On Representations of the Lorentz Group

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Abstract

The Lorentz group is a non-compact group. Consequently, it's representations cannot be expected to be equivalent to representations of a unitary group. Actually, they act on a large-component space and a separated small-component space, in some sense analogous to 4-vectors. In contrast to representations of compact groups state vectors carry the actual value of the non-compact variables, the boost-vector. In the non-boosted state the small components vanish and the large components transform according to a representation of the rotation subgroup. Application of a boost then generates small components, a process that preserves norms. However, the norm now has a growing positive contribution from the large-components and a negative contribution from the small-components, growing absolutely to keep the total unchanged. General transformations are described in detail. The freedom to assign boost directions to the phases of small components leads to a topological symmetry with flavor-generating representations for two-sheeted representations.

Symmetry *SO(4)*

There are six infinitesimal rotations for the group SO(4):

$$A_j = x_l \partial_k - x_k \partial_l \quad , \quad B_j = x_j \partial_0 - x_0 \partial_j \quad , (j, k, l) = (1, 2, 3) \text{ and cyclic permuted.} \tag{1}$$

They span the associated Lie algebra, which separates into a sum of two independent sub-algebras, generated by

$$J_j = \frac{1}{2}(A_j + B_j) \quad , \quad K_j = \frac{1}{2}(A_j - B_j) \quad , \tag{2}$$

which both obey the commutation relations of the Lie-algebra *o(3)* generating the group of orthogonal transformations in three dimensions.

$$[J_j, J_k] = J_l \quad , (j, k, l) \text{ as above.} \tag{3}$$

Correspondingly, *SO(4)* is a direct product of two groups, each homologous to *SO(3)* [1,2].

Therefore, its representations can be labeled by two integers or half-integers (*j, k*), which characterize the two factors of a particular representation. Due to the closed nature of *SO(4)* these representations are closed as well. Thus, one can apply the standard exponentiation procedure of Lie-groups to obtain an arbitrary transformation of a representation.

Every representation has a chirality parameter (*k-j*) chosen such that pure *j*-representations (*j*>0) have a negative value (left-handed chirality). Achiral representations have $j = k$. ???

Remark on the Dimension of Representations of SO(3)

Representations of *SO(3)* must be considered real. This can be seen by choosing a 'magnetic' basis in which a component, say J_3 is diagonal. A representation labeled by a half integer j is $(2j+1)$-dimensional and has the eigenvalues (of J_3) $m_j = -j, 1-j, \ldots, j-1, j$ with eigenfunctions $e^{i m_j \phi}$, where φ is the rotation angle about the 3-axis. Equivalently, we may choose the completely real functions $(e^{i m_j \phi}+e^{-i m_j \phi})/2$ and $(e^{i m_j \phi}-e^{-i m_j \phi})/(2i)$. The appearance of complex conjugated functions is owed to the fact that rotations about an axis are an Abelian subgroup with exclusively one dimensional representations. The eigenvalues of a transformation, which is real, are either real (1 or -1) or occur in complex conjugated pairs. Thus, two complex conjugated representations can be combined to a completely real two-dimensional representation now reducible, however.

If the vector spaces are considered as a field over the Euclidean *R4* we can extend them to a Hilbert space by introducing a local norm as the scalar product with the conjugated vector. Clearly, this Hilbert space is over the field of reals, and this holds true for the product representations of *SO(4)* as well. The group *SO(4)* preserves the Euclidean metric, and thus, is the isotropic symmetry group of *R4*.

A consequence of the field of reals is that a function $|\psi\rangle$, when multiplied by the imaginary unit i leads to a negative norm $i^2\langle\psi|\psi\rangle$, as we will encounter in the following. This situation is somewhat confusing when working with the complex conjugated representations mentioned above. In other words, the freedom of choosing among equivalent quantization axis must not be confused with a complex Hilbert space.

Proper Orthochronous Lorentz Group, *SO⁺(3,1)*

For the Lorentz group, the 'isotropic' symmetry group of the Minkowski space, there are several substantial modifications, due to the non-compact nature of the group $SO^+(3,1)$, introduced by the infinitesimal boost operators B. Obviously, we cannot expect to encounter unitary representations (which are compact). To see the problems, let us consider a special representation $(j,0)$ in which the second factor is the identity representation. We use the basis of magnetic quantum numbers in b-direction (boost). Starting from a non-boosted state, a boost transformation along the b-axis corresponds in *SO(4)* to a rotation in the two-dimensional space spanned by ψ and ${}^jD_{\hat{b}}\psi$, with

$$ {}^jD_{\hat{b}}=e^{i\pi\sigma_{\hat{b}}} \quad , \tag{4} $$

where the spin operator $\sigma_{\hat{b}}$ is the representation of the infinitesimal rotation J in expression (2) in b-direction. This π-rotation becomes now a discrete proper transformation (see below), which corresponds to a time reversal and an inversion of the b-axis. It mediates between 'separated' parts of the representation space. The term separated needs some explication. It corresponds, in some sense, to time- and space-nature of a *4*-vector in space-time. There exists no Lorentz transformation that converts a time-like vector in a space-like one. In this sense time- and space-components must be considered 'separated'. In our case we may introduce the term large p-components and (separated) small n-components in accordance with the wording used in Dirac's theory of the electron. The pre-scrips p- and n- stand for positive and negative (see below).

Clearly, for $(0,k)$-representations we must choose

$$ {}^kD_{\hat{b}}=e^{-i\pi\sigma_{\hat{b}}} \quad , \tag{5} $$

in accord with time direction in K (expression 2).

Writing the rotation in the $(0,b)$-plane from $SO(4)$ as

$$e^{i\phi\sigma_b{}^j}\psi_{m_j}=\cos\left(m_j\frac{\phi}{2}\right){}^j\psi_{m_j}+\sin\left(m_j\frac{\phi}{2}\right){}^jD_{\hat{b}}{}^j\psi_{m_j}\quad, \tag{6}$$

we can make the transition from $SO(4)$ to $SO^+(3,1)$ by recalling that we can first perform the rotation part ${}_rD$, followed by the pure-boost part ${}_bD$ due to commutation of the two operators. For the latter we start from (6) to arrive at

$$^jD_b(\phi){}^j\psi_{m_j}=\cosh\left(m_j\frac{\phi}{2}\right){}^j\psi_{m_j}-i\sinh\left(m_j\frac{\phi}{2}\right){}^jD_{\hat{b}}{}^j\psi_{m_j}\quad, \tag{7}$$

by the replacement $\varphi \rightarrow i\varphi$. We see that the 'separated' part of the representation acquires a phase that depends on the boost direction b due to the 3-vector nature of the exponent in expression (4), and it vanishes for non-boosted states. As an aside, we must have a simultaneous back-transformation:

$$^jD_b(\phi){}^jD_{\hat{b}}{}^j\psi_{m_j}=\cosh\left(m_j\frac{\phi}{2}\right){}^jD_{\hat{b}}{}^j\psi_{m_j}+i\sinh\left(m_j\frac{\phi}{2}\right)({}^jD_{\hat{b}})^2\,{}^j\psi_{m_j}\quad,\quad ({}^jD_{\hat{b}})^2=\pm 1\quad, \tag{8}$$

where the + sign applies for integer j, the – sign for half-integer j (two-valued representations).

To see that we can extend the concept of a norm, invariant under transformations, we consider (7) further and write down the norm of the original vector and the transformed one:

$$\langle{}^j\psi_{m_j}{}^j\hat{D}_b(\phi)|{}^jD_b(\phi){}^j\psi_{m_j}\rangle\ =$$

$$\cosh\left(m_j\frac{\phi}{2}\right)\cosh\left(m_j\frac{\phi}{2}\right)\langle{}^j\psi_{m_j}|{}^j\psi_{m_j}\rangle-\sinh\left(m_j\frac{\phi}{2}\right)\sinh\left(m_j\frac{\phi}{2}\right)\langle{}^j\psi_{m_j}{}^j\hat{D}_{\hat{b}}|{}^jD_{\hat{b}}{}^j\psi_{m_j}\rangle\quad. \tag{9}$$

The operator ${}^jD_{\hat{b}}$ is unitary and by multiplication with its conjugate transpose yields the unit operator, hence

$$\langle{}^j\psi_{m_j}{}^j\hat{D}_b(\phi)|{}^jD_b(\phi){}^j\psi_{m_j}\rangle=\left[\cosh^2\left(m_j\frac{\phi}{2}\right)-\sinh^2\left(m_j\frac{\phi}{2}\right)\right]\langle{}^j\psi_{m_j}|{}^j\psi_{m_j}\rangle\quad. \tag{10}$$

However, the operator ${}^jD_b(\phi)$ is the exponential of a Hermitean and thus non-unitary. We recognize that, for each m_j-component of ψ, we have an expression '$\cosh^2(x)$ - $\sinh^2(x)$' which equals one and is unchanged under a pure boost transformation. Obviously, the vectors of the non-boosted representation extend under application of a pure boost to a pair of $(2j+1)$-vectors. These two vectors cannot be combined to a single $(4j+2)$-vector, because the relation between them is fixed by the value of the boost.

It must be questioned whether the transformation behavior provided by expressions (7) and (8) can be termed representation in the regular sense. In this context the concept of little groups must be seriously questioned, because it supposes that translation of the in-homogeneous group yield one-dimensional irreducible representations which they do not in our case. p- and n-components are not independent! Their independence would destroy the preserved norm (10) under boost transformations.

A capital difference in comparing $SO^+(3,1)$ with $SO(4)$, is that the variable φ is now not periodic any more, but extends from zero to plus infinity. Negative values are taken care of by 'inverting' the boost direction (actually by a π-rotation, to stay with proper transformations). We emphasize this by replacing φ by the boost variable

$$|b|=\frac{\phi}{2}=\frac{1}{2}\,\mathrm{artanh}(v)\quad, \tag{11}$$

where $0 \leq v < 1$ is the speed of the boost. This allows us to give the vector b an extended meaning such that its norm characterizes the value (11) to which the state is boosted. Consequently, we can omit the variable φ in the operator ${}^{j}D_b$ (7).

We have the additional effect, that the m_j-component of fields acquire an additional phase of $\exp(im_j b)$, because, for a $(j,0)$ representation, a boost is always coupled with a synchronous coaxial rotation, see expression (2). When the speed of the boost approaches a value of one, this phase oscillates at ever increasing rate. For m_j=0 the n-component stays at zero and there is no oscillation. This signifies that states which travel at the speed of light (v=1, photons, gravitons) always have zero magnetic quantum numbers (m=0) with respect to the direction of propagation. In this case the little-group concept can be applied, because n-components vanish, we only have p-components. In these cases the wave-number 4-vectors are zero-vectors. Furthermore, to characterize these states we must take the limiting direction vector $\hat{b}$.

We can repeat this whole consideration for general (j,k) representations by the observation that one can, in boost direction, choose a magnetic quantum number basis in each factor of the representation which yield, under boost, components with total magnetic quantum numbers of $(m_j - m_k)$. The components of the corresponding p- and n-vectors now are $(2j+1)(2k+1)$-tuples, of which the n-components corresponding to $\Delta m = (m_j - m_k) = 0$ now stay at a value of zero. For integer values of $j+k$ we have $z_{jk} = 2\,min(j,k)+1$ of those. For half-integer $j+k$ there are no $\Delta m = 0$ states ($z_{jk} = 0$). Correspondingly, the representations of $(v \rightarrow 1)$-limiting states are characterized by a direction vector, a integer value of $j+k$, and z_{jk} longitudinal components. Actually, j and k must be smaller then one [3], which leaves us with gravitons and phonons, exclusively.

Transformation Behavior

A general state ψ is characterized by a boost vector b and the non-boosted state ψ_0 from which it is generated by the pure b-boost D_b. The subscript 0 indicates that this state is purely in the p-component space, with vanishing n-components

$$\psi = D_b \psi_0 \quad . \tag{12}$$

To keep track of the transformation behavior under general Lorentz transformations we take care that the transformation is such that we have the total pure boost as final transformation. Therefore, to rotate a state with final boost D_b by a rotation R we must write $D_R = D_R D_b D_{-b}$ such that we can operate on ψ_0 with $D_R D_b$:

$$D_R \psi = D_R D_b D_{-b} \psi = D_R D_b \psi_0 \tag{13}$$

We can apply the rotation R on the boost vector b to obtain the new boost vector p. There exist a whole set of rotations to achieve that, but only one of them transforms the pure boost B_b to the pure boost B_p, which we call

$$R_{p,b} b = p \tag{14}$$

This rotation has the rotation axis parallel to the vector product of b and p.

To proceed we make use of the decomposition possibilities of a general Lorentz transformation:

$$L = R B_b = R_{p,b} R_b B_b = B_p R_p R_{p,b} \quad , \tag{15}$$

where R_b and R_p are rotations about the axes b and p, respectively, by the same angle. Thus, one obtains R_b from R and $R_{p,b}$, L from the second equation of (15) and R_p from the third one. This completes the transformation (13):

$$D_R \psi = D_R D_b \psi_0 = D_p D_{R_p} D_{R_{p,b}} \psi_0 \quad . \tag{16}$$

Clearly, the rotated field $D_{R_p} D_{R_{p,b}} \psi_0$ is again purely time-like of which we know how to apply the pure boost D_p.

A similar consideration can be made for applying a pure boost

$$D_p \psi = D_p D_b D_{-b} \psi = D_p D_b \psi_0 \quad . \tag{17}$$

In this case one makes use of the formula for multiplication of two pure boost:

$$L = B_p B_b = R_{p\oplus b, b\oplus p} B_{b\oplus p} = B_{p\oplus b} R_{p\oplus b, b\oplus p} \quad . \tag{18}$$

Here we have to remember, that combining two boosts corresponds to adding the corresponding velocities according to the relativistic addition formula, symbolized by the circled plus sign, which is not commutative. Formula (18) also illustrates that the product of two pure boosts is not a pure boost (Thomas-Wigner Rotation). It again allows to start from the non-boosted ψ_0 and to apply the rotation before the final pure boost.

To summarize, a (j,k)-representations of the Lorentz group has a $(2j+1)(2k+1)$-dimensional p-component space and a separated space of n-components of the same dimension, which are characterized by a boost vector b. Applying the reverse boost, $-b$, produces a purely p-component situation, called the rest-system. The fact that a general Lorentz transformation can always be written as a product of a rotation with a final pure boost allows to perform the rotational part in the p-component space and finally apply the terminal boost (12).

The states have a norm (10) which is unchanged under Lorentz Transformations and under multiplication of the states by a arbitrary phase factor. However, one must acknowledge that this phase factor will change with the choice of the origin of the space-time coordinates, because a boost is always accompanied by a synchronous rotation (2). If the field possesses a local boost b, a change in the choice of the (4-)origin δx will produce a concurrent phase of

$$\Phi(\delta x) = e^{\pm\frac{\pi}{2} {}^jB\, \delta x} \Phi(0) \quad , \tag{19}$$

where ${}^jB$ is the j-specific (anti-hermitean) boost 4-vector-operator which is a zero-vector (see expressions (8) and (13) of Ref [3]). This new phase varies with the position of the field whenever the field changes it's (local) boost value with position.

It is important to acknowledge that states of a representation carry a parameter b, the characterization of the final pure boost. This is an important difference to representations of compact Lie groups, like e.g. the rotation groups $O(n)$, the states of which have no memory of any applied group-element transformations.

Discrete Transformations

For $O(4)$ the kernel $O(4)/SO(4)$ is the two-element inversion-group I, distinguishing improper transformations from the normal subgroup of proper ones. Under improper transformations the norm of a representation-vector must change it's sign. This can most easily be seen by inverting the quantization axis of spin wave-functions in $O(3)$. The spin operator is a pseudo-vector, the expectation value of which is not changed by the inversion. However, inversion changes the sign of the magnetic quantum number, and consequently the expectation value of the spin operator, unless the norm changes as well. For integer representations this amounts to multiplying the representation-vector with $\pm i$, and for half-integer ones to a sheet change. Twofold application of the same transformation leads to a change of the phase by -1.

For improper transformations the chirality changes sign [2]. In the case of non-chiral representations (j=k) one has an even and a odd representation [2]. For S$O(4)$ this is the only addition when going to the full group O(4).

For the Lorentz group the situation is somewhat richer because the kernel $O(3,1)/SO^{+}(3,1)$ now consists of the four elements {1, P, T, PT} and is isomorphic to the Abelian Klein four-group, K_4.

Furthermore, due to the ‘disconnected’ character of p- and n-components of a representation, we have a additional freedom. We have a oriented triple of boost directions ($\hat{b}_1, \hat{b}_2, \hat{b}_3$) and a associated triple of matrices (D_1, D_2, D_3) (4, 5) providing the phase between n- and p-components of a representation. Because of the ‘separated’ nature of p- and n-components the assignment of the two triples can be arbitrary, as long as the handedness is unchanged. Thus we have the additional symmetry, A_3, of even permutations of mutual assignment of b’s and D’s. For integer representations, $j+k = n = 0, 1, 2, \dots$ this yields no additional symmetry because it corresponds to a normal rotation, already included in the Lorentz transformations. This is no longer true for true half-integer representations $j+k = n+½$, because these representations are two-sheeted.

Case j = k

In this case a representation acquires a additional signature corresponding to one of the four representations of K_4.

Case j ≠ k, j+k = n (Integer)

Here the representations are chiral and change chirality under improper transformations (j↔k).This corresponds to the transition between particle and antiparticle. In addition there exists is a PT-parity.

Case j ≠ k, j+k = n + ½

In this situation the additional A_3-symmetry shows up as described above. It gives rise to the three representations known as flavor. As in the previous case one still has chirality as well as PT-parity, which is commonly considered an additional flavor characterization.

Conclusion

Representations of the Lorentz group $SO^{+}(3,1)$ are not unitary. Their states carry a boost parameter (3-vector). They have a structure consisting of a p-component (large) and a n-component (small). The n-component disappears for zero boost. States possess a norm which is invariant under Lorentz transformations and which has a positive and a negative contribution from p- and n-components, respectively. The two components are ‘separated’ in the sense, that there is no Lorentz transformation which can turn a only p-component vector into a only n-component vector.

The transformation behavior can be obtained by making use of the fact that a general Lorentz transformation can always be composed into a rotation followed by a final pure boost.

Improper transformations (particle-antiparticle relations) change the sign of the norm, such that the contribution of the large (small) components is negative (positive). Furthermore, representations carry a PT-parity.

In addition, due to the separated nature of p- and n-component, there is a topological symmetry derived from the assignment of boost directions to the three directions in the phase expressions (4) and (5) which leads to a three-valued flavor-signature for half-integer representations.

References

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