widetilde{G}^+(C_3^c)=\widetilde{S}_3^+$$

In what regards now the oriented triangle $C_3^\to$, we have here the following result:

Theorem 5.6. *The equations on the entries of a submagic* 3×3 *matrix*

$$u=\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}$$

producing the semigroup $\widetilde{\mathbb{Z}}_3^+=\widetilde{G}^+(C_3^\to)$ *are as follows:*

$$bd=db=cg=gc=fh=hf$$

$$ce=ec=ha=ah=id=di$$

$$af=fa=bi=ib=ge=eg$$

Proof. The adjacency matrix of the oriented 3-cycle $C_3^\to$ is as follows:

$$d=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$$

We have the following computation:

$$\begin{aligned}
Rdu&=R\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}\\
&=R\begin{pmatrix}d&e&f\\g&h&i\\a&b&c\end{pmatrix}\\
&=\begin{pmatrix}(a+b+c)d&(a+b+c)e&(a+b+c)f\\(d+e+f)g&(d+e+f)h&(d+e+f)i\\(g+h+i)a&(g+h+i)b&(g+h+i)c\end{pmatrix}\\
&=\begin{pmatrix}bd+cd&ae+ce&af+bf\\eg+fg&dh+fh&di+ei\\ha+ia&gb+ib&gc+hc\end{pmatrix}
\end{aligned}$$

We have as well the following computation:

$$\begin{aligned}
udC&=\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}C\\
&=\begin{pmatrix}c&a&b\\f&d&e\\i&g&h\end{pmatrix}C\\
&=\begin{pmatrix}c(a+d+g)&a(b+e+h)&b(c+f+i)\\f(a+d+g)&d(b+e+h)&e(c+f+i)\\i(a+d+g)&g(b+e+h)&h(c+f+i)\end{pmatrix}\\
&=\begin{pmatrix}cd+cg&ae+ah&bf+bi\\fa+fg&db+dh&ec+ei\\ia+id&gb+ge&hc+hf\end{pmatrix}
\end{aligned}$$

By cancelling the common terms, the equation $Rdu=udC$ reads:

$$\begin{pmatrix}bd&ce&af\\eg&fh&di\\ha&ib&gc\end{pmatrix}=\begin{pmatrix}cg&ah&bi\\fa&db&ec\\id&ge&hf\end{pmatrix}$$

But these relations, and their adjoints, give the formulae in the statement. □

The above result shows that $C_3^{\to}$, in analogy with C_3, has massive quantum partial symmetries. The same happens for C_n and $C_n^{\to}$ at higher n.

6. Oriented graphs with small number of vertices

We have seen in the previous sections that the finite graphs X tend to systematically have quantum partial symmetry, and with this being in contrast with what happens with the quantum group symmetries, whose lack is something that can happen.

We have no explanation for this phenomenon, but here is one more illustration for it:

Theorem 6.1. *All the oriented graphs on 2 vertices have quantum partial symmetry.*

Proof. Up to compementation and obvious symmetries of the problem, we have 4 adjacency matrices to be investigated, as follows:

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}\quad,\quad\begin{pmatrix}0&1\\0&0\end{pmatrix}\quad,\quad\begin{pmatrix}1&1\\0&0\end{pmatrix}\quad,\quad\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

(1) For the first matrix, the invariance equations are as follows:

$$\begin{pmatrix}p+q&0\\0&r+s\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}p&q\\r&s\end{pmatrix}=\begin{pmatrix}p&q\\r&s\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}p+r&0\\0&q+s\end{pmatrix}$$

By multiplying, these equations become:

$$\begin{pmatrix}p+q&0\\0&r+s\end{pmatrix}\begin{pmatrix}p&q\\0&0\end{pmatrix}=\begin{pmatrix}p&0\\r&0\end{pmatrix}\begin{pmatrix}p+r&0\\0&q+s\end{pmatrix}$$

Thus, the equations are as follows:

$$\begin{pmatrix}(p+q)p&(p+q)q\\0&0\end{pmatrix}=\begin{pmatrix}p(p+r)&0\\r(p+r)&0\end{pmatrix}$$

By using now the submagic condition, these equations become:

$$\begin{pmatrix}p&q\\0&0\end{pmatrix}=\begin{pmatrix}p&0\\r&0\end{pmatrix}$$

Thus the equations are $q=r=0$, and the submagic matrix must be as follows:

$$u=\begin{pmatrix}p&0\\0&s\end{pmatrix}$$

It follows that we have indeed quantum partial symmetry, because p,s can be here any two projections.

(2) For the second matrix, the invariance equations are as follows:

$$\begin{pmatrix}p+q&0\\0&r+s\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}p&q\\r&s\end{pmatrix}=\begin{pmatrix}p&q\\r&s\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}p+r&0\\0&q+s\end{pmatrix}$$

By multiplying, these equations become:

$$\begin{pmatrix}p+q&0\\0&r+s\end{pmatrix}\begin{pmatrix}r&s\\0&0\end{pmatrix}=\begin{pmatrix}0&p\\0&r\end{pmatrix}\begin{pmatrix}p+r&0\\0&q+s\end{pmatrix}$$

Thus, the equations are as follows:

$$\begin{pmatrix}(p+q)r&(p+q)s\\0&0\end{pmatrix}=\begin{pmatrix}0&p(q+s)\\0&r(q+s)\end{pmatrix}$$

By using now the submagic condition, these equations become:

$$\begin{pmatrix}qr&ps\\0&0\end{pmatrix}=\begin{pmatrix}0&ps\\0&rq\end{pmatrix}$$

Thus the equations are $qr=rq=0$, telling us that q,r must be in the center of the algebra. In particular, we have as solutions submagic matrices as follows:

$$u=\begin{pmatrix}p&0\\0&s\end{pmatrix}$$

It follows that we have indeed quantum partial symmetry, because p,s can be here any two projections.

(3) For the third matrix, the invariance equations are as follows:

$$\begin{pmatrix}p+q&0\\0&r+s\end{pmatrix}\begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}p&q\\r&s\end{pmatrix}=\begin{pmatrix}p&q\\r&s\end{pmatrix}\begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}p+r&0\\0&q+s\end{pmatrix}$$

By multiplying, these equations become:

$$\begin{pmatrix}p+q&0\\0&r+s\end{pmatrix}\begin{pmatrix}p+r&q+s\\0&0\end{pmatrix}=\begin{pmatrix}p&p\\r&r\end{pmatrix}\begin{pmatrix}p+r&0\\0&q+s\end{pmatrix}$$

Thus, the equations are as follows:

$$\begin{pmatrix}(p+q)(p+r)&(p+q)(q+s)\\0&0\end{pmatrix}=\begin{pmatrix}p(p+r)&p(q+s)\\r(p+r)&r(q+s)\end{pmatrix}$$

By using now the submagic condition, these equations become:

$$\begin{pmatrix}p+qr&q+ps\\0&0\end{pmatrix}=\begin{pmatrix}p&ps\\r&rq\end{pmatrix}$$

Thus the equations are $r=q=0$, and the submagic matrix must be as follows:

$$u=\begin{pmatrix}p&0\\0&s\end{pmatrix}$$

It follows that we have indeed quantum partial symmetry, because p,s can be here any two projections.

(4) Finally, for the fourth matrix we obtain $\widetilde{S}_3^+$, which is known to be infinite, as explained before, and we are done. $\square$

The exact computation of all the graph quantum semigroups appearing at $N=2$ is an interesting question, that we will not get into here. In relation now with our main finding, namely that all graphs seem to have quantum symmetry in our setting, here is as well a result at $N=3$, dealing with the unoriented case only:

Proposition 6.2. *All the unoriented graphs on* 3 *vertices have quantum partial symmetry.*

Proof. This is something that we already know, because at $N=3$ we only have the triabgle and its complement, having semigroup $\widetilde{S}_3^+$, and then the 1-edge graph and its complemenent, which by Proposition 3.4 have semigroup bigger than $\widetilde{S}_2^+$. □

Regarding the 1-edge graph, the precise result here is as follows:

Proposition 6.3. *The equations on the entries of a submagic* 3×3 *matrix*

$$u=\begin{pmatrix}a&b&c\\ d&e&f\\ g&h&i\end{pmatrix}$$

producing the semigroup $\widetilde{G}^+(\cdot\ -)$ *are as follows:*

$$dh=fb=gf=hc=-ib=-ge=-ec=-di$$

Proof. This is similar to the proof of Theorem 5.6. The adjacency matrix is:

$$d=\begin{pmatrix}0&0&0\\ 0&0&1\\ 0&1&0\end{pmatrix}$$

A direct computation gives the following formula:

$$Rdu=\begin{pmatrix}0&0&0\\ eg+fg&dh+fh&di+ei\\ hd+id&ge+ie&gf+hf\end{pmatrix}$$

We have as well the following computation:

$$udC=\begin{pmatrix}0&ce+ch&bf+bi\\ 0&fb+fh&ec+ei\\ 0&ib+ie&hc+hf\end{pmatrix}$$

Thus, we are led to the formulae in the statement. □

In any case, our conjecture is that all graphs should have quantum partial symmetry.

7. Infinite graphs and functional analysis questions

Generally speaking, most of the present constructions extend to $N=\infty$. We can talk about the compact quantum semigroup of partial permutations of $\mathbb N$, as follows:

$$C(\widetilde{S}_\infty^+)=C^*\left((u_{ij})_{i,j\in\mathbb N}\Big|u=\text{submagic}\right)$$

This is already quite remarkable, because in what regards S_∞^+ itself, this is something which cannot really be defined, due to technical functional analysis reasons. See [16].

Next in line, we can talk about the compact quantum semigroup of quantum partial automorphisms of an infinite graph X, as follows:

$$C(\widetilde{G}^+(X))=C(\widetilde{S}_\infty^+)\Big/\Big\langle R_i(du)_{ij}=(ud)_{ij}C_j\Big\rangle$$

Once again, this is quite remarkable, because as already mentioned above, $G^+(X)$ cannot be really defined for the empty graph $X=\mathbb N$, as explained in [16].

Our belief is that these constructions can help in connection with a number of questions, as for instance with the unification of the compact quantum group actions $G\curvearrowright X$ and the discrete quantum group actions $\Gamma\curvearrowright X$, in terms of actions of certain locally compact quantum subgroups $L\subset\widetilde{S}_\infty^+$. It is our belief that the systematic study at $N<\infty$ done in this paper is a useful thing, that can be of help, in connection with these questions.

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