A Marriage between the Inner-Outer Method and Averaging Methods in a special case, or at least an attempt at.

Alon Ray

May 4, 2021

Abstract

I had combined the methods of Averaging and Inner-Outer methods of solution in Perturbation theory of Ordinary Differential equations. Obviously one cannot solve analytically this ODE (i.e via elementary functions), perhaps through some sort of Special Functions' transformation which I haven't tried as of yet it may be feasible. What I have done is just take a simple specific example for the general ODE, I believe I was trying to find asymptotics for this specific example. This is my failed attempt for writing my thesis paper for M.Sc in Mathematics, my final attempt. Hope it's worth something to someone...

Contents

1

We try to combine two methods in Perturbation Theory: Averaging method and Inner-Outer solution to solve the following ODE problem:

$$\begin{aligned} \dot{x} &= f(x, y_1, y_2, z) \\ \dot{y}_1 &= \frac{1}{\epsilon} g(x, z) y_2 \\ \dot{y}_2 &= -\frac{1}{\epsilon} g(x, z) y_1 \\ \dot{z} &= -\frac{1}{\epsilon}(z - h(x, y_1, y_2)) \end{aligned} \tag{1}$$

where we deal with different cases for f, g, h. We shall give errors' estimations for the approximation solutions of the different cases, with appropriate initial conditions: $X(0) = x_0, Y_j(0) = y_{j,0}, Z(0) = z_0$. The dot differentiation is over t, the slow time variable is: $\tau = t/\epsilon$.

2

2.1

First, let's find a solution for when $g(x, z) = 1$. In that case we have for the subsystem of ODEs:

$$\dot{y}_1 = y_2/\epsilon, \quad \dot{y}_2 = -y_1/\epsilon$$

which its solution for the initial conditions: $y_1(0) = 1, y_2(0) = 0$ is $y_1(\tau) = \cos\tau, \quad y_2(\tau) = \sin\tau$, and the other subsystem of ODEs translate to:

$$\begin{aligned} \dot{x} &= f(x, \cos\tau, \sin\tau, z) \\ \dot{z} &= -\frac{1}{\epsilon}(z - h(x, \cos\tau, \sin\tau)) \end{aligned} \tag{2}$$

Now we use Inner-Outer solution method. We take $x(t, \epsilon) \approx p(t, \epsilon) + \epsilon q(\tau, \epsilon)$, and insert the ansatz back to the equation for $\dot{x}$, and get:

$$\dot{p} + q' = f(p + \epsilon q, \cos\tau, \sin\tau, z)$$

where $' = d/d\tau$.
For z, we use the ansatz: $z(t, \epsilon) = r(t, \epsilon) + s(\tau, \epsilon)$. We get that:

$$\epsilon\dot{z} = h(p + \epsilon q, \cos\tau, \sin\tau) - (r + s)$$

After comparing the first coefficients with respect to ϵ, we get:

$$\begin{aligned} \dot{p} &= f(p, r, y_1, y_2) \\ \epsilon\dot{r} &= (h(p, y_1, y_2) - r) \end{aligned} \tag{3}$$

and,

$$\begin{aligned} s'(\tau) &= h(p(\epsilon\tau) + \epsilon q(\tau), y_1, y_2) - (r(\epsilon\tau) + s(\tau)) - h(p(\epsilon\tau), y_1, y_2) + r(\epsilon\tau) \\ &= h(p(\epsilon\tau) + \epsilon q(\tau), y_1, y_2) - h(p(\epsilon\tau), y_1, y_2) - s(\tau) \end{aligned} \tag{4}$$

$$q'(\tau) = f(y_1, y_2, p(\epsilon\tau) + \epsilon q(\tau), r(\epsilon\tau) + s(\tau)) - f(y_1, y_2, p(\epsilon\tau), r(\epsilon\tau)) \tag{5}$$

and with the following suitable initial conditions:

$$p(0, \epsilon) = x_0$$

$$r(0,\epsilon) = z_0$$

$$\lim_{\tau\to\infty} q(\tau,\epsilon) = \lim_{\tau\to\infty} s(\tau,\epsilon) = 0$$

We, now expand $r(t,\epsilon) = r_0(t) + r_1(t)\epsilon + \ldots$, $s(\tau,\epsilon) = s_0(\tau) + s_1(\tau)\epsilon + \ldots$. And by the Inner-Outer method as described in the reference [1], we get the following ODE for $s_0(\tau)$:

$$s_0'(\tau) = h(x_0(0), \cos\tau, \sin\tau) - r_0(0) - s_0(\tau) \tag{6}$$

We find that the solution for $s_0(\tau)$ is:

$$s_0(\tau) = e^{-\tau}s_0(0) + e^{-\tau}\int_0^\tau e^s[h(x_0(0), \cos s, \sin s) - r_0(0)]ds$$

Now, we shall use the Averaging method as described in [1], for $s_0(\tau)$, and calculate the relative error between $x_0(\tau)$ and $z_0(\tau) = s_0(\tau)$, as defined in the reference [2], Lemma 4.3.1. We give the definition below for completeness.

Definition 2.1. Consider the vector field $\mathbf{f}(x,t) : \mathsf{R}^n \times \mathsf{R} \to \mathsf{R}^n$, T-periodic in the variable t, Lipschitz continuous in x on $D \subset \mathsf{R}^n$, $t > 0$; If the average:

$$\bar{\mathbf{f}}(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{f}(x,s)ds$$

exists and the limit is uniform in x on compact sets $K \subset D$, then $\mathbf{f}$ is called a KBM-vector field (KBM stands for Krylov, Bogliubov and Mitropolsky).

Definition 2.2. If $\mathbf{f}$ is a KBM-vector field and assuming $\epsilon T = o(1)$, as $\epsilon \to 0$, then on the time scale $1/\epsilon$ one has:

$$f_T(x,t) = \bar{f}(x) + \mathcal{O}(\delta_1(\epsilon)/(\epsilon T)),$$

where $f_T(x,t) = \frac{1}{T}\int_t^{t+T} f(x,s)$, which is called the local average by $\mathbf{f}$; and we define $\delta_1(\epsilon)$, the order function of $\mathbf{f}$ to be:

$$\delta_1(\epsilon) = \sup_{x\in D}\ \sup_{t\in[0,L/\epsilon)} \epsilon\left\|\int_0^t [\mathbf{f}(x,s) - \bar{\mathbf{f}}(x)]ds\right\|$$

According to Theorem 4.3.6 in [2] if we try to solve the following system of ODEs with the averaging method:

$$\dot{x}_0 = \epsilon f(x_0, \cos\tau, \sin\tau, z_0) \tag{7}$$
$$\dot{z}_0 = \epsilon \bar{f}(z_0)$$

where $\bar{f}(z_0) = \lim_{T\to\infty} 1/T \int_0^T f(z_0,t)dt$, and $z_0(t)$ belongs to an interior subset of D on the time scale $1/\epsilon$; then:

$$x_0(t) - z_0(t) = \mathcal{O}(\sqrt{\delta_1(\epsilon)})$$

as $\epsilon \to 0$ on the time scale $1/\epsilon$. (NB we should note that the initial conditions should match, i.e that $z_0(0) = x_0(0) = a$).

Now, we compute $\delta_1(\epsilon)$ for several cases.

First, take $f(x, y_1, y_2, z) = x + y_1^2 z$, and the domain $D = \{x : |x| < 100\}$. We find that:

$$\bar{f}(x_0) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left[x_0 + \cos^2(s)\big(e^{-s} z_0(0) + e^{-s} \int_0^s h(x, \cos r, \sin r) e^r dr\big)\right] ds \tag{8}$$

Suppose we can expand h by a Fourier Series, i.e $h(x, \cos r, \sin r) = a_0 + \sum_{n=1}^{\infty} a_n \cos nr + b_n \sin nr$. For $n \neq 2$ we get the following two identities:

$$\int_0^T \cos^2(s)\exp(-s) \int_0^s \exp(r)\sin(nr)drds = \tag{9}$$

$$= -\frac{1}{20(n^2+1)(n^2-4)n}\Big(2\exp(-T)\cos(2T)n^4 - 4\exp(-T)\sin(2T)n^4 +$$
$$+10\exp(-T)n^4 - 8\exp(-T)\cos(2T)n^2 + 16\exp(-T)\sin(2T)n^2 + 10\sin(nT)n^3 +$$
$$+5n^3\sin((n-2)T) + 5n^3\sin((n+2)T) - 12n^4 - 40\exp(-T)n^2 + 5\cos((n-2)T)n^2 +$$
$$+5\cos((n+2)T)n^2 + 10\cos(nT)n^2 + 10n^2\sin((n-2)T) - 10n^2\sin((n+2)T) + 10\cos((n-2)T)n -$$
$$-10\cos((n+2)T)n - 40\sin(nT)n + 28n^2 - 40\cos(nT) + 40\Big)$$

$$\int_0^T \cos^2(s)\exp(-s) \int_0^s \exp(r)\cos(nr)drds = \tag{10}$$

$$= \frac{1}{20(n^2+1)(n^2-4)n}\Big(2\exp(-T)\cos(2T)n^3 - 4\exp(-T)\sin(2T)n^3 + 10\exp(-T)n^3 +$$
$$-5\cos((n-2)T)n^3 - 5\cos((n+2)T)n^3 - 10\cos(nT)n^3 - 8\exp(-T)\cos(2T)n + 16\exp(-T)\sin(2T)n -$$
$$-10\cos((n-2)T)n^2 + 10\cos((n+2)T)n^2 + 10\sin(nT)n^2 + 5n^2\sin((n-2)T) + 5n^2\sin((n+2)T) +$$
$$+8n^3 - 40\exp(-T)n + 40\cos(nT)n + 10n\sin((n-2)T) - 10n\sin((n+2)T) - 40\sin(nT) + 8n\Big)$$

We can see that after dividing (9) and (10) by T and then taking the limit of $T \to \infty$, that they both vanishe.

For the computation of the integrals for $n = 2$, we get two numerical terms that aren't zero:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \cos^2(s)\exp(-s) \int_0^s \exp(r)\sin(2r)drds = -1/10 \tag{11}$$

$$\lim_{T\to\infty} \frac{1}{T} T \cos^2(s)\exp(-s) \int_0^s \exp(r)\cos(2r)drds = 1/20 \tag{12}$$

Notice that:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \exp(-s)\cos^2(s)z_0(0)ds = 0 \tag{13}$$

So, we get for this case of h that $\bar{f}(x_0) = x_0 + \frac{a_0}{2} + \frac{a_2}{20} - \frac{b_2}{10}$. And we get that: $\delta_1(\epsilon) = \mathcal{O}(\epsilon)$.
Now, assume that $H(s) = \exp(-s)H(0) + \exp(-s)\int_0^s h(x_0, \cos\alpha, \sin\alpha)\exp(\alpha)d\alpha$. By taking a derivative on both sides of the equation that defines $H(s)$, we get the ODE: $\frac{\partial}{\partial s}H(s) = h(s) - H(s)$; we assume that $H(s)$ is 2π periodic so we get that: $\int_0^{2\pi}[\frac{\partial}{\partial\theta}H(\theta) + H(\theta)]d\theta = \int_0^{2\pi} h(\theta)d\theta$, now since $H(s)$ is 2π-periodic we get that the first term on the LHS vanishes, so we have the following condition:

$$\int_0^{2\pi} H(\theta)d\theta = \int_0^{2\pi} h(\theta)d\theta = H(0)\int_0^{2\pi}\exp(-\theta)d\theta + \int_0^{2\pi}\exp(-\theta)\int_0^{\theta} h(\alpha)\exp(\alpha)d\alpha \tag{14}$$

Now assume that $H(s)$ can be expanded in a Fourier Series, i.e: $H(s) = a_0 + \sum_{n=1}^{\infty}(a_n\cos(ns) + b_n\sin(ns))$; we have the following four limits:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \cos^2(s)ds = 1/2; \tag{15}$$
$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \cos^2(s)\cos(2s)ds = 1/4;$$
$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \cos^2(s)\sin(ns)ds = 0;$$
$$\forall n \neq 2 \quad \lim_{T\to\infty}\frac{1}{T}\int_0^T \cos^2(s)\cos(ns)ds = 0.$$

So we get that here $\bar{f}(x_0) = x_0 + a_0/2 + a_2/4$. Below we show how to explicitly to calculate: $H(0)$.

$$\begin{aligned}
H(0) &= \int_0^{2\pi} h(\alpha)d\alpha/(1-\exp(-2\pi)) - \int_0^{2\pi}\exp(-\theta)\int_0^{\theta}\exp(\alpha)h(\alpha)d\alpha d\theta/(1-\exp(-2\pi)) = \\
&= \int_0^{2\pi} h(\alpha)d\alpha/(1-\exp(-2\pi)) - \int_0^{2\pi}\exp(\alpha)h(\alpha)\int_{\alpha}^{2\pi}\exp(-\theta)d\theta d\alpha = \\
&= \int_0^{2\pi} h(\alpha)d\alpha/(1-\exp(-2\pi)) - \int_0^{2\pi}\exp(\alpha)h(\alpha)\Big[\exp(-\alpha) - \exp(-2\pi)\Big]/(1-\exp(-2\pi)) = \\
&= [\exp(-2\pi)/(1-\exp(-2\pi))]\int_0^{2\pi}\exp(\alpha)h(\alpha)d\alpha
\end{aligned}$$

Again we get that $\delta_1(\epsilon) = \mathcal{O}(\epsilon)$.
Now, suppose that: $f(x, y_1, y_2, z) = x + (a_0 + \sum_{n=1}^{\infty} a_n\cos(n\tau) + b_n\sin(n\tau))\cdot z$.

So we want to calculate:

$$\bar{f}(x_0) = x_0 + \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\sum_{n=0}^{\infty} a_n \cos(ns) + b_{n+1} \sin(ns) \right) \Bigg(z_0(0) \exp(-s) + \exp(-s) \cdot \tag{16}$$

$$\int_0^s \exp(r) h(r) dr \Bigg) ds$$

Assume also that $h(r)$ can be expanded by a Fourier Series, i.e: $h(r) =$

$\tilde{a}_0 + \sum_{n=0}^{\infty} \tilde{a}_n \cos(nr) + \tilde{b}_n \sin(nr)$. So we get the following limits:

$$\frac{1}{T}\int_0^T \exp(r)\cos(nr)\int_r^T \cos(ns)\exp(-s)dsdr = \tag{17}$$

$$= -\frac{1}{4T(n^2+1)^2 n}\Big(\cos(2nT) - n^3 + \cos(2nT)n + 4\sin(nT)\exp(-T)n^2 -$$

$$2Tn^3 - \sin(2nT)n^2 - 4\cos(nT)\exp(-T)n + 3n - 2nT - \sin(2nT)\Big) \xrightarrow[T\to\infty]{}$$

$$\frac{1}{2(n^2+1)}.$$

$$\frac{1}{T}\int_0^T \exp(r)\sin(nr)\int_r^T \cos(ns)\exp(-s)dsdr = -\frac{1}{4T(n^2+1)^2 n}\Big(\sin(2Tn)n^3 -$$

$$4\sin(nT)\exp(-T)n^3 + 2Tn^4 + \sin(2nT)n + 4\cos(nT)\exp(-T)n^2 - 5n^2 + \cos(2nT)n^2 + 2Tn^2 +$$

$$\cos(2nT) - 1\Big) \xrightarrow[T\to\infty]{} \frac{-n}{2(n^2+1)}.$$

$$\frac{1}{T}\int_0^T \exp(r)\cos(nr)\int_r^T \sin(ns)\exp(-s)dsdr = -\frac{1}{4T(n^2+1)^2 n}\Big(\cos(2nT) - n^3 + \cos(2nT)n$$

$$+4\sin(nT)\exp(-T)n^2 - 2Tn^3 - \sin(2nT)n^2 - 4\cos(nT)\exp(-T)n + 3n - 2nT - \sin(2nT)\Big) \xrightarrow[T\to\infty]{}$$

$$\frac{1}{2(n^2+1)}.$$

$$\frac{1}{T}\int_0^T \exp(r)\sin(nr)\int_r^T \sin(ns)\exp(-s)dsdr = \frac{1}{4T(n^2+1)^2 n}\Big(\cos(2nT)n^3 - 4\cos(nT)\exp(-T)n^3 +$$

$$3n^3 + \cos(2nT)n - 4\sin(nT)\exp(-T)n^2 + 2Tn^3 - \sin(2nT)n^2 - n + 2nT - \sin(2nT)\Big) \xrightarrow[T\to\infty]{}$$

$$\frac{1}{2(n^2+1)}.$$

$$\frac{1}{T}\int_0^T \exp(r)\int_r^T \cos(ns)\exp(-s)dsdr = \Big(n^2\sin(nT) - \sin(nT)\exp(-T)n^2 + \cos(nT)\exp(-T)n +$$

$$\sin(nT) - n\Big)/(Tn(n^2+1)) \xrightarrow[T\to\infty]{} 0.$$

$$\frac{1}{T}\int_0^T \exp(r)\int_r^T \sin(ns)\exp(-s)dsdr = \Big[-\cos(nT)n^2 + \cos(nT)\exp(-T)n^2 + n\exp(-T)\sin(nT) -$$

$$\cos(nT) + 1\Big]/(T(n^2+1)n) \xrightarrow[T\to\infty]{} 0.$$

$$\frac{1}{T}\int_0^T \exp(r)\sin(nr)\int_r^T \exp(-s)dsdr = -\Big(\sin(nT)n + n^2\exp(-T) - n^2 + \cos(nT) - 1\Big)/(T(n^2+1)n)$$

$$\xrightarrow[T\to\infty]{} 0$$

$$\frac{1}{T}\int_0^T \exp(r)\cos(nr)\int_r^T \exp(-s)dsdr = \frac{-[\cos(nT) - n\exp(-T) - \sin(nT)]}{T(n^2+1)n} \xrightarrow[T\to\infty]{} 0.$$

$$\frac{1}{T}\int_0^T \exp(r)\int_r^T \exp(-s)dsdr = \frac{(\exp(T)T - \exp(T) + 1)\exp(-T)}{T} \xrightarrow[T\to\infty]{} 1.$$

So all in all, we get that:

$$\bar{f}(x_0) = x_0 + \sum_{n=1}^{\infty} \frac{1}{2(n^2+1)} a_n \tilde{a}_n - \sum_{n=1}^{\infty} \frac{n}{n^2+1} (a_n \tilde{b}_n + \tilde{a}_n b_n) + \sum_{n=1}^{\infty} \frac{1}{2(n^2+1)} b_n \tilde{b}_n + a_0 \tilde{a}_0 \tag{18}$$

Since all of the terms are constants with respect to x_0, we still get: $\delta_1(\epsilon) = \mathcal{O}(\epsilon)$.

References

[1] Perturbation: Theory and Methods - Prof. James Murdock, First Edition.

[2] Averaging Methods in Nonlinear Dynamical Systems - Jan A. Sanders, Ferdinand Verhulst, James Murdock, Second Edition.

[3]