Magnetic symmetry of geometrical optics

Miftachul Hadi[1,2]

[1)] *Physics Research Centre, Badan Riset dan Inovasi Nasional (BRIN), Puspiptek, Serpong, Tangerang Selatan 15314, Banten, Indonesia.*

[2)] *Institute of Mathematical Sciences, Kb Kopi, Jalan Nuri I, No.68, Pengasinan, Gn Sindur 16340, Bogor, Indonesia. E-mail: instmathsci.id@gmail.com*

We show that there exists a magnetic monopole in the $U(1)$ geometrical optics as a consequence of the magnetic symmetry in a $(4+d)$-dimensional unified space where the magnetic symmetry is a consequence of the extra internal symmetry. This magnetic symmetry restricts the gauge potential. The restricted (decomposed) gauge potential is made of the scalar potential as the unrestricted electric part and the vector potential as the restricted magnetic part. We also show that the refractive indices can be formulated in relation to the decomposed gauge potential. We treat the curvature in the curvature-refractive index relation of the $U(1)$ geometrical optics as an Abelian curvature form in the fibre bundle.

Keywords: *$(4+d)$-dimensional unified space, extra internal symmetry, magnetic symmetry, magnetic monopole, gauge potential, $U(1)$ geometrical optics, refractive index, Abelian curvature form, fibre bundle.*

Dirac proposed, due to symmetrical reasoning in the Maxwell's theory of electromagnetism, there exist a magnetic symmetry appears as a charged magnetic monopole[1,2]. Inspired by Dirac's idea, *the magnetic symmetry* was formulated as *a self-consistent subset* of *a local, a non-Abelian $SU(2)$* gauge theory in a $(4+d)$-dimensional unified space[3]. A $(4+d)$-dimensional unified space is a $(3+1)$-dimensional external space-time, i.e. a $(3+1)$-dimensional curved space-time, plus a d-dimensional internal (isometry group, G) space[4].

The formulation of a self-consistent subset, a local, a non-Abelian $SU(2)$ gauge theory in a $(4+d)$-dimensional unified space[3], roughly speaking, looks like a local, a non-Abelian $SU(N)$ Yang-Mills theory[5] where the choice of the simple Lie group $G=U(1)$ reduces the Yang-Mills theory to the Maxwell's theory[6]. *The role of a magnetic symmetry of the $SU(2)$ local gauge theory in a $(4+d)$-dimensional of unified space, is to restrict the gauge potential*[3].

To the best of our knowledge, the geometrical optics is formulated without including the magnetic symmetry[7–13]. The geometrical optics (the eikonal equation) can be derived from the Maxwell equations[8]. Because of the Maxwell's theory is a local[14], an Abelian $U(1)$ gauge theory and the eikonal equation can be derived from the Maxwell equations, *we argue that the geometrical optics is also a local, an Abelian $U(1)$ gauge theory, there exist a magnetic symmetry and a magnetic monopole in the geometrical optics.*

The gauge theory can be viewed as the general theory of relativity in a higher-dimensional unified space which consists of a $(3+1)$-dimensional external space-time plus a d-dimensional internal (isometry group, G) space[3]. If the metric tensor, $g_{\mu\nu}$ $(\mu,\nu=1,2,..,4+d)$, in a $(4+d)$-dimensional unified space has a d-dimensional isometry group whose Killing vector fields span the internal space, *the general theory of relativity becomes the gauge theory of the isometry group in curved space-time*[3].

Let us choose the d Killing vector fields, ξ_i $(i=1,2,..,d)$ to satisfy the canonical commutation relations of the isometry group, G[3]

$$[\xi_i,\xi_j]=f_{ij}^{\ \ k}\ \xi_k \tag{1}$$

By definition, these Killing vector fields must also satisfy[3]

$$\pounds_{\xi_i}\ g_{\mu\nu}=0,\quad (i=1,2,..,d) \tag{2}$$

where $\pounds_{\xi_i}$ is the Lie derivative along the direction of ξ_i.

The existence of the Killing vector fields means the existence of *an isometric mapping. A mapping of the space-time onto itself is an isometric mapping, if the Lie derivative of the metric tensor associated with it, as shown in eq.(2), vanishes*[15]. It means that the space-time have *an intrinsic symmetry*[15] (2). If we relate the intrinsic symmetry (2) with *the homogeneity principle*[16], *the homogeneity of space implies identical metric properties at all points of the space.* An exact definition of the homogeneity of space involves considering set of coordinate transformations that transform the space into itself, i.e. *leave its metric unchanged*[10]. We will call, from now on, an intrinsic symmetry (2) as *an internal symmetry.*

A magnetic monopole is formulated using the generalized gauge theory (i.e. the d Killing vector fields are *not kept to be orthonormal*) which has *an extra internal symmetry made of some additional Killing vector fields*, which are *internal* and which *commute* with the already existing fields, ξ_i. Let us assume that there exists only one such *vector field* denoted by m. By assumption[3] that m commutes with the already existing fields, ξ_i, we obtain

$$[m,\xi_i]=0,\quad (i=1,2,..,d) \tag{3}$$

and in analogy with the definition (2)

$$\pounds_m\ g_{\mu\nu}=0 \tag{4}$$

where $\pounds_m$ is the Lie derivative along the direction of m. The existence of the additional Killing vector field, m, means the existence of *an extra internal symmetry*[15] (4).

Since m is assumed to be internal[3], we have

$$m=m^i\ \xi_i \tag{5}$$

We see from eq.(5), because m and ξ_i are vectors then m^i should be a scalar. The commutation relation (3) tells us that *the multiplet*, $\hat{m}$, made of *the components*, m^i

$$\hat{m} = \begin{pmatrix} m^1 \\ m^2 \\ . \\ . \\ m^d \end{pmatrix} \tag{6}$$

must form an adjoint representation of the group[3]. Roughly speaking, the multiplet is nothing but vector representation.

As a consequence of the extra internal symmetry (4) there exist a magnetic symmetry[3], which can be written mathematically, as below

$$D_\mu \hat{m} = \partial_\mu \hat{m} + g\vec{B}_\mu \times \hat{m} = 0 \tag{7}$$

where $\vec{B}_\mu$ is *the gauge potential* of the isometry group, G. What about the value of the multiplet, $\hat{m}$? Let us assume for simplicity that the d Killing vector fields, ξ_i, are orthonormal to each other with respect to the metric tensor, $g_{\mu\nu}$. With this simplification, the magnetic symmetry (7) implies, among others, that the multiplet, $\hat{m}$, must have *a constant length*[3]

$$\hat{m}^2 = \text{constant} \tag{8}$$

which we can choose to be *the unit* without loss of generality.

The extra internal symmetry (4) or in turn the magnetic symmetry (7) restricts the gauge potential[3]. To see how the magnetic symmetry (7) restricts the gauge potential, $\vec{B}_\mu$, let us consider, for simplicity, the case when *the isometry group is* $SU(2)$. In case of $SU(2)$, the magnetic symmetry (7) can be solved exactly for $\vec{B}_\mu$ as follow[3]

$$\vec{B}_\mu^{SU(2)} = A_\mu^{SU(2)}\ \hat{m}^{SU(2)} - \frac{1}{g}\hat{m}^{SU(2)} \times \partial_\mu \hat{m}^{SU(2)} \tag{9}$$

where $A_\mu^{SU(2)}$ is the (Abelian) component of $\vec{B}_\mu^{SU(2)}$. We see that the gauge potential (9) is made of two parts, i.e. *the unrestricted part, a scalar*, $A_\mu^{SU(2)}$, and the other part which is *completely determined* by the magnetic symmetry, *a vector*, $\hat{m}^{SU(2)}$. We will call $A_\mu^{SU(2)}$, *electric* and $\hat{m}^{SU(2)}$, *magnetic*[3]. The restricted gauge potential, $\vec{B}_\mu^{SU(2)}$, with $A_\mu = 0$ and $\hat{m}^{SU(2)} = \hat{r}^{SU(2)}$ describes precisely *the Wu-Yang magnetic monopole*[17]. Here, $\hat{r}^{SU(2)}$ as $\hat{m}^{SU(2)}$ must have a constant length which we can choose to be unit without loss of generality.

The corresponding field strength, $\vec{G}_{\mu\nu}^{SU(2)}$, of the $\vec{B}_\mu^{SU(2)}$ (9) is[3]

$$\begin{aligned} &\vec{G}_{\mu\nu}^{SU(2)} \\ &= \partial_\mu \vec{B}_\nu^{SU(2)} - \partial_\nu \vec{B}_\mu^{SU(2)} + g\vec{B}_\mu^{SU(2)} \times \vec{B}_\nu^{SU(2)} \end{aligned} \tag{10}$$

We see that the third term on the right hand side of eq.(10) is *the non-Abelian (non-commutative) term what produces the non-linear term* in the equation[18]. This term is the main difference compare to the field strength of the Maxwell's theory which is Abelian.

In a $(4+d)$-dimensional unified space, for the geometrical optics approximation (short wavelength, $\lambda \to 0$[10]), the four-vector potential is replaced by the gauge potential, $\vec{B}_\mu$, and the related field strength, $\vec{G}_{\mu\nu}$, can be represented respectively as[12,13,19]

$$\vec{B}_\mu = a_\mu\ e^{i\psi} \tag{11}$$

and

$$\vec{G}_{\mu\nu} = \partial_\mu \vec{B}_\nu - \partial_\nu \vec{B}_\mu \tag{12}$$

where *a phase* (*an eikonal*), $\psi(x,y,z,t)$, and a slowly varying function of coordinates and time, *an amplitude*, a_μ[9], are represented in a $(4+d)$-dimensional unified space. Here, a space-time covariant derivative, ∇_μ[11], is replaced by the covariant derivative of a $(4+d)$-dimensional unified space, ∂_μ. We see from eq.(11), the amplitude, a_μ, has the same dimension as the displacement from equilibrium[20], the oscillating variable[21], the gauge potential $\vec{B}_\mu$.

In case of *a steady monochromatic wave*, the frequency[22] is constant and the time dependence of the eikonal, ψ, is given by a term $-f_\theta t$ where f_θ is a notation for (angular) frequency[9]. Let us introduce ψ_1, a function, which is also called *eikonal*[9]. The relation between ψ_1 and ψ can be expressed as[9]

$$\psi_1 = \frac{c}{f_\theta}\psi + ct \tag{13}$$

where the eikonal, ψ_1, is a function of coordinates only[9] and c is the speed of light in vacuum. The eikonal in a $(3+d)$-dimensional unified space (without time) is denoted by $\psi_1(x,y,z,d)$.

The equation of ray propagation in a transparent medium can be written in relation with the refractive index, n, as below[8,9]

$$|\vec{\nabla}\psi_1| = |\vec{n}| = n \tag{14}$$

where n is *a scalar*, $\vec{\nabla}$ is a notation for gradient. Because ψ_1 is a function of coordinates only, then the refractive index is also a function of coordinates only. More precisely, the refractive index is *a smooth continuous function* of the position[23]. In a 3-dimensional space, the refractive index is denoted by $n(x,y,z)$.

The equation (14) is called *the eikonal equation*[8,9], i.e. *a type of the first order linear partial differential equation.* The analysis of partial differential equation for steady state is very important e.g. for formulating the Atiyah-Singer index theorem, an effort for finding the existence and uniqueness of solutions to linear partial differential equations of elliptic type on closed manifold[24]. Because, as we will see, the refractive index is related with the curvature and we can apply the magnetic symmetry to the formulation of the refractive index, so probably, we could apply the magnetic symmetry of geometrical optics to the curvature part of the Atiyah-Singer index theorem. Progress work was reported[25].

Let us formulate the eikonal, ψ_1, in the $(4+d)$-dimensions of unified space. Because the eikonal, ψ_1, is a function of coordinates only (it is time-independent), so it becomes a $(3+d)$-dimensional eikonal, $\psi_1(x,y,z,d)$, which lives in a $(4+d)$-dimensional unified space. The gradient operator, $\vec{\nabla}$, in eq.(14) is replaced by the covariant four-gradient, ∂_μ. So, eq.(14) becomes

$$|\partial_\mu \psi_1| = |\vec{n}_\mu| = n \tag{15}$$

where μ runs from 1 to $4+d$ by considering that the time components of ψ_1 and n are zero.

We see from eq.(15), the refractive index is *a scalar, a real number. The zeroth rank tensor (scalar)* of the refractive index describes *an isotropic linear optics*[26]. But, the refractive index can be not simply a scalar[27]. The refractive index can also be *a second rank tensor* which describes that the electric field component along one axis may be affected by the electric field component along another axis[27]. The second rank tensor of the refractive index describes *an anisotropic linear optics*[26].

Our works[28,29] show that *the second rank tensor of the refractive index is a consequence of the fourth rank totally covariant tensor of Riemann-Christoffel curvature*, $R_{\mu\nu\rho\sigma}$. Naturally, it means that the fourth rank totally covariant tensor of Riemann-Christoffel curvature describes an anisotropic linear optics. *In this article, we will work with the refractive index as a scalar related to the second rank tensor of Ricci curvature where we apply the magnetic symmetry to the refractive index.* This work could be extended to the second rank tensor of refractive index related with the fourth rank totally covariant tensor of Riemann-Christoffel curvature.

In analogy with $SU(N)$ Yang-Mills theory, the choice of the simple Lie group, $G = U(1)$ reduces a local, a non-Abelian $SU(2)$ gauge theory in a $(4+d)$-dimensional unified space to the Maxwell's theory, a local, an Abelian $U(1)$ gauge theory in a $(4+d)$-dimensional unified space. We obtain from eqs.(9), (10) that *the restricted $U(1)$ gauge potential* and its related field strength can be represented respectively as

$$\vec{B}_\mu^{U(1)} = A_\mu^{U(1)}\ \hat{m}^{U(1)} - \frac{1}{g}\hat{m}^{U(1)} \times \partial_\mu \hat{m}^{U(1)} \tag{16}$$

and

$$\vec{G}_{\mu\nu}^{\ U(1)} = \partial_\mu \vec{B}_\nu^{\ U(1)} - \partial_\nu \vec{B}_\mu^{\ U(1)} \tag{17}$$

where $A_\mu^{U(1)}$ is the electric potential part of the $U(1)$ gauge potential, a scalar, which is not restricted by the magnetic symmetry (7) and $\hat{m}^{U(1)}$ is the multiplet or the magnetic potential part of the $U(1)$ gauge potential, a vector, which is completely determined by the magnetic symmetry (7). Both, $A_\mu^{U(1)}$ and $\hat{m}^{U(1)}$, live in a $(4+d)$-dimensional unified space.

In analogy with the Wu-Yang monopole in the $SU(2)$ Yang-Mills theory, if we set $A_\mu^{U(1)} = 0$ and $\hat{m}^{U(1)} = \hat{r}^{U(1)}$ in the restricted $U(1)$ gauge potential, $\vec{B}_\mu^{U(1)}$ (16), then the restricted $U(1)$ gauge potential, $\vec{B}_\mu^{U(1)}$, describes *the gauge potential of magnetic monopole of the $U(1)$ Maxwell's theory.*

As we mention previously that the eikonal equation of the geometrical optics can be derived from the Maxwell equations, it has a consequence that the field strength of the geometrical optics, $\vec{G}_{\mu\nu}$ (12), and the field strength of the Maxwell's theory, $\vec{G}_{\mu\nu}^{\ U(1)}$ (17), in principle are *the same* i.e. *both are fields*[30]. In turn, it has a consequence that we can replace $\vec{G}_{\mu\nu}$ by $\vec{G}_{\mu\nu}^{\ U(1)}$. In other words, we can replace the gauge potential, $\vec{B}_\mu$ (11), by the restricted $U(1)$ gauge potential, $\vec{B}_\mu^{\ U(1)}$ (16). If we replace $\vec{B}_\mu$ by $\vec{B}_\mu^{U(1)}$ then eq.(11) becomes

$$\vec{B}_\mu^{\ U(1)} = a_\mu\ e^{i\psi} \tag{18}$$

Eq.(18) expresses *the $U(1)$ gauge potential of the geometrical optics* in a $(4+d)$-dimensional unified space. This $U(1)$ gauge potential of the geometrical optics (18) is the same as the restricted $U(1)$ gauge potential of the Maxwell's theory in a $(4+d)$-dimensional unified space (16). The related field strength of the geometrical optics has the same form as the field strength of the Maxwell's theory (17) where the gauge potential of the geometrical optics is given by (18).

Eq.(18) can now be written as

$$\vec{B}_\mu^{U(1)}\ \underline{a}^\mu = a_\mu\ \underline{a}^\mu\ e^{i\psi} = a^2\ e^{i\psi} = e^{i\psi} \tag{19}$$

where $\underline{a}^\mu$ is a complex conjugate of a complex vector amplitude, a_μ, and a is a scalar amplitude[31] which we can take its value as 1. Using Euler's formula, eq.(19) can be written as

$$\cos\psi + i\sin\psi = \vec{B}_\mu^{U(1)}\ \underline{a}^\mu \tag{20}$$

Eq.(20) shows us that $\vec{B}_\mu^{U(1)}\underline{a}^\mu$ is *a complex function.* To simplify the problem, we take the real part[32] of (20) only, we obtain

$$\cos\psi = \mathrm{Re}\ (\vec{B}_\mu^{U(1)}\ \underline{a}^\mu) \tag{21}$$

where ψ in eq.(21), i.e. phase (eikonal or "gauge") is *an angle.* This angle has value

$$\psi = \arccos\left[\mathrm{Re}\left(\vec{B}_\mu^{U(1)}\ \underline{a}^\mu\right)\right] \tag{22}$$

Substituting eqs.(22), (16) into eq.(13), we obtain

$$\frac{c}{f_\theta}\arccos\left\{\mathrm{Re}\left[\left(A_\mu^{U(1)}\ \hat{m}^{U(1)} - \frac{1}{g}\hat{m}^{U(1)} \times \partial_\mu \hat{m}^{U(1)}\right)\underline{a}^\mu\right]\right\} + ct = \psi_1 \tag{23}$$

If we substitute eq.(23) into eq.(15), then the eikonal equation (15) becomes

$$\left|\partial_\nu\left(\frac{c}{f_\theta}\arccos\left\{\mathrm{Re}\left[\left(A_\mu^{U(1)}\hat{m}^{U(1)} - \frac{1}{g}\hat{m}^{U(1)} \times \partial_\mu \hat{m}^{U(1)}\right)\underline{a}^\mu\right]\right\} + ct\right)\right| = n \tag{24}$$

where n is a dimensionless quantity, a scalar, a real number, i.e. a function of $(3+d)$-coordinates which lives in a $(4+d)$-dimensional unified space.

Eq.(24) shows us that the refractive index is decomposed into the unrestricted electric part and the restricted magnetic part by the magnetic symmetry (7). This magnetic symmetry is a consequence of the extra internal symmetry of a $(4+d)$-dimensional unified space (4). In analogy with the Maxwell's theory, if we set $A_\mu^{U(1)} = 0$ and $\hat{m}^{U(1)} = \hat{r}^{U(1)}$ in the restricted gauge potential then the restricted gauge potential, $\vec{B}_\mu^{U(1)}$ (16), describes the gauge potential of magnetic monopole of the $U(1)$ geometrical optics as below

$$\vec{B}_\mu^{U(1)} = -\frac{1}{g}\,\hat{r}^{U(1)} \times \partial_\mu \hat{r}^{U(1)} \tag{25}$$

The form of the gauge potential of magnetic monopole of the $U(1)$ geometrical optics is precisely the same as the gauge potential of magnetic monopole of the $U(1)$ Maxwell's theory.

The equation of ray propagation in *a steady state* can also be derived from *the Fermat's principle*[9]

$$\delta\psi_1 = 0 \tag{26}$$

We obtain from the Fermat's principle (26), the curvature-refractive index relation as below[9,33,34]

$$\frac{1}{R} = \hat{N}\cdot\frac{\vec{\nabla}n}{n} \tag{27}$$

where n is *a scalar*, the same as n in the eikonal equation (14), (15), $\hat{N}$ is the unit vector along the principal normal, R is the radius of curvature. $1/R$ is the curvature of a 1-dimensional space, $\kappa(R) = 1/R$. We see that eq.(27) is *a type of the first order non-linear partial differential equation.* The nonlinearity is due to the form of $n^{-1}\,\vec{\nabla}n$. *Physically, eq.(27) says that the rays are bent in the direction of increasing the refractive index*[9,35].

The dimension of curvature of eq.(27) can be extended to any arbitrary number of dimensions[36]. In the $(4+d)$-dimensions of unified space, eq.(27) can be written as

$$R_{\mu\nu} = g\ N_\mu\ \partial_\nu \ln n \tag{28}$$

where $R_{\mu\nu}$ is the second rank tensor of Ricci curvature[15,37], a function of the metric tensor $g_{\mu\nu}$ and $g = |(\det\ g_{\mu\nu})|$, is a scalar, a real number. Why do we formulate the curvature in eq.(28) as the second rank tensor of Ricci curvature? It is because of the related refractive index in eq.(28) is the zeroth rank tensor, a scalar.

Why do we need to formulate a curvature in a fibre bundle? Actually, the fibre bundle and the gauge theory are developed independently. Until it was realized that the curvature in the fibre bundle and the field strength in Yang-Mills theory are identical[38]. *Simply speaking, the curvature in the fibre bundle is the field strength in the gauge theory.* Because we treat the geometrical optics as the $U(1)$ gauge theory so we need to formulate the curvature in the curvature-refractive index relation of the $U(1)$ geometrical optics as *the Abelian curvature form* in the fibre bundle. Probably, this is another reason why we do really need to formulate the curvature in the curvature form of the fibre bundle instead of the Riemann-Christoffel curvature tensor. The curvature form in the fibre bundle is able to be an Abelian (or a non-Abelian) which is not for the Riemann-Christoffel curvature tensor[39].

The curvature form, $\Omega_{\rho\sigma}$, can be written as[40,41]

$$\Omega_{\rho\sigma} = \sum R_{\rho\sigma\mu\nu}\ du^\mu \wedge du^\nu \tag{29}$$

where $R_{\mu\nu\rho\sigma}$ is the fourth rank tensor of Riemann-Christoffel curvature, u^μ, u^ν are local coordinates and $\wedge$ is a notation of wedge product. If we reformulate eq.(28) using the eq.(29) and a relation of $R_{\mu\nu} = g^{\rho\sigma} R_{\rho\sigma\mu\nu}$, we obtain

$$\Omega_{\rho\sigma} = \sum g\ g_{\rho\sigma}\ N_\mu\ \partial_\nu \ln n\ du^\mu \wedge du^\nu \tag{30}$$

Eq.(30) shows the relation between the scalar refractive index and the curvature form in a $(4+d)$-dimensional unified space formulated in the fibre bundle.

Let us introduce the general form of *the curvature matrix*, Ω, which can be written as below[40]

$$\Omega = d\omega - \omega\wedge\omega \tag{31}$$

where ω is *the connection matrix.* We see that eq.(31) is identical with eq.(10). Both equations are non-Abelian, non-linear equations, where the curvature matrix, Ω, is identical with the field strength, $G_{\mu\nu}^{SU(2)}$ and the connection matrix, ω, is identical with the gauge potential, $\vec{B}_\mu^{SU(2)}$. Roughly speaking, in case of the $U(1)$ gauge theory[42], we have

$$\Omega = d\omega \tag{32}$$

Is there a relationship between the curvature matrix, Ω (31), and the curvature form, $\Omega_{\rho\sigma}$ (29)? Yes, there is[43]. If $\omega_{\rho\sigma}$ and $\Omega_{\rho\sigma}$ denote *the components* of the connection and the curvature matrices, ω and Ω, respectively[40,41,44], then we can write

$$\Omega_{\rho\sigma} = d\omega_{\rho\sigma} - {\omega_\rho}^\tau \wedge \omega_{\tau\sigma} \tag{33}$$

In case of the $U(1)$ gauge theory, from eqs.(32),(33), we have

$$\Omega_{\rho\sigma} = d\omega_{\rho\sigma} \tag{34}$$

Eq.(34) is the equation of Abelian curvature form.

Substituting eq.(34) into eq.(30), we obtain

$$d\omega_{\rho\sigma} = \sum g\ g_{\rho\sigma}\ N_\mu\ \partial_\nu \ln n\ du^\mu \wedge du^\nu \tag{35}$$

where n is given below

$$\left|\partial_\nu\left(\frac{c}{f_\theta}\arccos\left\{\mathrm{Re}\left[\left(A_\mu^{U(1)}\hat{m}^{U(1)} - \frac{1}{g}\hat{m}^{U(1)}\times\partial_\mu\hat{m}^{U(1)}\right)\underline{a}^\mu\right]\right\} + ct\right)\right| = n \tag{36}$$

as in eq.(24). Eq.(35) shows explicitly the relation between a scalar refractive index and the Abelian curvature form.

We see from eqs.(35), (36), the refractive index is decomposed into the unrestricted scalar electric potential, $A_\mu^{U(1)}$, and the restricted vector magnetic potential, $\hat{m}^{U(1)}$. The decomposed refractive index also contains an information of the magnetic monopole as a topological object. *So, what is the consequence of the decomposed refractive index to the Abelian curvature form? Is the Abelian curvature form also decomposed? Does the Abelian curvature form also contain an information of the topological object? If the answer of these questions are positive, what is the topological object in the Abelian curvature form? What does it mean physically?*

ACKNOWLEDGMENT

Thank to Professor Yongmin Cho for introducing me the restricted gauge potential theory. Thank to Richard Tao Roni Hutagalung, Andri Sofyan Husein, Roniyus Marjunus for fruitful discussions. Thank to Reviewer for reviewing this manuscript, for many questions and suggestions. Special thank to beloved ones, Juwita Armilia and Aliya Syauqina Hadi, for much love and great hope. To Ibunda and Ayahanda, may Allah bless them with Jannatul Firdaus.

[1] P.A.M. Dirac, *Quantised Singularities in the Electromagnetic Field*, Proc. Roy. Soc. **A 133** 60 1931.

[2] P.A.M. Dirac, *The Theory of Magnetic Poles*, Physical Review Volume 74, Number 7 October 1, 1948.

[3] Y.M. Cho, *Restricted gauge theory*, Physical Review D, Volume 21, Number 4, 15 February 1980.

[4] The internal space is an abstract space where the magnetic symmetry "lives". We can not "see" this internal space due to the symmetry we are assumed (Y.M. Cho, Pong Soo Jang, *Unified Geometry of Internal Space with Spacetime*, June 1975).

[5] Cho restricted gauge theory is a self consistent subset of a non-Abelian $SU(2)$ gauge theory which tries to describe the *infrared regime* of the Yang-Mills gauge theory (Sedigheh Deldar, Ahmad Mohamadnejad, *Quark Confinement in Restricted $SU(2)$ Gauge Theory*, https://arxiv.org/pdf/1208.2165.pdf, 2012).

[6] David Tong, *Gauge Theory*, http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html, 2018.

[7] Kazuo Ota Cottrell, Jong Ping Hsu, *Gauge independence of the eikonal equation in Yang-Mills gravity*, Eur. Phys. J. Plus (2015) **130**: 147.

[8] Max Born, Emil Wolf, *Principles of Optics*, Pergamon Press, 1993.

[9] L.D. Landau, E.M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press, 1984.

[10] L.D. Landau, E.M. Lifshitz, *The Classical Theory of Fields*, Fourth Revised English Edition, Butterworth-Heinemann, 1987.

[11] A.B. Balakin, A.E. Zayats, *Non-minimal Wu-Yang monopole*, https://arxiv.org/pdf/gr-qc/0612019.pdf, 2006.

[12] Alexander B. Balakin, Alexei E. Zayats, *Non-minimal Einstein-Maxwell theory: the Fresnel equation and the Petrov classification of a trace-free susceptibility tensor*, https://arxiv.org/pdf/1710.08013.pdf, 2018.

[13] A.B. Balakin, A.E. Zayats, *Ray Optics in the Field of a Nonminimal Dirac Monopole*, Gravitation and Cosmology, 2008, Vol.14, No.1, pp.86-94.

[14] Ashok Das, *Lectures of Quantum Field Theory*, 2008. Mark Srednicki, *Quantum Field Theory*, 2007. Kerson Huang, *Quarks, Leptons and Gauge Fields*, 1992. I.J.R. Aitchison, A.J.G. Hey, *Gauge Theories in Particle Physics*, 2002. Ta Pei Cheng, Ling-Fong Li, *Gauge Theory of Elementary Particle Physics*, 2000. Franz Gross, *Relativistic Quantum Mechanics and Field Theory*, 1999.

[15] Moshe Carmeli, *Classical Fields: General Relativity and Gauge Theory*, John Wiley and Sons, Inc., 1982.

[16] We see from eq.(2), the metric is invariant under translation and rotation transformations. It has a consequence that the space is homogeneous and isotropic, respectively.

[17] Y.M. Cho. *Abelian Dominance in Wilson Loops*, https://arxiv.org/abs/hep-th/9905127 21 Jun 2000.

[18] Michael Atiyah, *Mathematics in the 20th Century*, Bull. London Math. Soc. **34** (2002) 1–15. DOI: 10.1112/S0024609301008566.

[19] We treat the four-vector potential, $\vec{B}_\mu$, is the same as wave field, ϕ, (any component of $\vec{E}$ or $\vec{H}$) given by a formula of the type $\phi = ae^{i\psi}$, where the amplitude a is a slowly varying function of coordinates and time and phase (eikonal), ψ, is a large quantity which is "almost linear" in coordinates and the time (L.D. Landau, E.M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press, 1984.). Both $\vec{B}_\mu$ and ϕ are solutions of the wave equation.

[20] H.J. Pain, *The Physics of Vibrations and Waves*, John Wiley and Sons Limited, 3rd Edition, 1983.

[21] Wikipedia, *Amplitude*.

[22] The time derivative of phase, ψ, gives the angular frequency of the wave, $\partial\psi/\partial t = -\omega$ and the space derivatives of ψ gives the wave vector, $\vec{\nabla}\psi = \vec{k}$, which shows the direction of the ray propagation through any point in space (L.D. Landau, E.M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press, 1984).

[23] G. Molesini, *Geometrical Optics*, Encyclopedia of Condensed Matter Physics, https://www.sciencedirect.com/topics/physics-and-astronomy/geometrical-optics, 2005.

[24] Nigel Higson, John Roe, *The Atiyah-Singer Index Theorem*.

[25] Miftachul Hadi, *On the geometrical optics and the Atiyah-Singer index theorem*, https://vixra.org/abs/2108.0006, 2021.

[26] Roniyus Marjunus, *Private communications*.

[27] Karsten Rottwitt, Peter Tidemand-Lichtenberg, *Nonlinear Optics: Principles and Applications*, CRC Press, 2015.

[28] Miftachul Hadi, *Linear and non-linear refractive indices in Riemannian and topological spaces*. RINarxiv, https://rinarxiv.lipi.go.id/lipi/preprint/view/18, 2020.

[29] Miftachul Hadi, Utama Alan Deta, Andri Sofyan Husein, *Linear and non-linear refractive indices in curved space*, Journal of Physics: Conference Series **1796** (2021) 012125.

[30] Y.M. Cho, *Private communication*.

[31] See e.g. Charles W. Misner, Kip S. Thorne, John Archibald Wheeler, *Gravitation*, W.H. Freeman and Company, 1973, p.573.

[32] The refractive index is often described as a real value. However, in a lossy material, the attenuation of the electric field is described through an imaginary part of the refractive index (Karsten Rottwitt, Peter Tidemand-Lichtenberg, *Nonlinear Optics: Principles and Applications*, CRC Press, 2015).

[33] If we compare the Fermat's principle, $\delta\psi_1 = 0$, and the least action principle, $\delta S = 0$, we see that the eikonal, ψ_1, looks similar with the action, S.

[34] Soma Mitra, Somenath Chakrabarty, *Fermat's Principle in Curved Spacetime, No Emission from Schwarzschild Black Holes as Total Internal Reflection and Black Hole Unruh Effect*, https://arxiv.org/pdf/1512.03885.pdf, 2015.

[35] Light changes its direction of propagation when it encounters *an inhomogeneity* in the medium. The curvature of the path is used to quantify this change of direction. This curvature is defined as the ratio of the change in the direction of propagation to the length measured along the curved path (K. Iizuka, *Engineering Optics*, 2008, p.104).

[36] The Riemann-Christoffel curvature tensor and the subsequent formulas, such as Ricci curvature tensor, Ricci curvature scalar, are valid in spaces of arbitrary number of dimensions (Moshe

Carmeli, *Classical Fields: General Relativity and Gauge Theory*, John Wiley and Sons, Inc., 1982.)

[37] Richard L. Faber, *Differential Geometry and Relativity Theory: An Introduction*, Marcel Dekker, Inc., 1983.

[38] Chen Ning Yang, *Topology and Gauge Theory in Physics*, International Journal of Modern Physics A, Vol. 27, No. 30 (2012) 1230035. DOI: 10.1142/S0217751X12300359.

[39] The Christoffel symbol does not transform as a tensor, but rather as an object in the jet bundle (Wikipedia, *Christoffel symbols*). If the non-linear term (non-Abelian term) of the Christoffel symbol happens to be zero in one coordinate system, it will in general not be zero in another coordinate system (Anonymous referee).

[40] Shiing-Shen Chern, *What is Geometry?* The American Mathematical Monthly, Vol. 97, No. 8, Special Geometry Issue (Oct., 1990), pp. 679-686.

[41] Shiing-Shen Chern, Wei-Huan Chen, Kai Shue Lam, *Lectures on Differential Geometry*, World Scientific, 2000.

[42] Mikio Nakahara, *Geometry, Topology and Physics*, Adam Hilger, 1991.

[43] Shing Tung Yau, *Private communication*.

[44] See e.g. Wikipedia, *Curvature form*, Shiing-Shen Chern, *What is Geometry?* The American Mathematical Monthly, Vol. 97, No. 8, Special Geometry Issue (Oct., 1990), pp. 679-686.