Vortic and thermodynamic fine structures of *real Schur flows*

Jian-Zhou Zhu (朱建州)@sccfis.org

Abstract

A two-component-two-dimensional coupled with one-component-three-dimensional (2C2Dcw1C3D) flow may also be called a real Schur flow (RSF), as its velocity gradient is uniformly of real Schur form. The thermodynamic and 'vortic' fine structures of 2C2Dcw1C3D flows are exposed and, in particular, the Lie invariances of the decomposed vorticity 2-forms of RSFs in d-dimensional Euclidean space $\mathbb{E}^d$ for any interger $d \geq 3$ are also proved. The two Helmholtz theorems of the complementary components of vorticity found recently in 3-space RSF is not coincidental, but underlied by a general decomposition theorem, thus essential. Many Lie-invariant fine results, such as those of the combinations of the entropic and vortic quantities, including the invariances of the decomposed Ertel potential vorticit 3-formsy (and their multiplications by any interger powers of entropy), then follow.

1 Introduction

The *Taylor columns* are ubiquitous in rotating fluid systems (c.f., e.g., Ref. [1] for a historical account). A 'column' indicates the two-dimensional structure, and, when referring to the velocity field in the rotation plane, it means the total 3-space velocity has a gradient matrix *uniformly* of the *real Schur form* (RSF), particularly for the formal Taylor-Proudman limit of compressible rotating flows.[2] On the other hand, any real matrices are *similar* to the Schur form, and, has appeared in studies of *local* flow patterns [3] and dynamics.[4] The flow with the velocity gradient being globally of uniform RSF may be called a *real Schur flow* (also RSF without ambiguity in the context). The RSF has a two-component-two-dimensional coupled with one-component-three-dimensional (2C2Dcw1C3D) structure,[4] thus is distinct from the traditional two-dimensional passive scalar or two-dimensional-three-component (2D3C) flow used recently with an attempt to bridge the two-dimensional and three-dimensional turbulence.[5] 2C2Dcw1C3D flow appears more naturally in between the 2D3C and the most general three-component-three-dimensional (3C3D) flows.

Compressible RSF deserves further theoretical attention: such a basic (anisotropic) flow may be considered as the corner stone of any flows, including the isotropic turbulence. For example, a helical RSF has been argued to be 'fastened' by the helicity through the Taylor-Proudman effect and has been proposed to play the role of *chiral base flow* for understanding the helicity effect on the compressibility of a turbulence,[6] which however may have only scratching on the surface. The other properties of RSF may also be useful for understanding general turbulence and deserve further clarification. Here, we focus on the most fundamental thermodynamic structures needed to maintain the 2C2Dcw1C3D dynamics and 'vortic' (related-to-vortex-dynamics) properties from 2C2Dcw1C3D character of the velocity field. In the latter, unlike the conventional constructions (e.g., Refs. [7, 11, 9, 10, 8]), a set of Lie invariances will be derived from studying which and how many independent components of the vorticity 2-form are invariant.

An internal motivation is to address the concerns: whether the two Helmholtz/frozen-in laws found in Ref. [4] is merely a coincidence, or there is something deeper or more essential underlying it. And, how about the structures of other thermodynamic variables (most recently Ref. [8] and references therein)? The former questions lead to considering general barotropic ideal RSF in d-dimensional Euclidean[12] space $\mathbb{E}^d$ for $d \geq 3$,[13] while the last question will also be addressed with more general situations.

2 Analysis

Differential forms are special covariant tensors and the vorticity 2-form is simply the antisymmetric second order one which can also be represented with a matrix. For example, the vorticity exterior derivative of the 1-form $U := \sum_{i=1}^{d} u_i dx_i$, corresponding to the velocity vector $\boldsymbol{u} := \{u_1, u_2, ..., u_d\}$, reads, with the index '$_{,i}$' $\leftrightarrow$ '∂_{x_i}', for $d = 4$ in co-ordinate form

$$\begin{aligned} dU \quad &= (u_{2,1} - u_{1,2})dx_1 \wedge dx_2 + (u_{3,1} - u_{1,3})dx_1 \wedge dx_3 + \\ &+(u_{4,1} - u_{1,4})dx_1 \wedge dx_4 + (u_{3,2} - u_{2,3})dx_2 \wedge dx_3 + \\ &+(u_{4,2} - u_{2,4})dx_2 \wedge dx_4 + (u_{4,3} - u_{3,4})dx_3 \wedge dx_4, \end{aligned} \tag{1}$$

whose matrix representation, for the components associated to the bases $dx_i \wedge dx_j$, writes

$$\begin{pmatrix} 0 & \frac{u_{1,2}-u_{2,1}}{2} & \frac{u_{1,3}-u_{3,1}}{2} & \frac{u_{1,4}-u_{4,1}}{2} \\ \frac{u_{2,1}-u_{1,2}}{2} & 0 & \frac{u_{2,3}-u_{3,2}}{2} & \frac{u_{2,4}-u_{4,2}}{2} \\ \frac{u_{3,1}-u_{1,3}}{2} & \frac{u_{3,2}-u_{2,3}}{2} & 0 & \frac{u_{3,4}-u_{4,3}}{2} \\ \frac{u_{4,1}-u_{1,4}}{2} & \frac{u_{4,2}-u_{2,4}}{2} & \frac{u_{4,3}-u_{3,4}}{2} & 0 \end{pmatrix}. \tag{2}$$

The matrix representation of $\nabla \boldsymbol{u}$ in $\mathbb{E}^d$ is

$$G = \begin{pmatrix} u_{1,1} & u_{2,1} & u_{3,1}| & u_{4,1}\| & u_{5,1} & u_{6,1} & \cdots & u_{d,1} \\ u_{1,2} & u_{2,2} & u_{3,2}| & u_{4,2}\| & u_{5,2} & u_{6,2} & \cdots & u_{d,2} \\ \underline{u_{1,3}} & \underline{u_{2,3}} & \underline{u_{3,3}}| & u_{4,3}\| & u_{5,3} & u_{6,3} & \cdots & u_{d,3} \\ \underline{\underline{u_{1,4}}} & \underline{\underline{u_{2,4}}} & \underline{\underline{u_{3,4}}} & \underline{\underline{u_{4,4}}}\| & u_{5,4} & u_{6,4} & \cdots & u_{d,4} \\ u_{1,5} & u_{2,5} & u_{3,5} & u_{4,5} & u_{5,5} & u_{6,5} & \cdots & u_{d,5} \\ u_{1,6} & u_{2,6} & u_{3,6} & u_{4,6} & u_{5,6} & u_{6,6} & \cdots & u_{d,6} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ u_{1,d} & u_{2,d} & u_{3,d} & u_{4,d} & u_{5,d} & u_{6,d} & \cdots & u_{d,d} \end{pmatrix}. \tag{3}$$

The three-dimensional (3D) case corresponds to the left-upper 3×3 block, the RSF of which is arranged to have its left-upper 2×2 block corresponding to the two conjugate complex eigenvalues of G, thus two vanishing left-lower elements (in blue color) indicating a two-component-two-dimensional coupled with one-component-three-dimensional (2C2Dcw1C3D [4]) flow: when all eigenvalues are real, $u_{1,2}$ also vanishes, which is a stronger condition but which in general does not lead to stronger results in our discussions, thus will not be particularly discussed; similarly is for $d > 3$. [The RSF is nonunique, depending on how the order of the eigenvalues or the corresponding coordinates are arranged, which however is not essential.] G can be decomposed into symmetric and anti-symmetric parts, $D = (G + G^T)/2$ and $A = (G - G^T)/2$, the latter, given in Eq. (2) for $d = 4$, may be viewed as a representation of the vorticity 2-form $\Omega = dU$.

Let's start with the equation in $\mathbb{E}^3$ for the RSF with density ρ, pressure p, velocity $\boldsymbol{u}$ and, for simplicity, the dissipation term $D(\boldsymbol{u}) = \nu \nabla^2 \boldsymbol{u}$ with constant kinetic viscosity ν:[14]

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{4}$$

$$\partial_t \boldsymbol{u}_h + \boldsymbol{u}_h \cdot \nabla_h \boldsymbol{u}_h = -\rho^{-1} \nabla_h p + \nu \nabla_h^2 \boldsymbol{u}_h, \tag{4b}$$

$$\partial_t u_3 + \boldsymbol{u}_h \cdot \nabla_h u_3 + u_3 u_{3,3} = -\rho^{-1} p_{,3} + \nu \nabla^2 u_3, \tag{4c}$$

where x_1 and x_2 are the '$_h$orizontal' coordinates and the corresponding $\boldsymbol{u}_h := \{u_1, u_2\}$ is independent of the 'vertical' coordinate x_3, i.e.,

$$\boldsymbol{u}_{h,3} \equiv 0. \tag{5}$$

Higher-dimensional case can be similarly formulated. For the barotropic case, $(\nabla p)/\rho = \nabla \Pi$, and the isothermal (constant-temperature) relation $p = c^2 \rho$ results in $\nabla \Pi = c^2 \nabla \ln \rho$, where c is the sound speed.

In Ref. [4], the author noticed that the incompressible RSF would have very restrictive structures, allowing even no periodic solution, but richer structures can present in the compressible one. Indeed, we see no reason to exclude periodic solutions for compressible RSF and actually preliminary numerical tests have shown the realizability of RSF turbulence in a cyclic box, but specific analysis of the latter, especially the numerical investigation of its detailed statistical dynamics, however belongs to another communication for different purpose.[15] Note that the selective cylindrical condition for the horizontal flow also presents in the Taylor-Proudman limit of fast rotating compressible flows,[2] as remarked earlier, thus such a 2C2Dcw1C3D flow is quite of physical sense, instead of being purely artificial. However, the Taylor-Proudman limit has extrally $\nabla_h \cdot \boldsymbol{u}_h = 0$ which is not required by our general RSF.

2.1 Thermodynamic structures

The RSF has not only the defining characteristics in the velocity but also some particular thermodynamic structures. Below, we will derive the results from the 2C2Dcw1C3D velocity field for the 3-space dynamics.

Taking derivative with respect to x_3 in Eq. (4b) for the horizontal momentum, Eq. (5) requires [16]

$$[(\nabla_h p)/\rho]_{,3} = 0. \tag{6}$$

2.1.1 Barotropic structures

The barotropic Eq. (6) writes $\Pi_{,13} = \Pi_{,23} = 0$, i.e.,

$$\Pi = \mathscr{P}_3(x_3) + \mathscr{P}_h(x_1, x_2). \tag{7}$$

We may consider the RSF in a box of dimension $L_z \times L_2 \times L_3$, cyclic in each direction, or with $L \to \infty$ in some direction(s) with the field vanishing sufficiently fast. Introducing

$$\langle \mathscr{P}_h \rangle_{12} := L_1^{-1} L_2^{-1} \int_0^{L_2} \int_0^{L_1} \mathscr{P}_h(x_1, x_2) dx_1 dx_2,$$

$$\langle \mathscr{P}_3 \rangle_3 := L_3^{-1} \int_0^{L_3} \mathscr{P}_3(x_3) dx_3$$

and

$$\langle \Pi \rangle_{123} := L_1^{-1} L_2^{-1} L_3^{-1} \int_0^{L_1} \int_0^{L_2} \int_0^{L_3} \Pi dx_1 dx_2 dx_3,$$

we have

$$\langle \Pi \rangle_3 = \mathscr{P}_h(x_1, x_2) + \langle \mathscr{P}_3 \rangle_3, \tag{8a}$$

$$\langle \Pi \rangle_{12} = \mathscr{P}_3(x_3) + \langle \mathscr{P}_h \rangle_{12}, \tag{8b}$$

$$\langle \Pi \rangle_{123} = \langle \mathscr{P}_h \rangle_{12} + \langle \mathscr{P}_3 \rangle_3 = \langle \Pi \rangle_{12} + \langle \Pi \rangle_3 - \Pi. \tag{8c}$$

Rewriting Eq. (4) as $\partial_t \ln\rho = -\boldsymbol{u}\cdot\nabla\ln\rho - \nabla\cdot\boldsymbol{u}$ and taking particularly

$$\Pi = c^2\ln\rho = c^2(\langle\ln\rho\rangle_{12} + \langle\ln\rho\rangle_3 - \langle\ln\rho\rangle_{123})$$

for the isothermal case, we have the partial-integral-differential equation

$$\partial_t \ln\rho = \langle\boldsymbol{u}\cdot\nabla\ln\rho\rangle_{123} - \langle\boldsymbol{u}\cdot\nabla\ln\rho + u_{3,3}\rangle_{12} - \langle\boldsymbol{u}\cdot\nabla\ln\rho + u_{1,1} + u_{2,2}\rangle_3, \tag{4a}$$

which summarizes Eqs. (4,5 and 6) and which, as we will see, applies also in more general nonbarotropic RSFs.

2.1.2 Nonbarotropic structures

For an ideal gas with, say, $p = \rho\mathcal{R}T$ with $\mathcal{R}$ being a constant, Eq. (6) reads

$$[\frac{\nabla_h(\rho T)}{\rho}]_{,3} = [(T\nabla_h\ln\rho) + \nabla_h T]_{,3} = 0. \tag{9}$$

Eq. (9) indicates that $\frac{\nabla_h(\rho T)}{\rho}$ is a function of only x_1 and x_2 and should have such separation of the variables

$$\rho = r(x_3)/R(x_1,x_2) \tag{10}$$

that the nominator and demominator can cancel the common factor $r(x_3)$. In other words, we have the same structure, especially Eq. (4a), as in the isothermal case. And, by taking (10) into (9), we further have

$$T(x_1,x_2,x_3) = \mathcal{T}(x_1,x_2) + R(x_1,x_2)\tau(x_3), \tag{11}$$

where $\mathcal{T}$ and τ and other variables are time dependent. They characterize the fine structures of T, which again may be exploited for numerical simulations or for check of the precision of the data: this may not appear to be obvious due to simultaneous appearance of x_1, x_2 and x_3 in the second term of the right-hand side (RHS) of Eq. (11), thus we make some elaborations in the following.

From Eq. (11), we have for four (superscript) values of x_3

$$\frac{T(x_1,x_2,x_3^2) - T(x_1,x_2,x_3^1)}{T(x_1,x_2,x_3^4) - T(x_1,x_2,x_3^3)} = \frac{\tau(x_3^2)-\tau(x_3^1)}{\tau(x_3^4)-\tau(x_3^3)}; \tag{12}$$

and, for two sets of (superscript) values, respectively, of x_1, x_2 and of x_3, we have

$$\frac{T(x_1^1,x_2^1,x_3^2) - T(x_1^1,x_2^1,x_3^1)}{T(x_1^2,x_2^2,x_3^2) - T(x_1^2,x_2^2,x_3^1)} = \frac{R(x_1^1,x_2^1)}{R(x_1^2,x_2^2)}. \tag{13}$$

Now the RHS of Eq. (12) is independent of x_1 and x_2 and the RHS of Eq. (13) is independent of x_3, both presenting clear fundamental physical properties that may be used for numerical simulation and check.

2.1.3 Adiabatic ideal Lie-invariant structures

In Ref. [4], the author found an interesting 'vortic' property which is that the frozen-in/Helmholtz theorem of vorticity is decomposed into two invariant laws for the 'horizontal' and 'vertical' vorticities. Logically speaking, it was not clear whether such a frozen-in decomposition was coincidental, happening due to other implicitly used condition(s) such as the dimension number, or essential. Since the general material/frozen-in invariance is described by the Lie invariance, particularly that for the barotropic ideal vorticity $(\partial_t + L_u)dU = 0$,[9] which applies in any dimension number, we thus are curious whether it is underlied by some more essential Lie-invariant theorem for flows in general d-dimensional space.

In particular, for adiabatic ideal flow, we have the 0-form/scalar entropy S which is Lie-invariant

$$(\partial_t + L_u)S = 0. \tag{14}$$

Thus, $S^s dU$ with any integrer s is still a Lie-invariant 2-form and may be called the "'s-entropic' or 'sentropic' vorticity", and the frozen-in decomposition of vorticity in Ref. [4] can be extended to such 2-forms. Similarly, the Ertel 'potential vorticity' 3-form $dS \wedge dU$ [11, 17, 18, 10, 8] and the "'sentropic' potential vorticity' $S^s dS \wedge dU$ are also accordingly decomposible. But, again, these entropy relevant results, already involving the vorticity, are so far only for flows in $\mathbb{E}^3$, and the essentiality should be examined in $\mathbb{E}^d$ for general $d > 3$, which is the target of Sec. 2.2.

2.2 Vortic structures

In terms of differential forms, the inviscid [$\nu = 0$ in Eqs. (4b,4c), say] equations of the horizontal and total velocities read,

$$\partial_t U_h + L_{\boldsymbol{u}_h} U_h = -d_h(\Pi - \boldsymbol{u}_h^2/2), \tag{15}$$

$$\partial_t U + L_{\boldsymbol{u}} U = -d(\Pi - \boldsymbol{u}^2/2), \tag{16}$$

which also applies for general *ideal* real Schur flows in $\mathbb{E}^d$ with $d \geq 3$, the precise meaning of U_h for $d > 3$ to be further clarified below. Thus, with the interchanges of the exterior derivative with the (partial) time derivative and the Lie derivative, and, with the replacement of $L_{\boldsymbol{u}_h}$ by $L_{\boldsymbol{u}}$ (Lemma 1), we have the Lie-invariance laws for the vorticity 2-forms $\Omega_h = dU_h$ and $\Omega = dU$,

$$\partial_t \Omega_h + L_{\boldsymbol{u}} \Omega_h = 0, \tag{17}$$

$$\partial_t \Omega + L_{\boldsymbol{u}} \Omega = 0. \tag{18}$$

We already can see that the two frozen-in laws of decomposed vorticities (for which the spatial derivatives of $\boldsymbol{u}_h$ and u_z are respectively responsible), found in Ref. [4] for $d = 3$, correspond to Eq. (17) and, according to the linearity of Lie derivative operator, the substraction by it of Eq. (18), but now also for compressible barotropic RSF. [Such a result however does not guarantee a direct extension to $d > 3$, because, as said, in the latter case the precise meaning of the index $_h$ needs special clarification.] The objective of frozen-in decomposition is then translated to Lie-invarant decomposition which can be formulated for general d as the following:

Definition 1 *A Lie-invariant decomposition of a (Lie-invariant) vorticity 2-form Ω into $M \geq 1$ components of a barotropic ideal flow in $\mathbb{E}^d$ is that*

$$\Omega = \sum_{i=1}^{M} \Omega_i, \text{ with } (\partial_t + L_{\boldsymbol{u}})\Omega_i = 0, \tag{19}$$

where Ω_is are linearly independent.

Remark 1 *Obviously, $M \leq d(d-1)/2$ (no larger than the number of the elements of the A matrix), and when $M = 1$, $\Omega_1 = \Omega$. Actually, we will see that $M \geq [\frac{d+1}{2}]$, where [...] denotes the integer part.*

2.2.1 Observation

As remarked in Sec. 2.2, concerning the left-upper 3×3 RSF block (with $u_{1,2} = u_{1,3} = 0$, in blue color) designated with single underline and single right 'wall' for $\mathbb{E}^3$ in Eq. (3), the vorticity 2-form, corresponding to the anti-symmetric part of the left-upper 2×2 block, of the horizontal velocity $\boldsymbol{u}_h$ is Lie-carried by $\boldsymbol{u}_h$, and thus also by the whole 3-space $\boldsymbol{u}$ (Lemma 1). Since the whole $\Omega = dU$ is Lie invariant respect to $\boldsymbol{u}$, the vorticity 2-form component corresponding to anti-symmetric part of the right column of the matrix is also accordingly Lie invariant, from simple substraction.

Now in the 4-space with the left-upper 4×4 RSF block, designated with double underlines and double right walls, with extrally $u_{1,4} = u_{2,4} = 0$ (in blue color), the vorticity 2-form component corresponding to the anti-symmetric part of the left-upper 2×2 block is Lie-carried by the 4-space $\boldsymbol{u}$ (twice applications of Lemma 1). Thus, the vorticity 2-form component corresponding to the anti-symmetric part of the rest two columns on the right, but not the third column as in 3-space, is also accordingly Lie invariant, again, from simple substraction.

Then, in the 5-space, similar to the 3-space case, with the left-upper 5×5 RSF block with extrally $u_{1,5} = u_{2,5} = u_{3,5} = u_{4,5} = 0$. We see both the vorticity 2-form components, corresponding respectively to the anti-symmetric parts of the left-upper 2×2 block and to the anti-symmetric part of the third and fourth columns, and their sum as a component of the 5-space vorticity 2-form, are all Lie-carried by $\boldsymbol{u}$ in $\mathbb{E}^5$; thus, again from simple substraction, the vorticity 2-form component corresponding to the anti-symmetric part of the fifth column is also Lie invariant.

The above have implied an inductive procedure with ‘trivial’ (see below for preciseness) extension of dimension and substraction, by which we see how the 6- and higher-space RSF flows carry linearly independent Lie-invariant vorticity 2-form components:

Theorem 1 *(Lie-invariant vorticity decomposition). With $d+1$ understood to be adding an extra spatial dimension the velocity component of which is constant, i.e., appending an extra column and row of zeros to the bottom and right of the d-space velocity gradient matrix G, there are $M = [\frac{d+1}{2}]$ linearly independent Lie-invariant (with respect to the ideal barotropic flow in $\mathbb{E}^d$) vorticity 2-form components, each of which subsequently corresponding to the anti-symmetric part of the two columns of G associated to $\Omega_i = dU_i$ for*

$$U_i := u_{2i-1}dx_{2i-1} + u_{2i}dx_{2i}. \tag{20}$$

Remark 2 *When $i \neq d/2$, each U_i and Ω_i are respectively perpendicular and parallel to the dth coordinate, thus the notations U_{h_i} and Ω_{h_i}, corresponding to Eqs. (15,17), would also be justified.*

2.2.2 Proof of Theorem 1

Lemma 1 *The (component of) vorticity 2-form Lie-invariant with respect to a k-space velocity is also invariant when the space is trivially extended to $k+1$ dimensions, where ‘trivially’ refers to the property that the velocity components responsible for the vorticity 2-form do not depend on the extended spatial coordinate.*

With $\boldsymbol{u}$ extended “trivially” from a k-space velocity to a $(k+1)$-space $\boldsymbol{u}'$, the proof of Lemma 1 is straightforward by the result of $L_{\boldsymbol{u}}dU_{(i)} = L_{\boldsymbol{u}'}dU'_{(i)}$ derived from $L_{\boldsymbol{v}} = \iota_{\boldsymbol{v}}d + d\iota_{\boldsymbol{v}}$, where $\iota_{\boldsymbol{v}}$ is the interior product with the vector $\boldsymbol{v}$. Thus, we are ready to prove Theorem 1:

Proof. With the application of Lemma 1, the induction-fashion argument made in Sec. 2.2.1 can be adapted with simple replacements of the dimension numbers, 3, 4 and 5 there, with $k = 2n-1$, $2n$ and $2n+1$ for $n \geq 2$, to constitute our proof.

Finally, the linear independence of the $[\frac{d+1}{2}]$ components in Eq. (19) is obvious by the fact that the numbers of bases, $dx_m \wedge dx_n$, involved in different Ω_is are different, thus the complete proof follows. □

Remark 3 *Any linear combinations of Ω_is are also Lie-invariant. It is very tempting to say that Ω_is constitute the linearly independent Lie-invariant basis of the invariant vorticity (component) of RSF, but there can be more* *vanishing components than those left-lower corners of RSF G in Eq. (3), which then may lead to some very special Lie-invariant vorticity component(s) not representable by those Ω_is: that's why we said in Remark 1 that M could be larger than $[\frac{d+1}{2}]$.*

2.3 Structures of united thermodynamics and vortics

Leaving the remarks concerning those in Sec. 2.1.3 for higher-order 'mixed' forms, such as the Ertel potential vorticity, to Sec. 3 and staying still within second-order forms, we have, with Ω_i given in Theorem 1 and s introduced in Sec. 2.1.3, the following:

Corollary 1 *The 'sentropic' vorticity 2-forms, $S^s\Omega_i$, are Lie invariant in the barotropic ideal flows.*

3 Further discussions

The conventional construction of higher-order Lie invariants then can be applied to the decomposed objects; that is, we have, with U_i and $\Omega_i = dU_i$ in Theorem 1, the following:

Corollary 2 *Any linear-sum and wedge-product combinations of Ω_is, and, particularly "'sentropic' potential vorticities" defined by the wedge products of S, dS and Ω_i, say, $S^s dS^m \wedge (\Omega_i + \Omega_j)^n$, are also locally invariant.*

Other higher-order pure thermodynamic quantities (Ref. [8] and references therein) accordingly have delicate structures, which however appears not demonstrable in an explicit way so far and is left for future study.

Various new Cauchy invariants equations associated to the decomposed components can be further established by further applying the main theorem of Besse & Frisch,[10] which will explicitly present the fine Lagrangian structures. As we remarked on the thermodynamic fine structures, such fine structures may also be useful for Lagrangian numerical techniques and for check of the preciseness of the simulations.

Using matrix decomposition or transformation techniques, one may perform various other decompositions of the flow and/or the vorticity, which, if being lack of the consideration on the dynamics (such as the Lie invariance respect to the flow discussed in the above), may be called 'kinetic'. For example, besides the well-known decomposition of G into the symmetric and antisymmetric parts (respectively, D and A), Ref. [3] shows that the RSF G can be further transformed into the canonical form and then decomposed into the shear part S and the canonical part N, the latter further being composed of the dilation part E, the part for the strain rate along some eigenvector Z and the rotation part Ψ. For the antisymmetric A representing the vorticity, its canonical form is (block) diagonal with some 2×2 antisymmetric block(s) and all other elements vanishing:[20] each 2×2 antisymmetric block represents the (rate of) rotation in the plane (c.f., equation 23 in Ref. [2] and the discussion following it), and, if more than one, all the rotation planes are orthogonal to each other. We call such rotations 'pure'. For example, if the left-lower and right-upper 2×2 blocks of the matrix (2) are zeros, then the (pure) rotations are respectively in the x_1-x_2 and x_3-x_4 planes in the corresponding canonical coordinate frame. However, the A of RSF G is in general not in the canonical form, thus containing 'entangled' rotations. For example, G being of RSF though, none of the nondiagonal elements of matrix (2) is indicated to be vanishing, thus the possibility of simultaneous rotations in x_1-x_2, x_3-x_4, x_1-x_3 and x_2-x_4 planes in the corresponding coordinate frame. The entanglement of the rotations may lead to ambiguity and even confusions about vorticity, as already presents in 3D case: more than one rotation planes are in general involved in the vorticity, thus very complicated flow pattern from the latter, unless in the canonical frame where there always is only one plane for the pure rotation.

We conclude by returning to remark on the 'dynamical' decomposition: the study of compressibility reduction of helical turbulence in Ref. [6] has made use of helical RSF as the chiral base flow but has not yet exploited the ideal vorticity frozen-in decomposition, however we expect that the latter may be intrinsic to the fundamental mechanisms of other issues of *fully* d-dimensional flows (not RSF), including 3D incompressible turbulence.

Acknowledgments

Paritally supported by NSFC (Nos. 11672102). Discussions with H. Ren, C.-X. Tang and W.-N. Zou to design the algorithms for accurate simulations of RSF have been very motivative to establish some of the results here.

References

[1] O. U. Velasco Fuentes, Kelvin' s discovery of Taylor columns, *European Journal of Mechanics B/Fluids* **28**, 469 (2009).

[2] J.-Z. Zhu, "Fast rotating flows in high spatial dimensions", arXiv: 1905.11783

[3] C.f., Eq. (9) of Z. Li, X.-W. Zhang & F. He, Evaluation of vortex criteria by virtue of the quadruple decomposition of velocity gradient tensor, Acta Phys. Sin. **60**, 094702 (2014); W. Zou, Private communications (2020).

[4] J.-Z. Zhu, Vorticity and helicity decompositions and dynamics with real Schur form of the velocity gradient, Phys. Fluids **30**, 031703 (2018).

[5] L. Biferale, M. Buzzicotti, and M. Linkmann, From two-dimensional to three-dimensional turbulence through two-dimensional three-component flows. Phys. Fluids **29**, 111101 (2017).

[6] J.-Z. Zhu, Compressible helical turbulence: fastened-structure geometry and statistics, Phys. Plasmas **28**, 032302 (2021).

[7] S. Gama & U. Frisch, Local helicity, a material invariant for the odd-dimensional incompressible Euler equations. In Theory of Solar And Planetary Dynamos (ed. M. R. E. Proctor, P. C. Matthews & A. M. Rucklidge), pp. 115–119. Cambridge University Press (1993).

[8] S. C. Anco & G. M. Webb, Hierarchies of new invariants and conserved integrals in inviscid fluid flow, Phys. Fluids **32**, 086104 (2020).

[9] V.I. Arnold & B.A. Khesin, Topological methods in hydrodynamics. Springer (1998).

[10] N. Besse & U. Frisch, Geometric formulation of the Cauchy invariants for incompressible Euler flow in flat and curved spaces, *J. Fluid Mech.* **825**, 412 (2017).

[11] A. V. Tur & V. V. Yanovsky, Invariants in dissipationless hydrodynamic media. J. Fluid Mech. **243**, 67 (1993).

[12] The Euclidean metric is needed for defining energy, among others, thus we use $\mathbb{E}^d$ instead of $\mathbb{R}^d$.

[13] For most recent simulations of high-dimensional quantum and classical flows, c.f., T. Miyazaki, W. Kubo, Y. Shiga, T. Nakano, T. Gotoh, Classical and quantum turbulence, Physica D **239**, 1359 (2010), and, A. Berera, R. D. J. G. Ho, and D. Clark, Homogeneous isotropic turbulence in four spatial dimensions, Phys. Fluids **32**, 085107 (2020), and references therein for a comprehensive bibliography on various aspects. Note in particular that progresses have also been made for d-dimensional fluid mechanics on the vortex dynamics [B. N. Shashikanth, Vortex dynamics in $\mathbb{R}^4$, J. Math. Phys. **53**, 013103 (2012)] and regularity issues [e.g., H.-J. Dong & D.-P. Du, Partial Regularity of Solutions to the Four-Dimensional Navier-Stokes Equations at the First Blow-up Time, Commun. Math. Phys. **273**, 785 (2007); J. Liu & W. Wang, Boundary regularity criteria for the 6D steady Navier–Stokes and MHD equations, J. Differential Equations **264**, 2351 (2018), and references therein] and on the mutual enlightments between fluid dynamics and gravity/blackholes theory through the holographic duality [e.g., A. Adams, P.M. Chesler, and H. Liu, Holographic turbulence, Phys. Rev. Lett. **112**, 151602 (2014) and references therein.]

[14] This is the model usually used in incompressible flows; the more realistic Stokesian viscosity takes the dependence on the temperature and the divergence of velocity into account, but the difference is negligible unless the flow presents very strong shocks and/or very high temperatures. Other non-Newtonian models will make the situation even more complex. Nevertheless, our results are precise for the inviscid-limit ideal flows.

[15] Manuscript in preparation.

[16] In the Stokesian viscosity model with the dissipation term $D(\boldsymbol{u}) = \mu\nabla^2\boldsymbol{u} + \frac{\mu}{3}\nabla(\nabla\cdot\boldsymbol{u})$ for constant dynamical viscosity μ, we have instead $\partial_3(\frac{1}{\rho}\nabla_h\frac{\mu u_{3,3}}{3}) = \partial_3(\frac{1}{\rho}\nabla_h p)$, and more complex relations would be required for other non-Newtonian models.

[17] M. Fecko, Modern geometry in not-so-high echelons of physics: case studies. *Acta Physica Slovaca* **63**, 261 (2013).

[18] G. M. Webb, B. Dasgupta, J. F. McKenzie, Q. Hu, and G. P. Zank, Local and nonlocal advected invariants and helicities in magnetohydrodynamics and gas dynamics I: Lie dragging approach. J. Phys. A: Math. Theor. **47**, 095501 (2014); G. M. Webb, B. Dasgupta, J. F. McKenzie, Q. Hu, and G. P. Zank, Local and nonlocal advected invariants and helicities in magnetohydrodynamics and gas dynamics II: Noether' s theorems and Casimirs. J. Phys. A: Math. Theor. **47**, 095502 (2014).

[19] V. I. Oseledets, On a new way of writing the Navier-Stokes equation. The Hamiltonian formalism. Commun. Moscow Math. Soc. **44**, 210 (1989).

[20] J. W. Stander and N. A. Wiegman, Canonical forms for certain matrices under unitary congruence, Can. J. Math., 12 (1960).