Heuristics for Memorizing Trigonometric Identities

Timothy W. Jones

April 11, 2021

Abstract

Trigonometric identities are hard to memorize. Frequently a plus or minus is the rub. We give various heuristics that help refine guesses as to an identity and get, with a little work, it correct. Heuristics, for us, are plausible arguments using graphs, consistency (with other identities), test points, and transformations. We also specify the utility of each identity in the context of advanced mathematics – calculus – with the hope that meaning adds to memorable credence.

Introductions

If you are like me, frustrated at not being able to recall perfectly trigonometric identities, you might find this article helpful. We show how graphs, consistency with other identities, testing points, and transformations can help. The sense of heuristics in our title is best given by an example. A test point only shows the right and left side of an identity work for one point. It's not a proof that they hold for all points. But if the test point is well choosen it might resolve whether a plus or minus should be used; one works with the test point and one doesn't. We give plausibility arguments that make, hopefully, some of the nineteen typical trigonometric identities high school students are asked to memorize rivetingly obvious. If the program works one can puzzle out an identity – each – from scratch, confirm it with a test point, and have confidence that it is correct.

Graphs

A good heuristic for fine tuning the recall of some trigonometric identities is to picture the graph of the function involved – in the first quadrant. A heuristic says that the identity must be consistent with at least this part of the function(s) of concern. So, you are trying to recall

$$\tan(A + B).$$

Is it a plus or minus on top with the opposite on the bottom? Well the curve of the tan is increasing on $[0, \pi/2)$, so if positive angles are added, the net must be an increase. With this reasoning we arrive at

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

The numerator gets large and the denominator gets smaller; the combination gives a clear increase.

The like reasoning yields

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

This heuristic has the added benefit of forcing (or rewarding) clear recalls of the graphs of the tan function. We can apply the same heuristic to arrive at

$$\sin(A \pm B) = \sin(A)\cos(B) + \cos(A)\sin(B). \tag{1}$$

The sin function, like tan is increasing from 0 to $\pi/2$. Of course, one has to remember everything but the plus or minus. But, speaking for myself, that is the part I frequently forget.

The cosine is a decreasing function on $[0, \pi/2]$. So once you have the terms right, you can get the $\pm$ right:

$$\cos(A \mp B) = \cos(A)\cos(B) \mp \sin(A)\sin(B).$$

Of course a good student knows how all the trigonometric identities are derived. One starts with deriving $\cos(A - B)$ and then inferring, using the oddness of cos, $\cos(A + B)$. Things must be consistent with all angles, so, using co-function properties, one gets from

$$\sin(A + B) = \cos(\pi/2 - (A + B)) = \cos((\pi/2 - A) - B)$$
$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B), \tag{2}$$

using the oddness of sine again.

Add test points

The double angle formulas follow with $\cos 2A$ having three forms: $\cos^2 A - \sin^2 A$; $2\cos^2 A - 1$; and $1 - \sin^2 A$. The latter two allow for power reducing formulas by solving for the square forms:

$$\cos^2 A = \frac{1 + \cos 2A}{2} \tag{3}$$

and

$$\sin^2 A = \frac{1 - \cos 2A}{2}. \tag{4}$$

Which one gets the plus and which the minus can be remembered using 0 as a test point. The power reducing formula for tan follows from these two. We know sin goes on top, so

$$\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}.$$

Testing $A = 0$, a root of tan and $\tan^2$ confirms the plus and minus are correct.

Transformations

One can, using graphs again, observe that a trigonometric function squared is positive and the right hand side of (3) and (4) with its vertical (plus 1) transformations renders positive functions. One can also infer that squaring changes the periodicity of the original function; the horizontally shrinking transformation in $\cos 2A$ is given by a contraction in cos period; it is halved. Also the division by 2, a vertical shrinking, may be understood as again the action of squaring less than 1 numbers: $1/2 \times 1/2 = 1/4$ with $1/4 < 1/2$.

An interesting exercise is to use a calculator to try to model the squares of cos, sin, and tan with transformations of cos. Generally books cover function transformations in an earlier chapter, but curiously drop it in favor of specific terms used for periodic – these trigonometric – functions. All transformations of $f(x)$ are given by $\pm Af(\pm Bx \pm C) \pm D$; and something like $-2\cos(\pi x + 4) - 7$ is clearly of this form. In my opinion transformations find their natural home with trig functions; this should be made clear and the transition to amplitude, phase shift, and period should occur as a stress to how important trig transformations are – they get their own names!

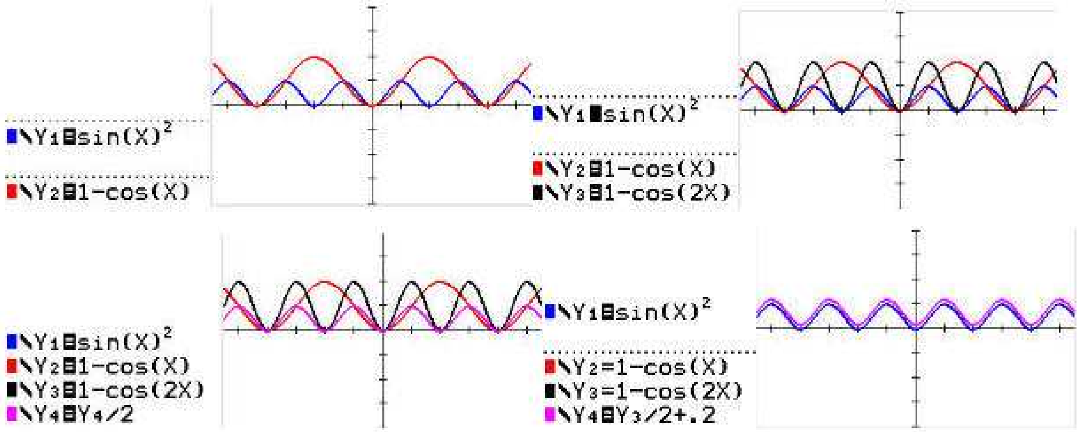

Figure 1: Modeling the square of sin with cosine in increments.

Optimistically speaking, one might first say I need to express, say, $\sin^2$ without a square power using a transformed single trigonometric function and then look at the $\sin^2$ and say I need to move it up by 1, change its period (or shrink it horizontally), and then half its amplitude (vertically shrink it). Figure 1 gives images from a graphing calculator that perhaps adds action and suspense, a memorable experience, to the recall of these powerful power reducing formulas.

Consistency by default

The identity for $\cos(A-B)$ begets the other identities of the same type and the double angle formulas. So, one can verify doubles by sums, and get power reducing identities with some algebra from the doubles. That is kind of the problem: every time you need an identity you can prove all of them from one of them, but it is time consuming. Consistency (or derivations in sequence) should be the tool of last resort.

The half angle formulas do follow from the power reducing angles by simple algebraic manipulation – just take the square root of both sides. For example, using

$$\cos^2 A = \frac{1+\cos 2A}{2}$$

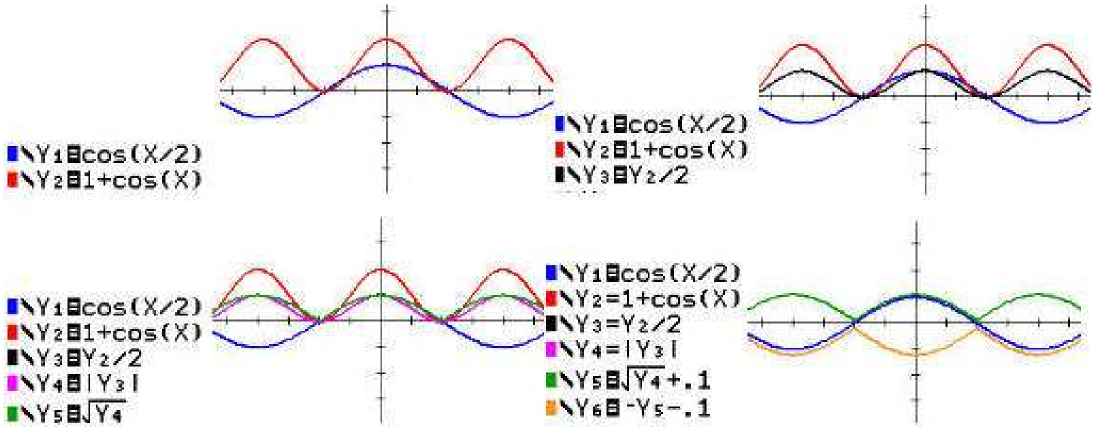

Figure 2: We see clearly why the plus(green)/minus(orange) is needed for $\cos(X/2)$(blue) and can figure out domains for each.

we substitute $A/2$ and solve for $\cos A$ by taking square roots:

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}. \tag{5}$$

Perhaps watching your TI-84 Plus CE draw the following might make the ideas hit home better, maybe quicker later: $Y_1 = \cos(X/2)$; $Y_2 = 1+\cos(X)$; $Y_3 = Y_2/2$; $Y_4 = \text{abs}(Y_3)$; $Y_5 = \sqrt{Y_4}$; and $Y_6 = -Y_5$. From this sequence you can see that $1+\cos(X)$ is all positive but has maximum and minimums in the right places to model $\cos(X/2)$, but it needs to be flattened with a vertical shrink: divide by 2. This gives right x values for maximums and minimums. Composition is not a transformation, but the rational power of a square root is 1/2, so maybe to get a half angle equal converted to something with a regular times one angle we need to take square roots. We try that with the graph of Y_5 – when the original $\cos(X/2)$ is positive. We add the negative half with Y_6 and our model is complete: Figure 2. This exercise rivets in the need for the $\pm$ sign in front of the radical. Perhaps slow evolution to a correct model adds credence; you believe it.

Test points can clarify choices for pluses and minuses for the two half angle formulas for cos and sin:

$$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}}.$$

At $A = 0$ we need 0 on both sides and 1 with (5).

Consistency is how sum to product and product to sum formulas can be calculated. The sum to product formulas follow from the sum and difference formulas for sine and cosine. And the product to sum follow from the sum to product. For the last two, we will give sample derivations:

$$\sin(A + B) + \sin(A - B)$$

$$= \sin(A)\cos(B) + \cos(A)\sin(B) + \sin(A)\cos(B) - \cos(A)\sin(B)$$

$$2\sin(A)\cos(B),$$

so,

$$\sin(A)\cos(B) = \frac{1}{2}[\sin(A + B) + \sin(A - B)]. \qquad (6)$$

That's a sum rendered as a product. Using $A = (a + b)/2$ and $B = (a - b)/2$ with (6), this becomes

$$\sin(a) + \sin(b) = 2[\sin((a + b)/2)\cos((a - b)/2)],$$

a product rendered as a sum.

One other identity is disambiguated with an appeal to consistency: the sum of two angles for tan can't have a minus on top or $\tan(2A) \equiv 0$; that's inconsistent.

Motivation

Perhaps the most annoying things about trigonometric identities from the perspective of a high school student is their lack of motivation. The only clear and convincing need to have trigonometric identities seems to be to get additional exact values for trig functions. You can get exact values for 15^o, for example, from $\cos(60^o - 45^o)$ being equal, via a trig identity, to a linear combination of known exact values. But this in itself seems rather superfluous given the calculator can give approximations to any radian or degree you like quickly – albeit not exact.

You can try to motivate something like the double angle formulas by saying you do get a value for double the angle, but the angles always used are from a short list: 30, 45, 60, and 90 and their multiples. You would never bother to calculate $\cos 60$ using $\cos^2 30 - \sin^2 30$; you have memorized that the value is 1/2.

To attempt to bring home the point, consider how it follows from the development in a good high school algebra book to look at all compositions of polynomials with trig functions. Given $p(x) = x^2-2x+5$, consider $p(\cos x) = \cos^2 x - 2\cos x + 5$; or given $f(x) = e^x$, consider $f(\sin(x))$. Why aren't there identities or helps or considerations for these compositions. You would expect that trig functions would be folded into the list of previous functions considered: polynomials, rational, radical, exponential, and logarithmic with each arithmetic combination and composition having identities. Why is it that the power reducing identities can just reduce to arguments with $2x$? What about reducing to expressions with just x? Why is $2x$ good enough? And why bother to reduce a square at all?

It is curious that just exactly the trig identities given in high school courses match what is needed in calculus – no more, no less.

As an example, consider the heuristic that the slope of the line tangent to cos at $x = 0$ looks like 0 (a TI-84 confirms this) and the value of sin at $x = 0$ is 0. The slope of the lines tangent to the curve of cos are negative and decreasing (approaching 0) and we notice that $-\sin$ mirrors this. Let's hypothesize that there is a function which gives the slopes of sin and it is cos and for cos it is $-\sin$. How could we prove this? We consider

$$\lim_{\Delta t \to 0} \frac{\sin(t + \Delta t) - \sin(t)}{\Delta t} = \text{slope of sin at } t.$$

Viola: we need an identity for $\sin(t + \Delta t)$. The applications of differential and integral calculus are so varied and time tested and periodic phenomenon worthy of being modeled and understood so pervasive (other superlatives) that trig identities like this one are shown to be important.

Another pervasive need for trig identities comes from integration using substitution. Consider that you can't readily do many integrals without a re-write of the integrand that transforms a function to another function with a change of argument. For example,

$$\int \sqrt{\frac{1 + \cos 2A}{2}} \text{ dA}$$

can't be integrated except by rewriting this integral, via a trig identity: it's equal to

$$\int \cos(A/2) \text{ dA}.$$

The power reducing and half angle identities all allow for integration.

In the next section, we list the nineteen trig identities and give ways to remember them, resurrect them, and maybe appreciate them by way of a salient example use from calculus or other later math courses.

List with helps

Harry Lorayne attempts to make it memorable.

1. $\cos(A-B)=\cos(A)\cos(B)+\sin(A)\sin(B)$

   Get the sign right by noting cos is a decreasing function, so when you back up (subtracting a B) the result should be increasing; it is when a plus is used.

   Also consistency with double angle terms: the one never to forget is $\cos(2A)=\cos^2(A)-\sin^2(A)$, as it allows for two other forms via the Pythagorean identity and these in turn allow for power reduction formulas and thence to half angle as well.

   This identity is the start up one in the sequence to all others. It can be proven using a unit circle and two equated ways of getting the distance of a chord between $(\cos(A-B),\sin(A-B)$ and $(1,0)$.

2. $\cos(A+B)=\cos(A)\cos(B)-\sin(A)\sin(B)$

   Get the sign right by noting cos is a decreasing function, so when you go forward (adding a B) the result should be decreasing; a minus is used.

   Also consistency with double angle terms: the one never to forget is $\cos(2A)=\cos^2(A)-\sin^2(A)$, as it allows for two other forms via the Pythagorean identity and these in turn allow for power reduction formulas and thence to half angle as well. Hence, all but sum to product and product to sum identities are derived from it. But sum to product result from adding sum formulas, so in away all come from $\cos(A-B)$.

   This is derived by using $\cos(A-(-B))$ and the oddness of sin.

3. $\sin(A+B)=\sin(A)\cos(B)+\cos(A)\sin(B)$

   Get the sign right by noting sin is an increasing function, so when you go forward (adding a B) the result should be increasing: a plus is used.

Also consistency with double angle terms: $\sin(2A) = 2\sin(A)\cos(A)$.

The fastest derivation of this and the previous identity is using complex numbers:

$$(\cos A + i\sin A)(\cos B + i\sin B) = (\cos(A+B) + i\sin(A+B)).$$

$$(\cos A\cos B - \sin A\sin B) + i(\sin A\cos B + \cos A\sin B) = (\cos(A+B) + i\sin(A+B)).$$

implies

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

and

$$\sin(A+B) = \sin A\cos B + \cos A\sin B.$$

4. $\sin(A-B) = \sin A\cos B - \cos A\sin B$

   Sine is an increasing function, so minus (subtraction) is suggested.

5. $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

   Perhaps consistency with previous identities is the fastest way to get this identity. It is derived by multiplying the equivalent $\sin(A+B)$ over $\cos(A+B)$ by $\cos A\cos B$. As $\sin(A+B)$ has a plus between its two RHS terms, the plus occurs in the RHS of the numerator for

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B} \tag{7}$$

   and, as $\cos(A+B)$ has a minus between its two RHS terms, the minus occurs in the RHS of the denominator for (7).

   If one remembers all but these operators, infer the plus (top) and minus (bottom) from knowledge that the tan function is increasing and a plus on top and a minus on bottom makes the RHS increasing.

   If one remembers the double angle formula for tan, then heuristically $\tan A + \tan A = 2\tan A$ shows $\tan A + \tan B$ must be the numerator and similarly with the denominator, consistency rules.

6. $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$

7. $\cos 2A = \cos^2 A - \sin^2 A$

8. $\cos 2A = 2\cos^2 A - 1$

$$\cos 2A = 2\cos^2 A - 1$$

9. $\cos 2A = 1 - 2\sin^2 A$

10. $\sin 2A = 2\sin A \cos A$

11. $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

12. $\cos^2 A = \frac{1+\cos 2A}{2}$

    Squares have a two (superscript) on the LHS and 2 2's on the RHS. Also you CC someone when you send out two and both "c" stand for cosine: i.e. cos occurs in both the square of cosine and the square of sine. Use $A = 0$ as a test point to determine the plus (addition) is used for cos and minus (subtraction) for sin. Also a vertical transformation is suggested, the plus one, as a square is always positive; that is we need to shift up to pull the function out of any negative territory.

13. $\sin^2 A = \frac{1-\cos 2A}{2}$

14. $\tan^2 A = \frac{1-\cos 2A}{1+\cos 2A}$

15. $\cos(A/2) = \pm\sqrt{\frac{1+\cos A}{2}}$

    The half angle formulas have a half on the LHS, the half angle, and a half on the RHS, the default index on the radical. Also, the reduction of a half angle is mirrored with the reduction of a square root. The vertical transformation of one must be addressed as the half angle still has negative values (its been horizontally stretched, not vertically shifted up), so a plus/minus is needed. A test point, $A = 0$ confirms the plus sign is needed for this half angle of cosine formula. As mentioned earlier, making a series of graphs with a calculator a few times will rivet this formula in to your head.

16. $\sin(A/2) = \pm\sqrt{\frac{1-\cos A}{2}}$

    The half angle formulas have a half on the LHS, the half angle, and a half on the RHS, the default index on the radical. Also, the reduction of a half angle is mirrored with the reduction of a square root. The vertical transformation of one must be addressed as the half angle still has negative values (its been horizontally stretched, not vertically shifted up), so a plus/minus is needed. A test point, $A = 0$ confirms the plus sign is needed for this half angle of cosine formula. As mentioned earlier, making a series of graphs with a calculator a few times will rivet this formula in to your head.

17. $\tan(A/2) = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$
18. $\tan(A/2) = \frac{1-\cos A}{\sin A}$
19. $\tan(A/2) = \frac{\sin A}{1+\cos A}$

S2P and P2S

Here are applications of sum to product and product to sum trig identities.

More applications

There are other identities and inverse trig functions that are used to integrate in calculus.

Conclusion

Move to complex number proofs of identities and maybe giving slope functions in route to power series: do polynomials via Taylor and then see periodicity to derivatives of trigs and make infinite series. Supposing correct get to DeMoivre and e^{ix}.

References

[1] R. Blitzer, *Algebra and Trigonometry*, 4th ed., Pearson, Upper Saddle River, NJ, 2010.

[2] Zwillinger, D. (2003). *CRC Standard Mathematical Tables and Formulas*, 31st ed. Boca Raton, FL: Chapman and Hall/CRC.