The Bilinski dodecahedron is a space-filling (tessellating) polyhedron

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Abstract: These are currently four well known isohedral space-filling convex polyhedra: parellelepiped (the most symmetric form being the cube), rhombic dodecahedron, oblate octahedron (also known as the square bipyramid) and the disphenoid tetrahedron. In this paper it is shown that a Bilinski dodecahedron is an isohedral space-filling tessellating polyhedron, thus bringing the number of these to five.

1 Introduction

Shapes that can be stacked together without gaps are deeply satisfying. They are also useful, for example in explaining the arrangement of atoms in crystals. In this paper it is shown that a Bilinski dodecahedron is an isohedral space-filling tessellating polyhedron, thus bringing the number of these to five.

2 Background

Shapes with flat surfaces and straight edges are called polyhedra.

Polyhedra which don't have any indents are known as convex polyhedra.

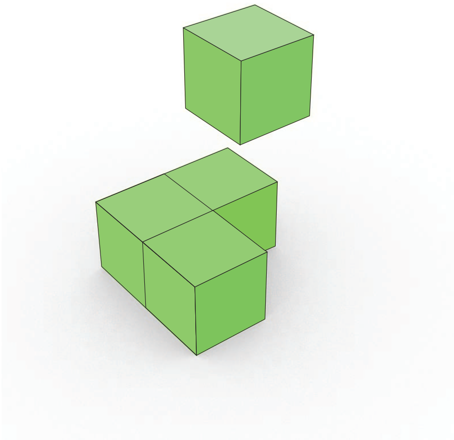
Convex polyhedra that stack together in a periodic space filling arrangement are known as ”convex uniform honeycomb” (which are also known as Andreini tessellations). There are currently 28 known convex uniform honeycomb.

There are a smaller subset of convex uniform honeycomb which only use a single type of polyhedron. These are called convex space-filling polyhedra.

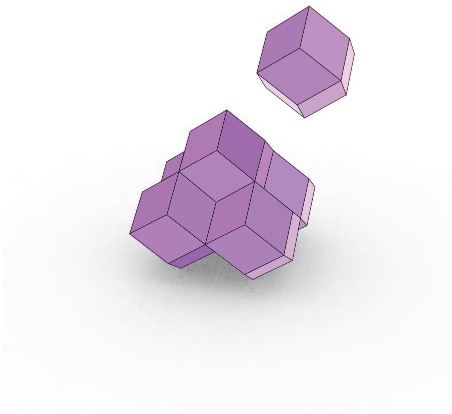
There is an even smaller subset of convex space-filling polyhedra where the polyhedron has faces of the same shape. These are called isohedral space-filling convex polyhedra. These are currently four well known isohedral space-filling convex polyhedra:

- parellelepiped (the most symmetric form being the cube)
- rhombic dodecahedron
- oblate octahedron (also known as the square bipyramid)
- disphenoid tetrahedron

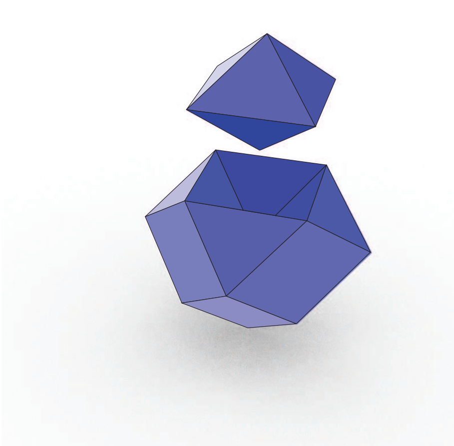
These are shown in figure 1:



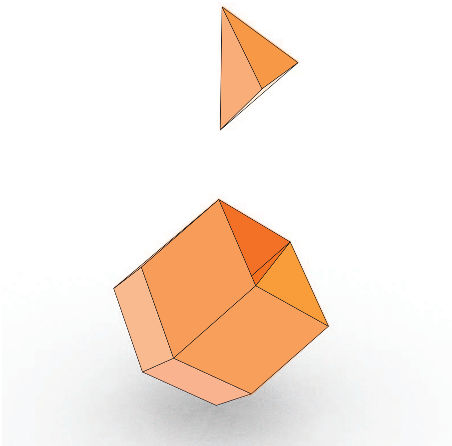
(a) cube



(b) rhombic dodecahedron



(c) oblate octahedron

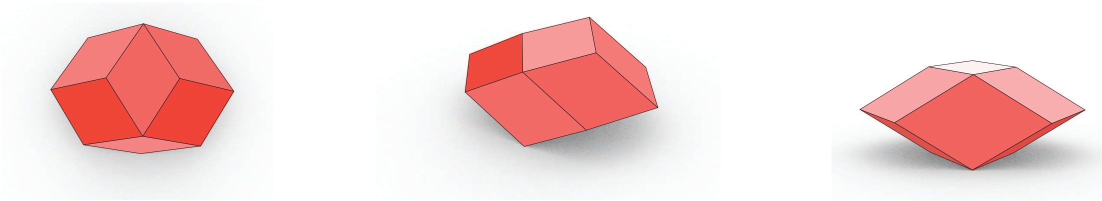


(d) disphenoid tetrahedron

Figure 1: isohedral space-filling polyhedra

3 The Bilinski dodecahedron

The Bilinski dodecahedron is a convex polyhedron with twelve congruent golden rhombus faces shown in figure 2:



(a) top (b) perspective (c) side

Figure 2

Bilinski dodecahedron history

The Wikipedia article on the Bilinski dodecahedron[1] summarises the history of this polyhedron:

> "This shape appears in a 1752 book by John Lodge Cowley, labeled as the dodecarhombus. [2],[3] It is named after Stanko Bilinski, who rediscovered it in 1960.[4] Bilinski himself called it the rhombic dodecahedron of the second kind.[5] Bilinski's discovery corrected a 75-year-old omission in Evgraf Fedorov's classification of convex polyhedra with congruent rhombic faces.[6]"

Bilinski dodecahedron construction

Each face of a Bilinski dodecahedron is a golden rhobus of the same size. A golden rhombus is a rhombus with the ratio of the distance between opposite corners being the 'golden ratio' which is calculated with the following equation:

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

A golden rhombus is shown in Figure 3:

The faces are labeled as horizontal, vertical and slanted as shown in figure 4. There are two horizontal face, two vertical faces and eight slanted faces (one horizontal face, one vertical face and three slanted faces are visible in figure 4).

To construct a Bilinski dodecahedron, first arrange the two horizontal and to vertical golden rhombi. Then the slanted golden rhombi will fit in gaps formed between the edges of the horizontal and vertical golden rhombi.

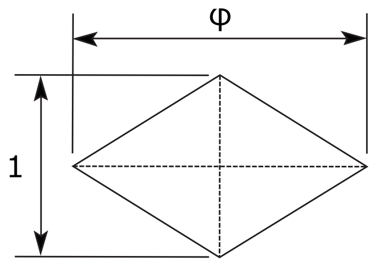


Figure 3: Golden rhombus

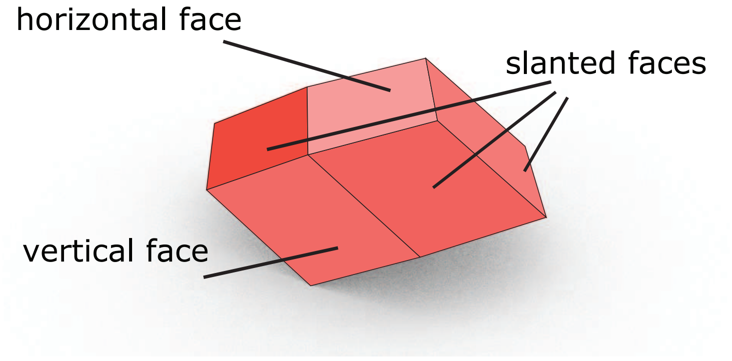


Figure 4: Bilinski dodecahedron with sides labelled

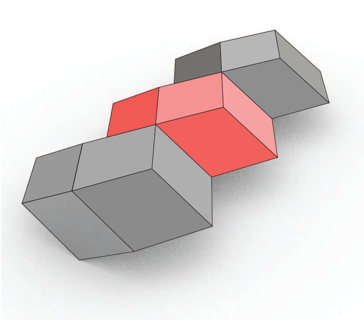
The dihedral angles of a Bilinski dodecahedron are as follows:

- Between horizontal faces and slanted faces: 144°
- Between vertical faces and slanted faces: 108°
- Between slanted faces (obtuse): 144°
- Between slanted faces (acute): 72°

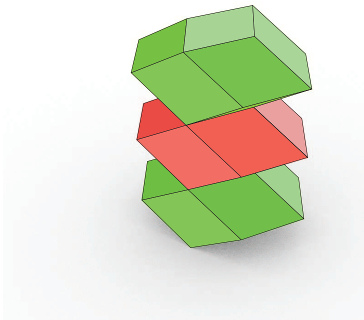
Showing that opposite faces of the Bilinski dodecahedron are parallel

The Bilinski dodecahedron comprises six pairs of opposite parallel faces, thus it is a parallelohedron. Figure 5 shows these six pairs of opposite parallel faces: (a) vertical faces, (b) horizontal faces, (c) first slanted faces, (d) second slanted faces, (e) third slanted faces, (f) fourth slanted faces:

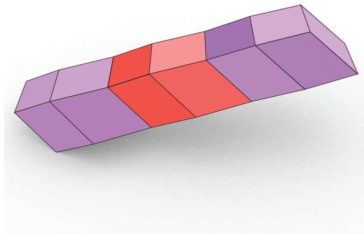
Figure 6 shows the Bilinski dodecahedra stacked with pointy ends touching.



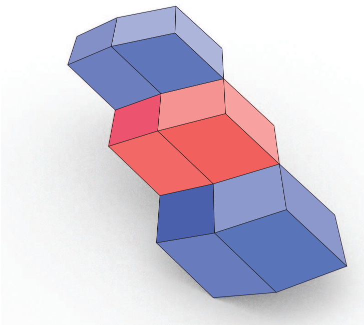
(a) vertical faces



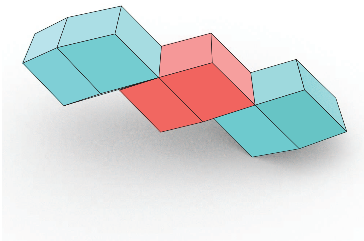
(b) horiztonal faces



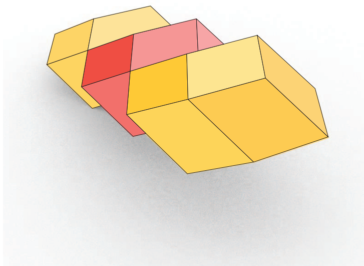
(c) first slanted faces



(d) second slanted faces



(e) third slanted faces



(f) fourth slanted faces

Figure 5: Bilinski dodecahedron stacked at opposite parallel faces

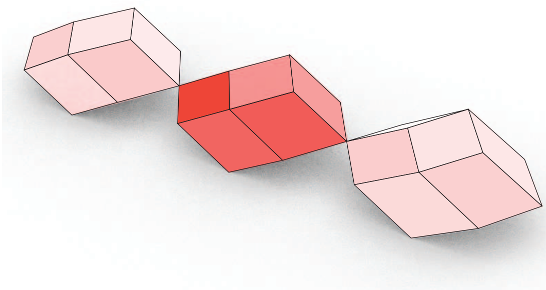


Figure 6: Bilinski dodecahedron stacked at pointy ends

Stacking the Bilinski dodecahedron in arrangements

Bilinski dodecahedra can be stacked in a first arrangement as shown in figure 7. This arrangement has Bilinski dodecahedra stacked on their horizontal and vertical surfaces and touching at the acute vertices (the pointy ends). Pairs of Bilinski dodecahedra with parallel slanted faces are the same colour: two red, two green, two grey and two pink.

A second arrangement identical to the first is shown in figure 8, with coloured pairs in dark blue, purple, light blue and gold:

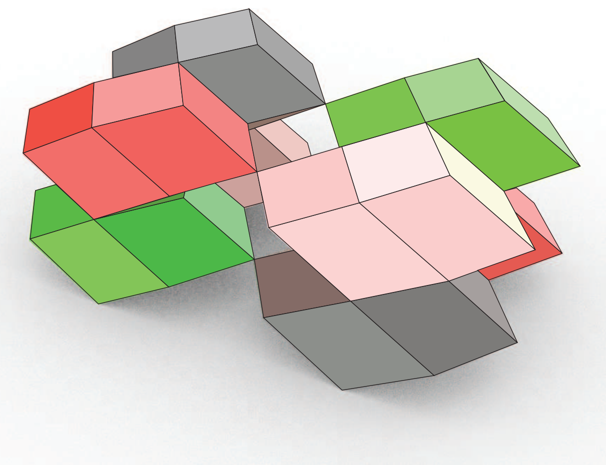


Figure 7: First arrangement

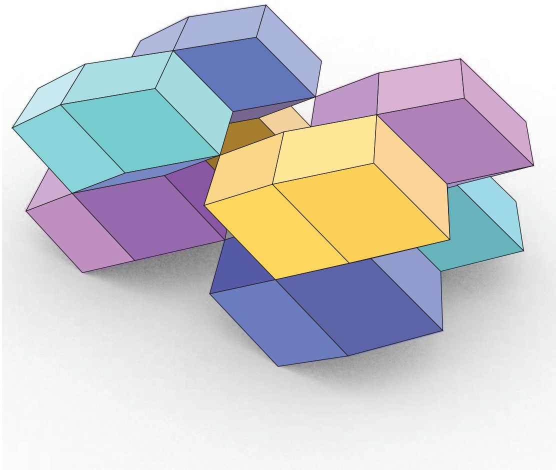


Figure 8: Second arrangement

The gaps in the first arrangement perfectly fit the second arrangement. Combining these two arrangements (Figure 9) shows that the Bilinski dodecahedron is space filling:

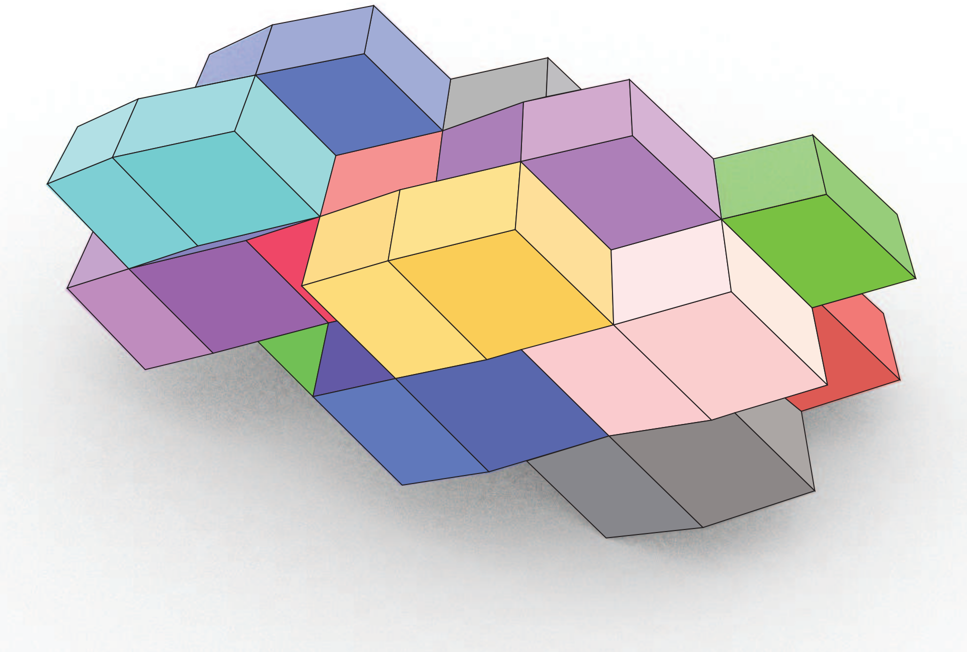


Figure 9: Bilinski dodecahedron stacked in space-filling arrangement

Notes on the re-discovery

The Bilinski dodecahedron is a space filling polyhedron. After I had written this paper, I realised that this was already known from Grünbaum[6].

I had erroneously assumed that various readily available web pages such as Wikipedia and Mathworld would have included this information. (I have now edited the Wikipedia pages on Bilinksi dodecahedron and paralleloheron to include this information.)

I think this is a familiar story to everyone who has explored maths. A disappointing feeling, but also validating to know that you were right but just a bit too late.

References

[1] https://en.wikipedia.org/wiki/Bilinski_dodecahedron

[2] Hart, George W. (2000), ”A color-matching dissection of the rhombic enneacontahedron”, Symmetry: Culture and Science, 11 (1–4): 183–199, MR 2001417,

[3] Cowley, John Lodge (1752), Geometry Made Easy; Or, a New and Methodical Explanation of the Elements of Geometry, London, Plate 5, Fig. 16. As cited by Hart (2000).

[4] Bilinski, S. (1960), ”Über die Rhombenisoeder”, Glasnik Mat. Fiz. Astr., 15: 251–263, Zbl 0099.15506

[5] Cromwell, Peter R. (1997), Polyhedra: One of the most charming chapters of geometry, Cambridge: Cambridge University Press, p. 156, ISBN 0-521-55432-2, MR 1458063

[6] Grünbaum, Branko (2010), ”The Bilinski dodecahedron and assorted parallelohedra, zonohedra, monohedra, isozonohedra, and otherhedra”, The Mathematical Intelligencer, 32 (4): 5–15, doi:10.1007/s00283-010-9138-7, hdl:1773/15593, MR 2747698