A Theorem and Conjecture on Nicomanchus' Theorem

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Abstract
In this paper, we prove that the sum of fifth powers of the first n natural numbers is a difference of two integers whose difference is a sum of the natural numbers. Furthermore, we give a general formula for all odd powers greater than one which serves as a conjecture.

Keywords: Faulhaber's theorem, Nicomanchus' theorem, Bernoulli formula.

1 Introduction

Sum of powers of the first n natural numbers has been of great interest to mathematicians over time. Jacob Bernoulli invented numbers that were greatly instrumental in creating a general formula for calculating the sum of natural powers of the first n natural numbers. This general formula is attributed to both Jacob Bernoulli and Johann Faulhaber because Johann Faulhaber knew at least the first 17 cases. Johann Faulhaber also notices that the sum of every odd power of the first n natural numbers is always divisible by the nth triangular number, which is called Faulhaber's thoeorem. Nicomanchus shows that every sum of cubes of the first n natural number is squared the sum of the natural numbers. (i.e.)

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$$

2 Main Result

Theorem : For all $n, p \in \mathbb{N}$, there exists $A \in \mathbb{N}$ such that

$$\sum_{k=1}^{n} k^{2p+1} = \left(A + \frac{n(n+1)}{2}\right)^2 - (A)^2 \tag{1}$$

Proof
P = 1

For p = 1 in (1), we can see that we have Nicomanchus' theorem where $A = 0$ (l.e.)

$$\sum_{k=1}^{n} k^3 = \left(0 + \frac{n(n+1)}{2}\right)^2 - (0)^2$$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

From (1), we see that

$$\sum_{k=1}^{n} k^{2p+1} = \left(A + \frac{n(n+1)}{2}\right)^2 - (A)^2$$

$$\sum_{k=1}^{n} k^{2p+1} = A^2 + 2\frac{n(n+1)}{2}A + \left(\frac{n(n+1)}{2}\right)^2 - A^2$$

$$\sum_{k=1}^{n} k^{2p+1} = 2\frac{n(n+1)}{2}A + \left(\frac{n(n+1)}{2}\right)^2 \tag{2}$$

P = 2

For p = 2 in (2), we have

$$\sum_{k=1}^{n} k^5 = 2\frac{n(n+1)}{2}A + \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{k=1}^{n} k^5 = n(n+1)A + \left(\frac{n(n+1)}{2}\right)^2 \tag{3}$$

we know that

$$(n-1)\left(\frac{n(n+1)}{2} + \frac{n(n+1)(n-1)}{6}\right)$$

is an integer for all natural n so, we may assume that

$$A = (n-1)\left(\frac{n(n+1)}{2} + \frac{n(n+1)(n-1)}{6}\right)$$

So,

$$A = (n-1)\left(\frac{n(n+1)}{2} + \frac{n(n+1)(n-1)}{6}\right)$$

$$A = \frac{n(n^2-1)}{2} + \frac{n(n^2-1)(n-1)}{6}$$

$$A = \frac{n^3-n}{2} + \frac{n^4-n^3-n^2+n)}{6}$$

$$A = \frac{3n^3 - 3n + n^4 - n^3 - n^2 + n}{6}$$

$$A = \frac{n^4 + 2n^3 - n^2 - 2n}{6} \tag{4}$$

putting (4) in (3), we have

$$\sum_{k=1}^{n} k^5 = n(n+1)\left(\frac{n^4 + 2n^3 - n^2 - 2n}{6}\right) + \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{k=1}^{n} k^5 = \frac{n^6 + 3n^5 + n^4 - 3n^3 - 2n^2}{6} + \frac{n^4 + 2n^3 + n^2}{4}$$

$$\sum_{k=1}^{n} k^5 = \frac{2n^6 + 6n^5 + 2n^4 - 6n^3 - 4n^2 + 3n^4 + 6n^3 + 3n^2}{12}$$

$$\sum_{k=1}^{n} k^5 = \frac{2n^6 + 6n^5 + 5n^4 - n^2}{12} \tag{5}$$

We can see that (5) is true according to Bernoulli or Faulhaber formula for the sum of fifth powers of the first n natural numbers and that, our assumption for A is true. Therefore,

$$\sum_{k=1}^{n} k^5 = \left((n-1)\left(\frac{n(n+1)}{2} + \frac{n(n^2-1)}{6}\right) + \frac{n(n+1)}{2}\right)^2 - \left((n-1)\left(\frac{n(n+1)}{2} + \frac{n(n^2-1)}{6}\right)\right)^2$$

3 Conjecture

Is (1) always true for all natural $p > 2$?

We can generate a proposed general formula that works for all $p > 2$ to concentrate on in order to prove this conjecture.

By Faulhaber's theorem, we know that if $p > 0$, $\left(\frac{n(n+1)}{2}\right)$ is always a factor of $\sum_{k=1}^{n} k^{2p+1}$ so, we can say that

$$\sum_{k=1}^{n} k^{2p+1} = \left(\frac{n(n+1)}{2}\right) Q_{2p+1}(n) \tag{6}$$

for some positive integer $Q_{2p+1}(n)$

By equating (2) and (6), we see that

$$2\left(\frac{n(n+1)}{2}\right) A + \left(\frac{n(n+1)}{2}\right)^2 = \left(\frac{n(n+1)}{2}\right) Q_{2p+1}(n)$$

$$2A + \left(\frac{n(n+1)}{2}\right) = Q_{2p+1}(n)$$

$$2A = Q_{2p+1}(n) - \left(\frac{n(n+1)}{2}\right)$$

$$A = \frac{1}{2}\left(Q_{2p+1}(n) - \left(\frac{n(n+1)}{2}\right)\right) \tag{7}$$

Putting (7) in (1), we have

$$\sum_{k=1}^{n} k^{2p+1} = \left(\frac{1}{2}\left(Q_{2p+1}(n) - \left(\frac{n(n+1)}{2}\right)\right) + \frac{n(n+1)}{2}\right)^2 - \left(\frac{1}{2}\left(Q_{2p+1}(n) - \left(\frac{n(n+1)}{2}\right)\right)\right)^2 \tag{8}$$

We can see that to prove (8) is true, it suffices to show that (7) is always an integer which in turn shows that (1) is true.

4 Conclusion

In this paper, we have proven that the sum of fifth power of the first n natural numbers is a difference of two squares whose difference is a sum of the natural number. Also, we have shown that the same statement might be true for all natural odd powers greater than five, which indicates that Nicomanchus' Theorem might be a special case of another theorem.