Limiting fluctuations in quantum gravity to diffeomorphisms

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Abstract

Within the background field formalism of quantum gravity, I show that if the quantum fluctuations are limited to diffeomorphic transformations, all the quantum corrections vanish on shell and the effective action is equivalent to the classical action. I also show that this choice of fields renders the path integral independent of the on-shell condition for the background field, and therefore incorporates a form of background independence. The proposed approach may provide insight into the development of a finite and background independent description of quantum gravity.

INTRODUCTION

The development of a quantum field theory of gravitation remains a challenge. Since the gravitational coupling constant has units of length, the quantum corrections correspond to higher-derivative terms with divergent coefficients [1–3]. In order to renormalize these terms, counterterms of a similar form must also be included in the bare classical action [4–7]. However, the addition of higher derivative terms leads to ghosts and a violation of unitarity in flat space perturbation theory [4, 5, 8].

On the other hand, it is well known that pure quantum gravity is finite at one loop order [1, 2, 9]. The one-loop divergence vanishes on shell, i.e., when the classical equations of motion are imposed on the background fields. Equivalently, it can absorbed by a background field redefinition [3, 10, 11] or suitable choice of gauge parameters [9, 12]. Unfortunately, this approach does not work when coupled to matter [1, 13] or at two-loop order, where a divergent term cubic in the curvature tensor remains on shell [14–17].

The dependence of the divergences on the choice of background fields is also problematic from the point of view of background independence, which is generally believed to be an essential component of quantum gravity [18]. In classical general relativity, background independence follows from diffeomorphism invariance and results in a lack of dependence on a fixed, non-dynamical background metric [19, 20]. In the background field method, background dependence arises through the perturbative expansion [3, 21, 22]. The expansion around a classical background leads to dependence of the effective action on the choice of on-shell conditions for the background fields [9, 12]. Ultimately, a proper incorporation of background independence should lead to an effective action which is independent of the choice of on shell conditions.

A related aspect is the dependence of the effective action on the choice of gauge and parameterization of the quantum field [9, 23–29]. While on shell the effective action is independent of the parameterization and gauge choice, off shell the effective action depends on both. This dependence has been shown both generally [9, 25, 30] and explicitly in several cases [27, 31–34]. The vanishing of the gauge and parameterization dependence on shell implies that if the quantum corrections are constrained to such variations, they would vanish on shell. Shifting the quantum corrections off shell would preclude nonrenormalizable higher derivative terms and ghosts and guarantee unitarity of the S-matrix.

In this Letter, I show that this situation can occur if the quantum fluctuations of the background field are constrained to diffeomorphic gauge transformations. In this case, all of the quantum corrections reduce to a delta function in the on shell condition, so the on-shell effective action is equal to the classical action with no quantum corrections. In principle, the resulting theory is finite and requires no renormalization. I also show that this choice of fields renders the effective action independent of the on shell condition for the background fields, and thus incorporates a form of background independence. Lastly, I discuss the implications of associating quantum fluctuations with gauge transformations and the relation to other renormalizable field theories.

APPROACH

The classical Einstein-Hilbert action for the gravitational field $g_{\mu\nu}$ is

$$S[g_{\mu\nu}] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}g^{\mu\nu}R_{\mu\nu}, \tag{1}$$

where G is Newton's constant and $R_{\mu\nu}$ is the Ricci tensor. The corresponding quantum theory is described by the effective action $\Gamma[\bar{g}_{\mu\nu}]$, a functional of the classical mean field $\bar{g}_{\mu\nu}$ [23–26, 28]. In the background field method, the effective action is obtained from a functional integral [23–26]

$$\exp\left(\frac{i}{\hbar}\Gamma[\bar{g}_{\mu\nu}]\right) = \int \mathcal{D}h_{\mu\nu}\exp\left(\frac{i}{\hbar}S[\bar{g}_{\mu\nu}+h_{\mu\nu}] - \frac{i}{\hbar}\int d^4x\sqrt{-\bar{g}}h^{\mu\nu}\frac{\delta\Gamma[\bar{g}_{\mu\nu}]}{\delta\bar{g}^{\mu\nu}}\right). \tag{2}$$

where $g_{\mu\nu}$ has been divided into the sum of $\bar{g}_{\mu\nu}$ and the quantum fluctuation $h_{\mu\nu}$ as [3, 21, 22]

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \tag{3}$$

At this stage, one normally proceeds by expanding $S[\bar{g}_{\mu\nu}+h_{\mu\nu}]$, adding gauge fixing and ghost terms, and performing the functional integral over $h_{\mu\nu}$ [23–26, 35]. However, there are several issues with this approach. First, it is not background independent. The linear term in the expansion is proportional to the classical equations of motion for the background fields, so in general the effective action depends on the choice of on-shell conditions. Second, the functional integral is not renormalizable. This can be seen mathematically by recognizing that Eq. (2) represents the functional Fourier transform of $\exp\left(\frac{i}{\hbar}S[\bar{g}_{\mu\nu}+h_{\mu\nu}]\right)$ with respect to the conjugate variables $h_{\mu\nu}$ and $\delta\Gamma[\bar{g}_{\mu\nu}]/\delta\bar{g}^{\mu\nu}$. Thus, in the absence of S, the functional integral over $h_{\mu\nu}$ results in a delta functional in the on shell condition as $\delta\left[\delta\Gamma[\bar{g}_{\mu\nu}]/\delta\bar{g}^{\mu\nu}\right]$. This implies that any dependence of $S[\bar{g}_{\mu\nu}+h_{\mu\nu}]$ on $h_{\mu\nu}$ will produce (nonrenormalizable) terms on shell. The only way to ensure that all quantum corrections vanish on shell is for $S[\bar{g}_{\mu\nu}+h_{\mu\nu}]$ to be independent of $h_{\mu\nu}$, that is $h_{\mu\nu}$ must correspond to a diffeomorphic gauge transformation

$$h_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \tag{4}$$

In this case, $S[\bar{g}_{\mu\nu}+h_{\mu\nu}] = S[\bar{g}_{\mu\nu}]$ and the expression for the effective action reduces to

$$\exp\left(\frac{i}{\hbar}\Gamma[\bar{g}_{\mu\nu}]\right) = \exp\left(\frac{i}{\hbar}S[\bar{g}_{\mu\nu}]\right)\int \mathcal{D}\xi_\mu \det\left(\frac{\delta h_{\mu\nu}}{\delta\xi_\mu}\right)\exp\left(-\frac{i}{\hbar}\int d^4x\sqrt{-\bar{g}}\nabla^\nu\xi^\mu\frac{\delta\Gamma[\bar{g}_{\mu\nu}]}{\delta\bar{g}^{\mu\nu}}\right),$$

where the determinant is the Jacobian of the diffeomorphism. Since $\delta h_{\mu\nu}/\delta\xi_\mu = \nabla_\nu$, the determinant is independent of ξ_μ and can be brought outside the integral. Then, performing integration by parts on the integrand and ignoring the boundary term,

$$\exp\left(\frac{i}{\hbar}\Gamma[\bar{g}_{\mu\nu}]\right) = \exp\left(\frac{i}{\hbar}S[\bar{g}_{\mu\nu}]\right)\det\left(\nabla_\nu\right)\int \mathcal{D}\xi_\mu \exp\left(\frac{i}{\hbar}\int d^4x\sqrt{-\bar{g}}\xi^\mu\nabla^\nu\frac{\delta\Gamma[\bar{g}_{\mu\nu}]}{\delta\bar{g}^{\mu\nu}}\right). \tag{5}$$

Recognizing the integral over ξ_μ as a functional delta function, this reduces to

$$\exp\left(\frac{i}{\hbar}\Gamma[\bar{g}_{\mu\nu}]\right) = \exp\left(\frac{i}{\hbar}S[\bar{g}_{\mu\nu}]\right)\det(\nabla_\nu)\delta\left[\nabla^\nu\frac{\delta\Gamma[\bar{g}_{\mu\nu}]}{\delta\bar{g}^{\mu\nu}}\right]. \tag{6}$$

Applying the scaling property for delta functions, ∇^ν inside the delta function cancels with the determinant, leaving

$$\exp\left(\frac{i}{\hbar}\Gamma[\bar{g}_{\mu\nu}]\right) = \exp\left(\frac{i}{\hbar}S[\bar{g}_{\mu\nu}]\right)\delta\left[\frac{\delta\Gamma[\bar{g}_{\mu\nu}]}{\delta\bar{g}^{\mu\nu}}\right]. \tag{7}$$

Therefore, on shell the effective action is equal to the classical action,

$$\frac{\delta\Gamma[\bar{g}_{\mu\nu}]}{\delta\bar{g}^{\mu\nu}} = 0, \qquad \Gamma[\bar{g}_{\mu\nu}] = S[\bar{g}_{\mu\nu}]. \tag{8}$$

This is the main result: when the quantum fluctuations are constrained to diffeomorphic transformations, all the quantum corrections vanish on shell. The resulting theory is finite to all orders and requires no renormalization.

Another appealing aspect of this approach is that it incorporates a form of background independence, namely the effective action is independent of the on shell condition for the background fields. To see this, consider the variation of the action

$$S[\bar{g}_{\mu\nu}+h_{\mu\nu}] = S[\bar{g}_{\mu\nu}] - \frac{1}{8\pi G}\int d^4x\sqrt{-\bar{g}}\bar{G}^{\mu\nu}(\nabla_\mu\xi_\nu), \tag{9}$$

where $\bar{G}^{\mu\nu} = \bar{R}^{\mu\nu} - \frac{1}{2}\bar{R}\bar{g}^{\mu\nu}$ is the on-shell condition for the background field. Integrating by parts, the variation can be written as a boundary term, which vanishes under reparameterization, and a term proportional to $\nabla_\mu \bar{G}^{\mu\nu}$, which is zero by the Bianchi identity. Therefore, when $h_{\mu\nu}$ is a diffeomorphism, $S[\bar{g}_{\mu\nu} + h_{\mu\nu}]$, and thus $\Gamma[\bar{g}_{\mu\nu}]$, is independent of the on shell condition for the background fields, which is a form of background independence.

From the equation of motion for the effective action $\delta\Gamma[\bar{g}_{\mu\nu}]/\delta\bar{g}^{\mu\nu} = -T_{\mu\nu}$, it follows that Eq. (8) applies to pure gravity with no external sources. When sources are present, the more general form in Eq. (6) must be used involving the Jacobian for diffeomorphisms. This determinant also arises in gauge fixing of conformal field theories. It can be exponentiated using the Faddeev-Popov determinant as [36–40]

$$\det\left(\frac{\delta h_{\mu\nu}}{\delta\xi_\mu}\right) = \int \mathcal{D}b_{\alpha\nu}\mathcal{D}c^\alpha \exp\left(i\int d^4x\sqrt{-\bar{g}}\bar{g}^{\mu\nu}c^\alpha\nabla_\mu b_{\alpha\nu}\right), \tag{10}$$

where $b_{\alpha\nu}$ is a symmetric and traceless tensor and c^α is an antisymmetric vector. Inserting this expression into Eq. (6), and assuming a conserved stress tensor ($\nabla^\nu T_{\mu\nu} = 0$),

$$\exp\left(\frac{i}{\hbar}\Gamma[\bar{g}_{\mu\nu}]\right) = \exp\left(\frac{i}{\hbar}S[\bar{g}_{\mu\nu}]\right)\int \mathcal{D}b_{\alpha\nu}\mathcal{D}c^\alpha \exp\left(i\int d^4x\sqrt{-\bar{g}}\bar{g}^{\mu\nu}c^\alpha\nabla_\mu b_{\alpha\nu}\right). \tag{11}$$

Thus, for gravity coupled to matter with a conserved stress tensor, the effective action is equal to the classical action with an additional reparameterization ghost. This ghost must be included to maintain consistency of the path integral measure. Physically, they represent the residual degrees of freedom when $h_{\mu\nu}$ is constrained to vector gauge transformations. The energy momentum tensor associated with the ghosts is [40, 41]

$$T_{\mu\nu} = c^\alpha\nabla_\mu b_{\alpha\nu} - \frac{1}{2}g_{\mu\nu}c^\alpha\nabla^\beta b_{\alpha\beta}, \tag{12}$$

which in flat space perturbation theory would give rise to an additional ghost-graviton vertex linear in the internal momenta [42]. The equations of motion for the ghosts are [38–40]

$$\nabla^\nu b_{\alpha\nu} = 0, \quad \nabla^\nu c^\alpha + \nabla^\alpha c^\nu = 0. \tag{13}$$

Importantly, since the ghost action vanishes on shell, it only impacts the vertices of internal loop diagrams and does not affect the S-matrix.

Lastly, it is worth discussing the implications of associating quantum fluctuations with gauge transformations. Normally, fields related by gauge transformations are regarded as physically equivalent, and therefore considered redundant in the path integral. To resolve this, one fixes the gauge and limits the integration to gauge-inequivalent fields. In the approach described here, the situation is reversed: only gauge-equivalent metrics are included. This raises the question of why only gauge-equivalent fields should be counted for gravitation. This clearly requires more investigation. However, it is worth noting that in a renormalizable field theory, the quantum corrections leave the form of the action invariant. Only fields, coupling constants, and masses are renormalized. In this sense, the proposed approach puts gravitation on a similar footing by constraining the quantum fluctuations in such a way that the form of the classical action is invariant after quantization.

SUMMARY

Within the background field method of quantum gravity, I have shown that if the quantum fluctuations are limited to diffeomorphic transformations, the quantum corrections to the effective action reduce to a delta function in the on shell condition. Therefore, on-shell the effective action is equal to the classical action with no quantum corrections. When coupled to matter with a conserved stress tensor, I show that an additional reparameterization ghost must be included for consistency. In addition, I have shown that this choice of fields renders the effective action independent of the on shell condition for the background field, which is a form of background independence. The proposed approach may provide insight into the development of a finite and background independent theory of quantum gravity.

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[1] G. 't Hooft and M. Veltman, Ann. Inst. Henri Poincaré **20**, 69 (1974).

[2] E. Alvarez, Rev. Mod. Phys. **61**, 561 (1989).

[3] H. W. Hamber, *Quantum Gravitation: The Feynman Path Integral Approach* (Springer, 2009).

[4] K. S. Stelle, Phys. Rev. D **16**, 953 (1977).

[5] E. Tomboulis, Phys. Lett. B **97**, 77 (1980).

[6] E. S. Fradkin and A. A. Tseytlin, Nucl. Phys. B **201**, 469 (1982).

[7] M. Asorey, J. L. Lopez, and I. L. Shapiro, Int. J. Mod. Phys. A **12**, 5711 (1997).

[8] I. L. Shapiro, Phys. Lett. B **744**, 67 (2015).

[9] R. E. Kallosh, O. Tarasov, and I. V. Tyutin, Nucl. Phys. B **137**, 145 (1978).

[10] G. 't Hooft, NATO Adv. Study Inst. Ser. B Phys. **5**, 263 (1974).

[11] G. 't Hooft, NATO Adv. Study Inst. Ser. B Phys. **44**, 323 (1979).

[12] D. Capper and J. Dulwich, Nucl. Phys. B **221**, 349 (1983).

[13] S. Deser and P. Nieuwenhuizen, Phys. Rev. D **10**, 401 (1974).

[14] M. H. Goroff and A. Sagnotti, Phys. Lett. B **160**, 81 (1985).

[15] M. H. Goroff and A. Sagnotti, Nucl. Phys. B **266**, 709 (1986).

[16] A. van de Ven, Nucl. Phys. B **378**, 309 (1992).

[17] Z. Bern, H. H. Chi, L. Dixon, and A. Edison, Phys. Rev. D **95**, 046013 (2017).

[18] D. Rickles, S. French, and J. Saatsi, *The structural foundations of quantum gravity* (Oxford University Press, 2006) pp. 196–239.

[19] C. Rovelli and M. Gaul, *Towards quantum gravity* (Springer, 2000) pp. 277–324.

[20] C. Rovelli, *Quantum gravity* (Cambridge university press, 2004).

[21] L. Abbott, Acta Phys. Pol. B **13**, 33 (1981).

[22] B. Giacchini, P. Lavrov, and I. Shapiro, Phys. Lett. B **797**, 134882 (2019).

[23] G. Vilkovisky, Nucl. Phys. B **234**, 125 (1984).

[24] A. O. Barvinsky and G. A. Vilkovisky, Phys. Rep. **119**, 1 (1985).

[25] I. L. Buchbinder, S. D. Odintsov, and I. L. Shapiro, *Effective Action in Quantum Gravity* (IOP Publishing, 1992).

[26] I. G. Avramidi, arXiv preprint **hep-th/9510140** (1995).

[27] N. Ohta, R. Percacci, and A. Pereira, J. High. Energ. Phys. **6**, 1 (2006).

[28] B. Giacchini, T. de Paula Netto, and I. Shapiro, Phys. Rev. D **102**, 106006 (2020).

[29] B. Giacchini, T. de Paula Netto, and I. Shapiro, J. High Energy Phys. **10**, 1 (2020).

[30] D. Goncalves, T. de Paula Netto, and I. L. Shapiro, Phys. Rev. D **D97**, 026015 (2018).

[31] G. de Berredo-Peixoto, A. Penna-Firme, and I. L. Shapiro, Mod. Phys. Lett. **A15**, 2335 (2000).

[32] M. Y. Kalmykov, Class. Quant. Grav. **12**, 1401 (1995).

[33] M. Kalmykov, K. Kazakov, P. Pronin, and K. Stepanyantz, Class. Quant. Grav. **15**, 3777 (1998).

[34] K. Falls, Phys. Rev. D **92**, 124057 (2015).

[35] P. Lavrov and I. Shapiro, Phys. Rev. D **100**, 026018 (2019).

[36] M. Green, J. Schwarz, and E. Witten, *Superstring Theory: Introduction, Vol. 1.* (Cambridge University Press, 1988).

[37] J. Polchinski, *An Introduction to the Bosonic String* (Cambridge University Press, 2005).

[38] A. Zamolodchikov and A. Zamolodchikov, *Lectures on Liouville Theory and Matrix Models* (2007).

[39] H. Erbin, *Notes on 2d quantum gravity and Liouville theory* (2015).

[40] H. Erbin, *String Theory: A Field Theory Perspective* (2021).

[41] J. Qualls, arXiv preprint **arXiv:1511.04074** (2015).

[42] G. 't Hooft and M. Veltman, *Diagrammar*, Tech. Rep. 73-9 (CERN, 1973).