An elementary proof of $0.999\cdots = 1$

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1 Abstract

One of the properties distinguishing irrational and rational numbers is the uniqueness (or the lack thereof) of their decimal representations. For example, the numbers π and 1 can be used as specimens of this phenomenon, as π has precisely one expression as decimals, but $1 = 1.0 = 1.00 = \ldots$. In this paper, we provide an elementary proof for the fact that $0.999\ldots$ is also a decimal representation of 1, using the Lebesgue measure.

2 Preliminaries

We will refer to these well-known [1] facts throughout this paper.

Lemma 2.1. *The following are true:*
(1) Given any interval $A \subseteq \mathbb{R}$, its Lebesgue measure satisfies $m(A) = 0$ iff A has only countably many points.
(2) The only intervals with countably many points are degenerate intervals (singletons) and the empty set. In particular, any non-vacuous open interval has positive measure.
(3) If A is not degenerate, then $m(A) = \max(A) - \min(A)$.

3 The main result

We are now ready to prove the

Theorem 3.1. $0.999\cdots = 1$.

Proof. We will argue by contradiction. Assume that $0.999\cdots < 1$. There is then an open interval $\emptyset \neq A \subseteq (0.999\ldots, 1)$. By Lemma 2.1(1), $m(A) > 0$. Let $(a_n)_{n\in\mathbb{N}}$ be the sequence

$$a_0 = 0, a_{n+1} = a_n + 9 \cdot 10^{-(n+1)}.$$

Clearly, $a_n \to 0.999\ldots$ and $1 - a_n = 10^{-n} \to 0$ when $n \to \infty$. Defining $A_n = (a_n, 1)$, we obtain a sequence

$$(0,1) \supset (a_1, 1) \supset (a_2, 1) \supset \cdots$$

of open intervals, such that $A \subseteq \bigcap_{n\in\mathbb{N}}(a_n, 1)$. However,

$$m(\bigcap_{n\in\mathbb{N}}(a_n, 1)) = \lim_{n\to\infty} m((a_n, 1)) = \lim_{n\to\infty} 10^{-n} = 0,$$

by 2.1(3). It follows from 2.1(2) that the intersection consists of at most one element, but then $0 \leq |A| \leq 1$, contradicting $m(A) > 0$. Thus, we conclude that $0.999\cdots = 1$. □

4 References

[1] Obviously true.