Irrationality Proofs: From e to $\zeta(n \geq 2)$

Timothy W. Jones

August 23, 2021

Abstract

We develop definitions and a theory for convergent series that have terms of the form $1/a_j$ where a_j is an integer greater than one and the series convergence point is less than one. These series have terms with denominators that can be used as number bases. The series for $e-2$ and $z_n = \zeta(n) - 1$ are of this type. Further, both series yield number bases that can represent all possible rational convergence points as single digits. As partials for these series are rational numbers, all partials can be given as single decimals using some a_j as a base. In the case of $e-2$, the last term of a partial yields such a base and partials form systems of nesting inequalities yielding a proof of the irrationality of $e-2$. Using limits in an unusual way we are able to give a second proof for the irrationality of $e-2$. A third proof validates the second using Dedekind cuts. In the case of z_n, using the z_2 case we determine that such systems of nesting inequalities are not formed, but we discover partials require bases greater than the denominator of their last term. We prove this property for the general z_n case and, using the unusual limit style proof mentioned, prove z_n is irrational. We once again validate the proof using Dedekind cuts. Finally, we are able to give what we consider a satisfying proof showing why both $e-2$ and z_n are irrational.

Introduction

Apery's $\zeta(3)$ is irrational proof [1] and its simplifications [3, 11] are the only proofs that a specific odd argument for $\zeta(n)$ is irrational. The irrationality of even arguments of zeta are a natural consequence of Euler's formula [2]:

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1} \frac{2^{2n-1}}{(2n!)} B_{2n} \pi^{2n}. \tag{1}$$

Apery also showed $\zeta(2)$ is irrational, and Beukers, based on the work (tangentially) of Apery, simplified both proofs. He replaced Apery's mysterious recursive relationships with multiple integrals. See Poorten [12] for the history of Apery's proof; Havil [7] gives an overview of Apery's ideas and attempts to demystify them. Also of interest is Huylebrouck's [8] paper giving an historical context for the main technique used by Beukers. Papers by Poorten and Beukers are in *Pi: A Source Book* [4] and *The Number* π [5] gives Beukers's proofs (condensed) and related material. Both the proofs of Apery and Beukers require the prime number theorem and subtle $\epsilon - \delta$ reasoning.

Thus we have all even $\zeta(n)$ proven irrational using a classic formula and exactly one odd; whereas, you would think that both evens and odds could be proven in the same way.

Attempts to generalize the techniques of the one odd success seem to be hopelessly elusive. Apery's and other ideas can be seen in the work of Rivoal and Zudilin [13, 16]. Their results, that there are an infinite number of odd n such that $\zeta(n)$ is irrational and at least one of the cases 5,7,9, 11 likewise irrational do suggest a radically different approach is necessary.

We claim all $\zeta(n \geq 2)$ can be proven to be irrational by using what we call decimal sets and well known and relatively simple properties of decimal bases: all integers greater than one can be used as a number basis [6, Sections 9.1-9.3]. We still need the lesser cousin of the prime number theorem, Bertrand's postulate [6], and some new use of limits which we validate using Dedekind cuts. Like Apery and Beukers, we use a known irrational number, e, (they used $\zeta(2)$) as a proving ground and reader familiarization strategy for our decimal techniques.

Decimals and series

We give definitions that make a connection between certain convergent infinite series and number bases. Here is the idea. Every partial sum for an infinite series of fractions is a rational number that can be given as a single decimal in a specific base. We use the symbol $.(p)_q^k$ to designate that the partial with upper index k has for its first digit in base q the number (symbol) p. For some k, this digit becomes fixed; we designate this with $.(p)_q^{k+}$. If a partial is not equal to a single digit in the number base q, it is between two such numbers. As all rational numbers in $(0, 1)$, designated henceforth with $\mathbb{Q}(0, 1)$, can be represented as single decimal digits, if we can show a series convergent point, known to be in $(0, 1)$, is not equal to any such single digit, then we will have shown it is irrational. We use denominators of the fractions of the series as a source for number bases.

Definition 1. A *plus one* series is a convergent infinite series with a convergent point less than one and terms of the form $1/a_j$ with a_j a strictly increasing sequence of integers all greater than one. Partials for such sums are given by s_k where k is the upper index; the infinite series convergent point is given by s.

Examples of plus one series are the telescoping series, geometric series as given by infinite decimals in a base (2 varieties), $e - 2$, and $\zeta(n) - 1$. We will designate the partials sums of these series with $s_k(tele)$, $.(a-1)\overline{(b-1)}_k$, $.\overline{a}_{(k,b)}$, $s_k(e-2)$, and s_k^n. With similar notation for their convergence points.

Definition 2. A plus one series with denominators a_j is said to be complete if

$$B\{a_j\}_{j=1}^{\infty} = \mathbb{Q}(0, 1),$$

where $B\{a_j\}_{j=1}^{\infty}$ is the union of all single decimal numbers formed with a_j as number bases.

The series $s(tele)$, $s(e-2)$, and s_k^n are complete; $.(a-1)\overline{(b-1)}_k$ and $.\overline{a}_{(k,b)}$ are not complete.

As the partial sums of a plus-one series are all rational, a complete plus-one series must have partials that can be given as a single digit decimal using some term's denominator. The question is which denominator is used.

Definition 3. A plus-one series having partials s_k is said to be k-less, k-equal, or k-greater if s_k can be represented as a single decimal in a smallest base a_r where $r < k$, $r = k$, or $r > k$, respectively. If no such a_r exists the series is termed k-null.

Partials of a complete plus-one series will always be k-less, k-equal, or k-greater.[1] Incomplete series might be k-null.

Finally, plus-one series are convergent series. They can converge to a rational or irrational number in $(0, 1)$.

Definition 4. A plus-one series with convergent point s is said to be series-k-plus if there exists a smallest base a_r that can represent s as a single decimal. Such series are said to be series-k-null if no such a_r exists.

The following theorem is not surprising, but hopefully it cements the ideas given.

Theorem 1. *A complete series-k-null plus-one series converges to an irrational number.*

Proof. Suppose such a series converges to a rational number. Then that rational number can be represented in some base a_r as a single decimal digit. But a series-k-null series has no such a_r, a contradiction. $\square$

	s_3^n	s_4^n	s_5^n	...
2^n	WF	WF	WF	...
3^n	WF	WF	WF	...
4^n	WF	WF	WF	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	RF			
$\vdots$				
$\vdots$				
		RF		

Table 1: A term base partial table for zeta(n).

[1] It is possible that some partials are k-greater, say, and some are k-less, but this is not the case for the plus-one series mentioned in this article. The telescoping series is k-less, but has an oscillating feature; it is also *way* less, less than half.

Term Base Partial Tables

A complete k-greater series allows a systematic depiction of partials with decimals in a set of bases that would seem to force an irrational convergence point. In Table 1 partial sums for z_n are given in the top row and terms for these series are given in the left column. As the partial sums can't be given as finite decimals in the bases of the terms that define the partial sum, they must be mixed or repeating decimals. But convergence to a rational number implies that partials should have accruing fixed decimals of the form $.(a-1)\overline{(b-1)}_R+$, where R indicates the number of times the decimal $(b-1)$ repeats and the plus indicates additional non-fixed digits. By completeness such a base, the right form, RF must exist. As this RF migrates per the k-greater property, all decimal bases are eventually the wrong form, WF.

In Table 2

What is immediately of interest is systematically representing partial sums in bases given by terms. Table 1 shows the terms of $\zeta(n)-1$ being used as bases for its partial sums. If the convergence point is rational, say a/b, then for decimal base b partials will need to develop fixed decimals of the form $.(a-1)\overline{(b-1)}_{f(k)}$. We will show that this can't happen.

$$p_k = \sum_{j=1}^{k} t_j.$$

Examples

The following examples are plus-one series.

Example 1. The telescoping series

$$s_k(tele) = \sum_{j=2}^{k} \frac{1}{j} - \frac{1}{j+1} = \sum_{j=2}^{k} \frac{1}{j(j+1)} = \frac{k-1}{2(k+1)}$$

converges:

$$s(tele) = \frac{1}{2}.$$

The formula for partials shows that it is a k-less series. The formula for partial sums also shows that even ks cause a cancellation and a further reduction in the denominator. It is easy to show that it is complete.

Example 2. A geometric series is given by the repeating decimal $.\overline{1}$ base 10. Its partials can be represented by powers of 10 given by their last term; this series converges to 1/9, a number that can't be represented as a single decimal in base 10; it is, then, series-k-null.

Using this series one can form a sequence of nesting intervals:

$$.1 < .\overline{1} < .2$$

$$.1 < .11 < .\overline{1} < .12 < .2$$

$$\vdots$$

$$.1 < .11 < \cdots < .\overline{1}_k < \cdots < .\overline{1} < \cdots < .\overline{1}_{k-1}2 < \cdots < .12 < .2,$$

where the subscripts indicate how many times the 1 digit is repeated. Given a finite decimal $a = .a_1a_2 \ldots a_k$ in base 10, we claim $a \leq .\overline{1}_k$ or $a \geq .\overline{1}_{k-1}2$. This follows as all finite decimals are either less than 1/9 or greater; no finite decimal in base 10 can represent 1/9.

This partitioning of finite decimals into two sets, all those less than 1/9 and all those greater, forms a cut, like a Dedekind cut, in the set of finite decimals base 10.

Example 3. *We further develop our examples in this section.* The second $.2\overline{9}$ has partials that can be represented by powers of ten given by their last term; this series converges to .3, a number represented using its first and any other terms[2] and so it is a series-k-plus series.

Both of these geometric plus-one series are incomplete: using the series terms one can't represent $\mathbb{Q}(0,1)$ as single decimals. One can only represent finite decimals in base 10, 2 and 5 [6]. It should be noted that finite decimals like .25 can be converted into a single decimal by increasing the base to 10^2: $.25 = .(25)_{100}$, where (25) is a single digit in base 100.

Example 4. The number $e - 2$, its infinite series, is a plus-one, k-equal, complete series. We will prove these properties in the next section and use them to give a proof of the irrationality of this series.

[2]We will use the phrase *using the term* as shorthand for *using the denominator of the term as a number basis.*

Properties and irrationality of $e-2$

Consider the series

$$e - 2 = \sum_{j=2}^{\infty} \frac{1}{j!} = \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots. \tag{2}$$

As $2 < e < 3$, this series is a one-plus series. We will show that it is a complete, k-equal series.

Lemma 1. *The series* (2) *is complete.*

Proof. We simply note

$$\frac{p(q-1)!}{q!} = \frac{p}{q} = .(p(q-1)!)_{q!}.$$

The decimal is a single decimal in base $q!$ as $p < q$ implies $p(q-1)! < q!$. □

Lemma 2. *The series* (2) *is k-equal.*

Proof. We need to show that if

$$s_k = \sum_{j=2}^{k} \frac{1}{j!},$$

then $s_k = .(x)_{k!}$. That is partials can be expressed as single decimals using the denominator of the last term in the partial as a number basis.

As $k!$ is a common denominator of all terms in this partial sum, $s_k = .(x)_{k!}$, for some x, $1 \le x < k!$. The following induction argument shows that $k!$ is the least such factorial possible.

Clearly $2!$ is the least such factorial for the first partial. Suppose $k!$ is the least factorial for the kth partial. Let

$$s_{k+1} = \frac{x}{k!} + \frac{1}{(k+1)!} = \frac{y}{a!} \tag{3}$$

for some positive integers a and y. If $a \le k$, then multiplying (3) by $k!$ gives an integer plus $1/(k+1)$ is an integer, a contradiction. So $a > k$, but $a = k+1$ works, so it is the least possible factorial. □

Lemma 3. *For each integer $k > 1$, there exists decimal digits x and $x+1$ base $k!$ such that*

$$.(x)_{k!}^{k} < e - 2 < .(x+1)_{k!}^{k}. \tag{4}$$

Proof. By Lemma 2, $s_k = .(x)_{k!}^{k}$. We have, using a geometric series,

$$0 < (e-2) - s_k = \sum_{j=k+1}^{\infty} \frac{1}{j!} = \frac{1}{k!}\left(\frac{1}{(k+1)} + \frac{1}{(k+1)(k+2)} + \dots\right)$$

$$< \frac{1}{k!}\left(\frac{1}{(k+1)} + \frac{1}{(k+1)^2} + \dots\right) = \frac{1}{k}\frac{1}{k!} < \frac{1}{k!}.$$

That is $0 < e - 2 - .(x)_{k!}^{k} < 1/k!$. Adding $.(x)_{k!}$ and noting $.(x)_{k!} + 1/k! = .(x+1)_{k!}$, we have (4). □

Lemma 3 implies the boundary decimals don't change with increasing partial upper index: the middle expression in (4) is independent of k. We next show that nesting intervals are formed.

Lemma 4. *For every k,*

$$.(x-1)_{k!} < .(x-1)_{(k+1)!} < e - 2 < .(x)_{(k+1)!} < .(x)_{k!}, \tag{5}$$

where the factorial in the subscript indexes the digit.

Proof. We can rewrite (5) as

$$\sum_{j=2}^{k} \frac{1}{j!} < \sum_{j=2}^{k+1} \frac{1}{j!} < e - 2 < \sum_{j=2}^{k+1} \frac{1}{j!} + \frac{1}{(k+1)!} < \sum_{j=2}^{k} \frac{1}{j!} + \frac{1}{k!}$$

The first inequality is immediate. The right inequality can be rewritten as

$$\sum_{j=2}^{k} \frac{1}{j!} + \frac{1}{(k+1)!} + \frac{1}{(k+1)!} < \sum_{j=2}^{k} \frac{1}{j!} + \frac{1}{k!}.$$

Subtracting the right hand side from the left, the summations cancel giving equivalent inequalities:

$$\frac{1}{k!} - \frac{2}{(k+1)!} > 0 \iff 1 > \frac{2}{k+1} \iff k+1 > 2 \iff k > 1.$$

□

Recalling our superscript conventions, here are some examples:

$$.(1)_2^{1+} < e - 2 < (1)_2^{1+};$$

that is $1/2 < e - 2 < 1$;

$$.(1)_2^{1+} < .(4)_6^{2+} < e - 2 < .(5)_6^{2+} < (1)_2^{1+};$$

that is $1/2 < 4/6 < e - 2 < 5/6 < 1$; and

$$.(1)_2^{1+} < .(4)_6^{2+} < .(17)_{24}^{3+} < e - 2 < .(18)_{24}^{3+} < .(5)_6^{2+} < (1)_2^{1+}; \tag{6}$$

which is $1/2 < 4/6 < 17/24 < e - 2 < 18/24 < 5/6 < 1$.

We can get the idea for a proof of the irrationality of $e - 2$ using (6). Suppose $e-2$ is a rational number, a single digit in base 24. It would have to be inside $(.(17)_{24}, .(18)_{24})$, that is the open interval with $.(17)_{24}$ and $.(18)_{24}$ endpoints – not possible.

Theorem 2. *$e - 2$ is irrational.*

Proof. Suppose $e-2$ is rational, then by Lemma 1 there exists a k such that $e - 2 = .(x)_{k!}$, but by Lemma 3 for some y

$$.(y)_{k!}^{(k-1)+} < e - 2 = .(x)_{k!} < .(y+1)_{k!}^{(k-1)+}, \tag{7}$$

but no single digit in base $k!$ can be between two other single digits in the same base, a contradiction. ☐

Neat, sweet, petite!

Theorem 2 uses a proof by contradiction. But we can also use the properties developed for a proof by elimination.

Theorem 3. *$e - 2$ is irrational.*

Proof. Letting $\Xi_{(k-1)!}$ be the set of all rational numbers in $(0, 1)$ expressible with denominators $2!, \ldots, (k-1)!$ and $s_k(e-2)$ be the partial sum of the series for $e - 2$ with upper index k,

$$s_k(e-2) \in \mathbb{R}(0,1) \setminus \Xi_{(k-1)!}. \tag{8}$$

This uses $e - 2$ is k-equal (not k-less) and, using its completeness, as k goes to infinity, we have

$$e - 2 \in \mathbb{R}(0,1) \setminus \mathbb{Q}(0,1) = \mathbb{H}(0,1),$$

where $\mathbb{H}(0,1)$ is the set of irrational numbers in $(0,1)$. That is $e - 2$ is irrational. ☐

One might be a little skeptical of this second proof. It is unaccustomed, but we can give yet another proof that might explain why this second proof works. It involves Dedekind cuts.

First, a quick refresh of Dedekind cuts is in order. Dedekind cuts are typically, actually exclusively (to my knowledge) used to prove the existence of a structure that gives the real numbers and their properties [10, 14]. Dedekind cuts construct irrational numbers using rational numbers. Irrational numbers are defined as $(A|B)$ where A and B are sets of rational numbers; $A \cup B = \mathbb{Q}$, $A \cap B = \emptyset$; and if $a \in A$ and $b \in B$, then $a < b$. It is easy to comprehend the idea, imagine a real number line and put a tick mark on it (a cut) and define A as all the rational numbers less than or equal to where the cut is and define B as all the rational numbers greater than this cut. Rational numbers have a least upper bound in A and irrational numbers have neither a least upper bound in A nor a greatest lower bound in B.

For our purposes, an irrational Dedekind cut consists of sets A and B with single decimal elements that give in the union of A and B all rational numbers in $(0, 1)$ with A having no least upper bound given by its elements.

Theorem 4. *$e - 2$ is irrational.*

Proof. Let

$$A_k = \{.(y)_{k!} | .(y)_{k!} \leq .(x-1)_{k!} = s_k(e-2)\}$$

and

$$B_k = \{.(y)_{k!} | .(y)_{k!} \geq .(x)_{k!} = s_k(e-2)\}.$$

So, using are earlier observations, $A_2 = \{.(1)_{2!}\}$,

$$A_3 = \{.(1)_{3!}, .(2)_{3!}, .(3)_{3!}, .(4)_{3!}\},$$

and

$$A_4 = \{.(1)_{4!}, .(2)_{4!}, .(3)_{4!}, \ldots, .(15)_{4!}, .(16)_{5!}, .(17)_{4!}\}.$$

Similarly, $B_2 = \{(1)_{2!}\}$[3]

$$B_3 = \{.(5)_{3!}\},$$

and

$$B_4 = \{.(18)_{4!}, .(19)_{4!}, .(20)_{4!}, .(21)_{4!}, .(22)_{4!}, .(23)_{4!}\}.$$

[3]Technically, this is excluded from $(0, 1)$: $1 \notin (0, 1)$. Another inconsistency occurs with using rational numbers in $(0, 1)$ versus all rational numbers per Landau, for example. One can add all rational numbers less than $1/2$ to each of these A_k sets and all rationals greater than 1 to B_k to get strict, but unnecessary conformity.

A Dedekind cut is defined as $(A|B)$, where

$$A = \bigcup_{k=2}^{\infty} A_k$$

and

$$B = \bigcup_{k=2}^{\infty} B_k.$$

By (4) ($e-2$ partials trapped), A has no least upper bound in A and B has no greatest upper bound in B; this Dedekind cut defines an irrational number, $e-2$. □

Does Theorem 4 justify the limit reasoning in Theorem 3? The expression (8) implies that $s_k(e-2)$ cuts $(0,1)$ into two sets of rational numbers. The rational numbers are from $\Xi_{k!}$. In the limit $\mathbb{Q}(0,1)$ is cut into A and B sets as given in Theorem 4. Theorem 3 is, I suggest, a faster Theorem 4.

Table 2 summarizes the properties in the above example series.

Partial Sums	k-less	k-equal	k-greater
Incomplete		$.\overline{1}$, $.2\overline{9}$ base 10	
Complete	Telescoping	$e-2$	z_n (to be shown)

Table 2: Example series with decimal properties.

	series-k-plus	series-k-null
Incomplete	$.2\overline{9}$	$.\overline{1}$
Complete	Telescoping	$e-2$, z_n?

Table 3: Correlation between series properties and rational and irrational convergence points.

If a k-equal, complete series converges to an irrational number, what can be said of a k-greater, complete series?

The series z_2 appears k-greater

We use the following symbols:

$$z_n = \zeta(n) - 1 = \sum_{j=2}^{\infty} \frac{1}{j^n} \text{ and } s_k^n = \sum_{j=2}^{k} \frac{1}{j^n}.$$

In this section we will use z_2 in hopes a finding a general pattern.

As with the series for $e-2$, we can form systems of inequalities for z_2 using its partial sums and the denominators of their terms as number bases. With upper index 3 we derive inequalities for bases 4 and 9:

$$(.1)_4^3 < (.3)_9^3 < s_3^2 = .(13)_{36}^3 < .(4)_9^3 < .(2)_4^3. \tag{9}$$

For upper index 4, we derive another set of inequalities:

$$.(1)_4^4 < .(3)_9^4 < .(6)_{16}^4 < s_4^2 = .(61)_{144} < .(7)_{16}^4 < .(4)_9^4 < .(2)_4^4. \tag{10}$$

Unlike the $e-2$ case, single fixed digits are not immediately created with each increment of the upper index. The inequalities don't immediately nest. Continuing with just the bases 4, 9, and 16, we observe

$$.(1)_4^5 < .(7)_{16}^5 < .(4)_9^5 < s_5^2 = .(1669)_{3600} < .(8)_{16}^5 = .(2)_4^5 < .(5)_9^5. \tag{11}$$

Base 16 and base 9 have been transposed and, on the right, base 16 and base 4 endpoints collide (i.e. are equal). The next two iterations are

$$.(1)_4^6 < .(7)_{16}^6 < .(4)_9^6 < s_6^2 = .(1769)_{3600} < .(8)_{16}^6 = .(2)_4^6 < .(5)_9^6 \tag{12}$$

and

$$.(4)_9^7 < .(8)_{16}^7 = .(2)_4^{7+} < s_7^2 = .(90281)_{176400} < .(5)_9^7 < .(9)_{16}^7 < .(3)_4^{7+}. \tag{13}$$

The left and right digits for base 4 have migrated to $.(2)_4$ and $.(3)_4$. As $.(2)_4 < z_2 < .(3)_4$, these left and right values for base 4 are fixed for $k \geq 7$. The decimal digits for this base are fixed, as indicated by the plus sign in the superscripts. The inequalities don't nest immediately and the nesting can change, even collapse.

But we do see a pattern of interest in these inequalities: this z_2 series seems to be, as indicated in Table 2, a k-greater series: the basis needed for the partials we've calculated exceed the denominator of the last term of the partial. We will show z_n (and z_2) has this property in Corollary 1, nota bene general n. We will also show z_n is complete in Lemma 5. These properties will enable us to give a proof that z_n (both odd and even n) have irrational convergence points.

The proof will not be as neat, sweat, or petite as that for $e-2$. It may be a helpful, although a bit ghoulish picture that gives the central image thus far: the inequalities (9), (10), (11), (12), and (13) can be likened to various fingers being used to squeeze a tube of toothpaste; no matter who squeezes it and no matter how hard or soft, the resulting toothpaste is not like the deformed tube, the base needed for a single digit is bigger. The partial sums divide (read cut) rational points in two: those below and those above.

Properties of z_n

First a definition.

Definition 5. *Let*

$$d_{j^n} = \{1/j^n, \ldots, (j^n-1)/j^n\} = \{.1, \ldots, .(j^n-1)\} \text{ base } j^n.$$

That is d_{j^n} consists of all single decimals greater than 0 and less than 1 *in base j^n. The decimal set for j^n is*

$$D_{j^n} = d_{j^n} \setminus \bigcup_{k=2}^{j-1} d_{k^n}.$$

The set subtraction removes duplicate values.

Example 5. For z_2, $d_4 = \{.1, .2, .3\}_4$, where each single decimal is in base 4. The denominator for the 2nd term is 9; $d_9 = \{.1, .2, \ldots, .8\}_9$; the third term's denominator is 16; $d_{16} = \{.1, .2, .3, .4 = .(1)_4 \text{ out}, \ldots, .(15)\}_{16}$. The elements in d_{16} shared with the earlier set d_4 are removed: so

$$D_{16} = \{.1, .2, .3, .5, .6, .7, .9, .(10), .(11), .(13), .(14), .(15)\}_{16},$$

where the subscript is used to designate the number basis for the single decimals contained in the set.

Definition 6.

$$\bigcup_{j=2}^{k} D_{j^n} = \Xi_k^n$$

Example 6. The union using the first three terms of z_2 is

$$\Xi_4^2 = D_4 \cup D_9 \cup D_{16},$$

where each single decimal is represented by the least base possible.

We next show this union of decimal sets gives all rational numbers in $(0,1)$, the series z_n are complete.

Lemma 5. *The series z_n are complete.*

Proof. Every rational $a/b \in (0,1)$ is included in a d_{b^n} and hence in some D_{r^n} with $r \leq b$. This follows as $ab^{n-1}/b^n = a/b$ and as $a < b$, per $a/b \in (0,1)$, $ab^{n-1} < b^n$ and so $a/b \in d_{b^n}$. □

Next we will show z_n is k-greater. We use, once again, the z_2 case (with partials s_k^2) to look for helpful patterns. Table 4 gives some evidence that the reduced fractions giving partial sum totals have much larger denominators than the denominators of their last term: $36 > 3^2$; $144 > 4^2$; $3600 > 5^2$; $3600 > 6^2$; $176400 > 7^2$. We saw this earlier: (9), (10), (11), (12), and (13).

Table 4 also suggests a strategy for proving this. Notice that the prime factorization of s_k^2's denominators (the third column) have powers of 2 and a prime greater than half the k-value also to a power. This may translate to the reduced fraction's denominator is at least twice something greater than half; that's more than k; with sufficient powers of 2 and the prime mentioned that's more than the last term's denominator – it's, then, *k-greater*. Details follow. Apostol's *Introduction to Analytic Number Theory* [2](Chapter 2, problem 21), solution in [9], gives the general technique used in this section.

The remainder of this section is the hardest part of the paper and the most easily skipped on first reading. Corollary 1 below, specially in light of our second proof of the irrationality of $e - 2$, gives all that is needed for the rest of the paper. I think it is plausible enough.

Lemma 6. *If $s_k^n = r/s$ with r/s a reduced fraction, then 2^n divides s.*

k	s_k^2	Prime factorization
3	$.(13)_{36}$	$36 = 2^2 3^2$
4	$.(61)_{144}$	$144 = 2^4 3^2$
5	$.(1669)_{3600}$	$3600 = 2^4 3^2 5^2$
6	$.(1769)_{3600}$	$3600 = 2^4 3^2 5^2$
7	$.(90281)_{176400}$	$176400 = 2^4 3^2 5^2 7^2$

Table 4: The reduced fractions (given as decimals) have denominators (basis) divisible by powers of 2 and a prime greater than $k/2$.

Proof. The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, a; the set $\{2^n, 3^n, \ldots, k^n\}$ will have a greatest power of 2, na. Also $k!$ will have a powers of 2 divisor with exponent b; and $(k!)^n$ will have a greatest power of 2 exponent of nb. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + (k!)^n/3^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{14}$$

The term $(k!)^n/2^{na}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $nb - na$ for 2. As all other terms but this term will have more than an exponent of 2^{nb-na} in their prime factorization, we have the numerator of (14) has the form

$$2^{nb-na}(2A + B),$$

where $2 \nmid B$ and A is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^n/2^{na}$. The denominator, meanwhile, has the factored form

$$2^{nb}C,$$

where $2 \nmid C$. This leaves 2^{na} as a factor in the denominator with no powers of 2 in the numerator, as needed. □

Lemma 7. *If $s_k^n = r/s$ with r/s a reduced fraction and p is a prime such that $k > p > k/2$, then p^n divides s.*

Proof. First note that $(k, p) = 1$. If $p|k$ then there would have to exist r such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of such a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 6. Consider

$$\frac{(k!)^n}{(k!)^n}\sum_{j=2}^{k}\frac{1}{j^n} = \frac{(k!)^n/2^n + \cdots + (k!)^n/p^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{15}$$

As $(k, p) = 1$, only the term $(k!)^n/p^n$ will not have p in it. The sum of all such terms will not be divisible by p, otherwise p would divide $(k!)^n/p^n$. As $p < k$, p^n divides $(k!)^n$, the denominator of r/s, as needed. □

Lemma 8. *For any $k \geq 2$, there exists a prime p such that $k < p < 2k$.*

Proof. This is Bertrand's postulate. □

Theorem 5. *If $s_k^n = \frac{r}{s}$, with r/s reduced, then $s > k^n$, that is z_n is k-greater.*

Proof. Using Lemma 8, for even k, we are assured that there exists a prime p such that $k > p > k/2$. If k is odd, $k - 1$ is even and we are assured of the existence of prime p such that $k - 1 > p > (k - 1)/2$. As $k - 1$ is even, $p \neq k - 1$ and $p > (k - 1)/2$ assures us that $2p > k$, as $2p = k$ implies k is even, a contradiction.

For both odd and even k, using Lemma 8, we have assurance of the existence of a p that satisfies Lemma 7. Using Lemmas 6, 7, and 8 we have $2^n p^n$ divides the denominator of r/s and as $2^n p^n > k^n$, the proof is completed. □

Corollary 1 restates Theorem 5.

Corollary 1.

$$s_k^n \notin \Xi_k^n \text{ or } s_k^n \in \mathbb{R}(0,1) \setminus \Xi_k^n$$

where $\mathbb{R}(0, 1)$ is the set of real numbers in $(0, 1)$.

The Irrationality of z_n

We can give a proof similar to the second $e - 2$ proof.

Theorem 6. *z_n is irrational.*

Proof. Using Lemma 5 (complete),

$$\lim_{k\to\infty} \Xi_k^n = \mathbb{Q}(0,1),$$

with Corollary 1 (k-greater) we have

$$\lim_{k\to\infty} \mathbb{R}(0,1) \setminus \Xi_k^n = \mathbb{R}(0,1) \setminus \mathbb{Q}(0,1) = \mathbb{H}(0,1), \tag{16}$$

where $\mathbb{H}(0,1)$ is the set of irrational numbers in $(0,1)$.

We have then

$$\lim_{k\to\infty} s_k^n \in \mathbb{R}(0,1) \setminus \Xi_k^n \implies z_n \in \mathbb{H}(0,1), \tag{17}$$

using $s_k^n \to z_n$ with (16). That is z_n is irrational. □

It really is spooky and creepy: it's not in the accustomed *proof by contradiction* form. What we would like is a proof by contradiction – like Theorem 2 for $e-2$. That not being possible, perhaps we could get a justification for this unsettling theorem using Dedekind cuts, like we did with the second proof of the irrationality of $e-2$. The best would have a clearer picture that unites all three parallel with $e-2$ proofs with a visual rendering of what's going on as the partials s_k^n approach z_n.

The next lemma moves us closer to this last goal.

Lemma 9. *After a finite number of updates, the digit of a decimal set used to approximate s_k^n is not updated again.*

Proof. Let $y_j = j^n$, $j = 2, \ldots, k$. That is y_j gives the denominators, the number bases, for the terms of the partial s_k^n.

Consider the following inequality:

$$\frac{x-1}{\text{any } y_j} < \frac{x-1}{\prod_{j=2}^{k} y_j} = s_k^n < \frac{x}{\text{any } y_j}, \tag{18}$$

where the denominators of each expression indexes the numerator.

These inequalities follow as a partial can be given as a fraction using the product of all the denominators in its terms; that gives the validity of the middle equation. This denominator can also express the best upper and lower bounds for each decimal basis using the partial's terms; cancellations yield the simpler forms. Per Corollary 1, the inequalities are pure.

As k increases the numerator for a number basis y_j might change to better approximate the particular s_k^n. As there are only a finite number of numerators (digits in a basis), this updating can take place only a finite number of times. $\square$

Using a sufficiently large k for the decimal sets in Ξ_k^n we are assured that the digit used in each decimal set for a lower and upper bound for all s_k^ns after k are fixed. The next definition captures this idea.

Definition 7. *Let*

$$A_{k+} = \{.(x-1)_{j^n}^k | j = 2, \ldots, k \textit{ and } k > max(\{K_j : j = 2, \ldots, k\})\},$$

where the subscript on the decimal indexes the digit and k value and K_j gives the k value specified by Lemma 9 for the decimal set D_{j^n}.

We continue to form a Dedekind cut.

Definition 8. *Let*

$$A = \bigcup_{k=3}^{\infty} A_{k+}$$

The B set of the Dedekind cut $(A|B)$ is similarly defined.

Example 7. *For z_2, using a spreadsheet,*

$$A = \{.(2)_4, .(5)_9, .(10)_{16}, .(16)_{25}, .(23)_{36}, .(31)_{49}, .(41)_{64}, \ldots\}$$

and

$$B = \{.(3)_4, .(6)_9, .(11)_{16}, .(17)_{25}, .(24)_{36}, .(32)_{49}, .(42)_{64}, \ldots\}.$$

Technically, as mentioned in an early footnote, for strict conformity with Landau (and others) all decimals less (greater) than each of the elements in A (B) are also included.

Theorem 7. *z_n is irrational.*

Proof. Given a $p/q \in \mathbb{Q}(0, 1)$, $p/q \in \Xi_k^n$, for some k (Lemma 5, complete). Using Lemma 9, there are three cases possible:

$$p/q = .(x-1)_q^{k+}$$

As s_k^ns are inside $(.(x-1), .x)$, p/q can't be a greatest lower bound (GLB) for A.

$$p/q < .(x-1)_q^{k+}$$

In this case $p/q \in A_{k+}$.

$$p/q > .(x-1)_q^{k+}$$

For this case $p/q \in B$ as the only possible value of p/q is $.(x)_q^{k+}$ or a greater such decimal. So in all cases p/q can't be a GLB for A in A. Convergence implies the GLB for A equals the LUB for B. So neither exists and the Dedekind cut $(A|B)$ defines an irrational number. □

Third proof

Fix the bases of Ξ_k^n. As these bases are used to represent s_k^n, their first digit will become fixed. As s_k^n is never equal to a single digit in bases from Ξ_k^n we know this must be the case. Suppose at $k + r_k$ all first digits of all bases in Ξ_k^n are fixed. This set of first digits forms a nested set of intervals with all s_k^n, $k > K_r$ inside. Next increase the k of Ξ_k^n so as to include as a single decimal value $s_{k+r_k}^n$. Now add a fixed number, say 100, to $s_{k+k_r}^n$. Repeat by expanding Ξ_k^n so as to get fixed decimals for this new s_k^n.

Finally, we are able to give a more satisfying proof that all z_n are irrational.

Lemma 10. *Every reduced rational p/q in $\mathbb{Q}(0,1)$ can be written uniquely in the form $.(p-1)\overline{(q-1)}$, base q.*

Proof. This follows immediately as $.0\overline{q-1}$ converges to $1/q$ and adding $.(p-1)$ to $1/q$ gives $.(p)$, base q. □

Lemma 11. *Partial sums of plus-one series, s_k, converging to a rational number, p/q reduced, can be expressed as $.(p-1)\overline{q-1}_R x$, where $\overline{q-1}_R$ are fixed decimals and x are other decimals.*

Proof. Using Lemma 10 we know the convergence point of such a series can be given as $.(p-1)\overline{q-1}$. Partial sums must get arbitrarily close to this number. Partials approximate this number by having an increasing number of matching fixed digits. □

Lemma 12. *A plus one series, $s_k \to s$, converges to a rational number if and only if the base used to represent its partial sums in the form $.(x-1)\overline{b-1}_R m$ uses the same base for all partial sums.*

Theorem 8. *z_n is irrational.*

Proof. Assume z_n is rational. Let $.(x-1)\overline{(b-1)_3}$ be the best approximation to s_3^n in the bases of Ξ_3^n; in general, let $.(x-1)\overline{(b-1)_k}$ be the best approximation to s_k^n in the bases of Ξ_k^n. As z_n is a complete one-plus series, a single basis will emerge. This basis will occur at a specific k value and persist for all Ξ_k^n afterwards; that is the basis used will become fixed.

But each s_k^n is such that $s_k^n = .(x-1)\overline{(b-1)_k(b-1)}$, an exact *approximation* of s_k^n using the denominator of the reduced fraction for s_k^n as a basis. The notation indicates that k initial decimals are all $(b-1)$ and the remaining decimals are all $(b-1)$ as well. Now eventually all s_k^n bases are incorporated into Ξ_{k+}^n, where the $+$ indicates some greater than k value. But, per Corollary 1 the primes in the bases of s_k^n increase without bound and the approximation being perfect (less than any $\epsilon > 0$), this bases should be chosen as the best approximation to s_k^n. But as a prime factor will exceed any given prime factor of a given fixed basis, the basis must differ from the one implied by our rationality assumption. We have a contradiction.

Note the error given by the tail is one over the base minus the finite decimal head. As this goes to zero with the increasing bases needed a difference with a fixed base form grows. One can say that the basis selected from Ξ_k^n is incorrect; it doesn't have the smallest error. □

With another definition, we can, at long last, get a picture of how partials for z_n are excluding rational values and converging to an irrational number.

Definition 9. *A k-zeta-nest is defined as two sequences of powers of consecutive prime numbers. The first is*

$$x_k = p_1^k p_2^k p_3^k \dots p_k^k$$

and the second sequence is

$$y_k = x_k p_{k+1}^k p_{k+2}^k \dots p_{k+k}^k.$$

Example 8. *A 3-zeta-nest is given by $x_3 = 2^3 3^3 5^3$ and $y_3 = 2^3 3^3 5^3 7^3 11^3 13^3$. A 4-zeta-nest:*

$$x_4 = 2^4 3^4 5^4 7^4$$

and

$$y_4 = 2^4 3^4 5^4 7^4 11^4 13^4 17^4 19^4.$$

Finally, we have the long sought after picture for the irrationality of z_n. Its a shuffling action.

Theorem 9. *z_n is irrational.*

Proof. The following systems of inequalities shows all z_n are irrational:

$$.(x-1)_{x_k} < s_{k_1}^n < z_n \leq .(x)_{y_k} < .(x)_{x_k},$$

where the decimal digits conform with Lemma 9. For a given n and k_1, for sufficiently large k, all $s_{k_1}^n$ will have an upper bound of $.(x)_{y_k}$. We define the decimal digit to be the least such upper bound. But as k increases this upper bound is transferred to $.(x)_{x_k}$ and the mixed inequality changes to a pure inequality. Any $p/q \in \mathbb{Q}(0,1)$ is thus eliminated as a convergence point for any z_n. □

Conclusion

How does this proof compare to the work of Beukers and Apery? Why do we get a general result here and not with their techniques? We will focus on Beukers's $\zeta(2)$ proof.

Beukers uses double integrals that evaluate to numbers involving partials for $\zeta(2)$. He uses

$$\int_0^1 \int_0^1 \frac{x^r y^s}{1-xy}\, dxdy = \text{various expressions related to } \zeta(2)$$

and uses this to calculate

$$\int_0^1 \int_0^1 \frac{(1-y)^n P_n(x)}{1-xy}\, dxdy,$$

where $P_n(x)$ is the nth derivative of an integral polynomial.

These calculations yield integers A_n and B_n in

$$0 < |A_n + B_n\zeta(2)|d_n^2 < \left\{\frac{\sqrt{5}-1}{2}\right\}^{5n} \zeta(2) < \left\{\frac{5}{6}\right\}^n, \tag{19}$$

where d_n designates the least common multiple of the set of integers $\{1, \ldots, n\}$. This last, assuming $\zeta(2)$ is rational, forces an integer between 0 and 1, giving a contradiction. An upper limit for d_n requires the prime number theorem.

These themes repeat for $\zeta(3)$ with the complexity of the expressions and manipulations (tricks needed) at least doubling. In both cases proofs by contradiction are used and require a unique trap for each case.

We don't use integrals to generate in effect an interval, a trap, like (19). We use relationships between terms and partials to generate partitions of $(0, 1)$ narrowing and leaving only irrational numbers for all n in the same way. We use inherent and simple properties of z_n's partials and terms, Corollary 1, to avoid intractable complexity inherent in such an artifice as multiple integrals.

The same problems are present in the recursive relationships of mi amigo, the Frenchman Apery. They are specific to each n and not general to all n by design.

References

[1] Apéry, R. (1979). Irrationalité de $\zeta(2)$ et $\zeta(3)$. *Astérisque* 61: 11-13.

[2] Apostol, T. M. (1976). *Introduction to Analytic Number Theory.* New York: Springer.

[3] Beukers, F. (1979). A Note on the irrationality of $\zeta(2)$ and $\zeta(3)$, *Bull. London Math. Soc.* 11: 268–272.

[4] Berggren, L., Borwein, J., Borwein, P. (2004). *Pi: A Source Book*, 3rd ed. New York: Springer.

[5] Eymard, P., Lafon, J.-P. (2004). *The Number* π. Providence, RI: American Mathematical Society.

[6] Hardy, G. H., Wright, E. M., Heath-Brown, R. , Silverman, J. , Wiles, A. (2008). *An Introduction to the Theory of Numbers*, 6th ed. London: Oxford Univ. Press.

[7] J. Havil (2012). *The Irrationals.* Princeton, NJ: Princeton Univ. Press.

[8] Huylebrouck, D. (2001). Similarities in irrationality proofs for π, $\ln 2$, $\zeta(2)$, and $\zeta(3)$, *Amer. Math. Monthly* 108(10): 222–231.

[9] Hurst, G. (2014). Solutions to Introduction to Analytic Number Theory by Tom M. Apostol. https://greghurst.files.wordpress.com/2014/02/apostol_intro_to_ant.pdf

[10] Landau, E.G.H. (1951). *Foundations of Analysis*. New York, NY: Chelsea Publishing Company.

[11] Nesterenko, Y. V. (1996). A few remarks on $\zeta(3)$, *Math. Zametki* 59(6): 865–880.

[12] van der Poorten, A. (1978/9). A proof that Euler missed... Apery's proof of the irrationality of $\zeta(3)$, an informal report. *Math. Intelligencer* 1(4): 195–203.

[13] Rivoal, T. (2000). La fonction zeta de Riemann prend une infinit de valeurs irrationnelles aux entiers impairs, *Comptes Rendus de l'Acadmie des Sciences, Srie I. Mathmatique* 331: 267-270.

[14] Rudin, W. (1976). *Principles of Mathematical Analysis*, 3rd ed. New York: McGraw-Hill.

[15] Sondow, J. (2006). A geometric proof that e is irrational and a new measure of its irrationality. *Amer. Math. Monthly* 113(7): 637–641.

[16] Zudilin, W. W. (2001). One of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational. *Russian Mathematical Surveys* 56(4): 747–776.