Quantum error correction inapplicable to really superpositioned states

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Abstract: QEC (Quantum Error Correction) assumes that qubit states are disturbed by errors primarily through interactions with their local environment states. The problem is that, when quantum states are superpositioned, QEC improperly assumes all the superpositioned terms share same local environment states and resulting errors though they are, in general, different term by term to which QEC with the fixed number of syndrome qubits is, except for some slightly superpositioned states such as two terms ones, not applicable. Further, fundamental difficulty to scale quantum parallelism regardless of implementation and physical details is derived from theory of Shannon.

1. Introduction

According to Shor [1], a qubit is disturbed by noise to cause errors as a result of interaction between states of the qubit and its local environment $|e_0\rangle$ as (Eq. (3.2) of [1]):

$$\begin{aligned} |e_0\rangle|0\rangle &\rightarrow |a_0\rangle|0\rangle + |a_1\rangle|1\rangle \\ |e_0\rangle|1\rangle &\rightarrow |a_2\rangle|0\rangle + |a_3\rangle|1\rangle \end{aligned} \quad (1)$$

which means an error operator on the qubit depends on its local environment state $|e_0\rangle$, which is basic error model assumed by all the QEC (Quantum Error Correction). Though some extensions from the error model considering, for example, some interaction between qubits, may be possible, such extensions are not essential. As such, hereafter, only the error model of [1] is considered. Then, a possible problem is that local environment states are rarely considered explicitly.

As long as QEC is applied to unentangled, that is, essentially classical, states, that is not an actual problem. However, to superpositioned states where local environment states are involved in entanglement, that is, local environment states and resulting error operators are different term by term, QEC is inapplicable because the fixed number of syndrome qubits is not enough to correct many (often exponentially many) variations of errors, which invalidates an improper and implicit (resulting from not explicitly considering local environment states) assumption that local environment states are same for all the terms.

A complication is that, as resulting states of Eq. 1: $|a_0\rangle|0\rangle + |a_1\rangle|1\rangle$ for the initial qubit value of $|0\rangle$ and $|a_2\rangle|0\rangle + |a_3\rangle|1\rangle$ for $|1\rangle$, can be fully independent, initial local environment states for them may also be different. That is, a local environment states around the qubit may be totally different to be $|e_0\rangle$ and $|e_1\rangle$ depending on the corresponding qubit value of $|0\rangle$ and $|1\rangle$ and actual error model of [1] is not Eq. (1) but:

$$\begin{aligned} |e_0\rangle|0\rangle &\rightarrow |a_0\rangle|0\rangle + |a_1\rangle|1\rangle \\ |e_1\rangle|1\rangle &\rightarrow |a_2\rangle|0\rangle + |a_3\rangle|1\rangle \end{aligned} \tag{2}$$

which makes QEC applicable to superpositioned states with two terms or tensor products of such states, for example, Shor codes of $|0\rangle$ and $|1\rangle$ but not $|0\rangle + |1\rangle$.

The following notations are used in this letter. $|Q\rangle_E$ represents quantum state just after encoding of some QEC, where state of qubits $|Q\rangle$ will, just before decoding, be disturbed to be $E|Q\rangle$ by an error operator E (resulting from some local environment states just after encoding). Let I, B, S and C correspondingly represent identity, bitflip, signflip, bitflip followed by signflip error operators on a qubit. Products between error operators are tensor product. No attempt is made to normalize vectors representing quantum states. Over bar is logical not ($\bar{0}$=1, $\bar{1}$=0).

Examples of states to which QEC of bitflip or Shor code is applicable/inapplicable are shown in section 2. Section 3 explains fundamental difficulty of quantum parallelism regardless of implementation and physical details. Section 4 concludes the letter.

2. Examples

With QEC of bitflip code, $|0\rangle$ may be encoded to be $|000\rangle_{BII}$, which will, just before decoding suffering from a single bitflip error on the first qubit, be $|100\rangle$, which will be decoded to be $|111\rangle$, which will, after syndrome observation, be $|111\rangle$, which will, by QEC based on the observed syndrome, correctly restored to be $|0\rangle$.

However, for a superpositioned case, $|0\rangle + |1\rangle$ may be encoded to be $|000\rangle_{BII}$+$|111\rangle_{III}$, which will, just before decoding, be $|100\rangle + |111\rangle$, which will be decoded to be $|111\rangle + |100\rangle$, syndrome observation of which will be $|111\rangle$ or $|100\rangle$ (equal probability), which will be, by QEC based on the observed syndrome, incorrectly corrected to be $|0\rangle$ or $|1\rangle$, not correctly to be $|0\rangle + |1\rangle$. Because of Eq. (2), the error on the two terms state $|000\rangle + |111\rangle$ to be $|100\rangle + |111\rangle$ may still be interpreted as a not-term-specific single qubit error, but, then, the error by the initial state involves signflip errors as $|000\rangle_{\frac{I+B-S-C}{2}II} + |111\rangle_{\frac{I+B-S-C}{2}II}$ not correctable by bitflip code.

For another example with less drastic error operator difference, if $|0\rangle + |1\rangle$ is encoded to be $|000\rangle_{\left(\frac{3I+B}{\sqrt{10}}\right)II} + |111\rangle_{III}$, final result will be $3/\sqrt{10}|0\rangle + |1\rangle$ or $|0\rangle$ (with probability ratio of 19:1), which is better (more similar to the correct result with higher probability) than the result of the previous example but still significantly (w.r.t. the amount of error operator difference) incorrect.

Similarly, with Shor code, if $|0\rangle$ is, with term specific single qubit errors, encoded to be:

$$(|000\rangle_{BII} + |111\rangle_{III})(|000\rangle_{III} + |111\rangle_{III})(|000\rangle_{III} + |111\rangle_{III}) \tag{3}$$

the state just before decoding will be:

$$(|100\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \tag{4}$$

but the same state will be obtained from a encoding result with a single qubit error shared by all the terms of:

$$(|000\rangle_{\frac{I+B-S-C}{2}II} + |111\rangle_{\frac{I+B-S-C}{2}II}) \tag{5}$$
$$(|000\rangle_{III} + |111\rangle_{III})(|000\rangle_{III} + |111\rangle_{III})$$

As such, because of Eq. (2), the state of Eq. (4) is correctable by Shor code.

However, with Shor code, if $|0\rangle + |1\rangle$ is encoded to be:

$$(|000\rangle_{BII} + |111\rangle_{BII})(|000\rangle_{III} + |111\rangle_{III})(|000\rangle_{III} + |111\rangle_{III}) + \tag{6}$$
$$(|000\rangle_{III} - |111\rangle_{III})(|000\rangle_{III} - |111\rangle_{III})(|000\rangle_{III} - |111\rangle_{III})$$

state just before decoding will be:

$$(|100\rangle + |011\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) + \\ (|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle) \tag{7}$$

decoded for bit flip correction to be:

$$(|1\rangle + |0\rangle)|11\rangle(|0\rangle + |1\rangle)|00\rangle(|0\rangle + |1\rangle)|00\rangle + \\ (|0\rangle - |1\rangle)|00\rangle(|0\rangle - |1\rangle)|00\rangle(|0\rangle - |1\rangle)|00\rangle \tag{8}$$

observed for bit flip correction to be:

$$(|1\rangle + |0\rangle)|11\rangle(|0\rangle + |1\rangle)|00\rangle(|0\rangle + |1\rangle)|00\rangle \tag{9}$$

or

$$(|0\rangle - |1\rangle)|00\rangle(|0\rangle - |1\rangle)|00\rangle(|0\rangle - |1\rangle)|00\rangle \tag{10}$$

after bit flip correction to be:

$$(|0\rangle + |1\rangle)|11\rangle(|0\rangle + |1\rangle)|00\rangle(|0\rangle + |1\rangle)|00\rangle \tag{11}$$

or

$$(|0\rangle - |1\rangle)|00\rangle(|0\rangle - |1\rangle)|00\rangle(|0\rangle - |1\rangle)|00\rangle \tag{12}$$

which will, after sign flip correction stages, be finally decoded to be $|0\rangle$ or $|1\rangle$, not correctly to be $|0\rangle + |1\rangle$.

3. Information theoretic considerations on fundamental difficulty of quantum parallelism

Typical quantum computing for *N*-ary function *f* with Boolean arguments and results is to construct a linear quantum circuit for qubits to accept $|q_1 q_2 \dots q_N 0\rangle$ and output $|q_1 q_2 \dots q_N f(q_1, q_2, \dots, q_N)\rangle$. Then, ignoring noise, if $\sum_{q_1,q_2,\dots,q_N \in \{0,1\}} c_{q_1 q_2 \dots q_N} |q_1 q_2 \dots q_N 0\rangle$ is input to the circuit, $\sum_{q_1,q_2,\dots,q_N \in \{0,1\}} c_{q_1 q_2 \dots q_N} |q_1 q_2 \dots q_N f(q_1, q_2, \dots, q_N)\rangle$ is output with a single computation without computing each $f(q_1, q_2, \dots, q_N)$ individually, which is the quantum parallelism (hereafter, for simpler explanations, $c_{q_1 q_2 \cdots q_N}$ are omitted.).

However, it is information theoretically obvious from theory of Shannon [2] that distinguishing (with reasonably small error probability) doubly exponentially many (w.r.t. *N*) fully superpositioned states of $\sum_{q_1,q_2,\dots,q_N \in \{0,1\}} |q_1 q_2 \dots q_N f(q_1, q_2, \dots, q_N)\rangle$ is a lot more difficult than distinguishing exponentially many unentangled states of $|q_1 q_2 \dots q_N f(q_1, q_2, \dots, q_N)\rangle$.

That is, correctly computing fully superpositioned state for *f* means, for functions almost identical to *f*, for example, $f'(q_1, q_2, \dots, q_N) = f(q_1, q_2, \dots, q_N)$ except that $f'(0,0,\dots,0) = \overline{f(0,0,\dots,0)}$, fully superpositioned state for *f*:

$$\sum_{q_1,q_2,\dots,q_N \in \{0,1\}} |q_1 q_2 \dots q_N f(q_1, q_2, \dots, q_N)\rangle = \\ |00 \dots 0 f(0,0,\dots,0)\rangle + \sum_{\substack{q_1,q_2,\dots,q_N \in \{0,1\} \\ (q_1,q_2,\dots,q_N) \neq (0,0,\dots,0)}} |q_1 q_2 \dots q_N f(q_1, q_2, \dots, q_N)\rangle \tag{13}$$

and f':

$$\sum_{q_1,q_2,\dots,q_N \in \{0,1\}} |q_1 q_2 \dots q_N f'(q_1, q_2, \dots, q_N)\rangle = \\ |00 \dots 0 \overline{f(0,0,\dots,0)}\rangle + \sum_{\substack{q_1,q_2,\dots,q_N \in \{0,1\} \\ (q_1,q_2,\dots,q_N) \neq (0,0,\dots,0)}} |q_1 q_2 \dots q_N f(q_1, q_2, \dots, q_N)\rangle \tag{14}$$

must be distinguishable, which is a lot harder than distinguishing:

$$|00 \dots 0 f(00 \dots 0)\rangle \tag{15}$$

and

$$|00 \dots 0 f'(00 \dots 0)\rangle = |00 \dots 0 \overline{f(00 \dots 0)}\rangle \tag{16}$$

So far, no implementation and physical details are considered, which means quantum parallelism is fundamentally difficult also physically regardless of implementation and physical details. That is, it is obvious that, assuming some random noise, quantum circuits merely designed to be able to distinguish states of Eqs. (15) and (16) cannot distinguish states of Eqs. (13) and

(14).

However, with improper physical error model that all the terms are affected by a same error operator, elementary bitflip error from fully superpositioned state of Eq. (13) was considered to be:

$$\sum_{q_1,q_2,\ldots,q_N\in\{0,1\}} |q_1q_2 \ldots q_N\overline{f(q_1,q_2,\ldots,q_N)}\rangle \tag{17}$$

which is as significantly different from the state of Eq. (13) as the unentangled state of Eq. (15) from the state of Eq. (16), which is why QEC was considered applicable to superpositioned states to enable scalable quantum parallelism. The assumed noise is highly regular and scarcely random, because only a single global noise is considered and exponentially many relative noises, such as relative phase noises, between exponentially many terms are assumed to be absolute zero.

Poor understanding on threshold theorems made the situation worse.

The classical threshold theorem [3] is that, if universal gates with error rate well below certain threshold can be constructed, error correcting universal gates with much smaller error rate can be constructed, which is directly applicable to quantum error correcting circuits for essentially classical unentangled states. Then, if construction of error correcting universal gates is performed recursively, doubly exponential (w.r.t. recursion level) error reduction is possible which should be more than enough to distinguish the states of Eqs. (13) and (14). As such, quantum threshold theorem [4] was expected to be applicable to fully entangled states to make quantum superposition scale. However, for the recursive construction, the construction must be possible at least once, which is, as discussed above, simply impossible for fully entangled states (note that, even though universal gates have the fixed numbers of input/output qubits, entanglement between which is limited, the qubits are, for quantum parallelism, aggressively (with exponentially many terms for exponential quantum parallelism) entangled with other qubits in a quantum computer), invalidating [4].

4. Conclusion

It is shown, information theoretically regardless of implementation and physical details, that scalable quantum parallelism is fundamentally difficult.

It is also shown that the error model of QEC by Shor [1] is improper and is not applicable to really superpositioned states necessary for practical scale quantum parallelism, where not only qubit states but also local environment states around qubits are involved in entanglement.

QEC is applicable only to unentangled or slightly entangled states and is not really so quantum but rather classical. It should be noted that, though signflip error do not occur in modern computers, it is not because the error is classically impossible but because, in modern computers, phase is not used to carry information. With modern communication, it is well known that relative phase error between polarization modes, which is a signflip error, causes PMD (Polarization Mode Dispersion), which can, unless properly equalized, cause bit errors

References

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