RELATIVISTIC ROTATIONAL ANGULAR MOMENTUM

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Abstract. This is an exercise without a conclusion, intended to examine a phenomenon that, in the relativistic world, seems most difficult to comprehend, let alone explain – relativistic rotational angular momentum. Many have tackled the problem, first cited by Ehrenfest in 1909, and known since then as the "Ehrenfest Paradox," primarily in terms of analyzing effects on the size and geometry of a relativistically rotating disk. Length contraction and time dilation interplay with aspects of general relativity and hyperbolic geometry according to the experts. Tackling this aspect is beyond me, so I try to examine a somewhat more predictable phenomenon, relativistic angular momentum as a result of relativistic mass increase in both a rotating rod and disk. No conclusions are drawn, but observations are offered as food for thought for the reader.

1. INTRODUCTION

Rotational motion has always posed intriguing phenomena that sometime seem to defy the laws of physics. For example, consider the stability of a bicycle in motion or a spinning gyroscope. Or the underwater rotating cylinders of Ionel Dinu that seem to mimic the behavior of bar magnets, whereby cylinders rotating in the same direction (clockwise-clockwise or counterclockwise-counterclockwise) will "repel" each other, while those rotating in opposite directions (clockwise-counterclockwise) will "attract." In fact, if left to spin on their own in either direction, they will align just as the north and south poles of a pair of bar magnets will. (https://sciencewoke.org/scientist/ionel-dinu/) Or Eric Laithwaite's famous 1983 video demonstration of how a spinning disk at the end of rod, too heavy to easily lift when stationary, becomes surprisingly "lightweight" when set to rotating rapidly. (https://www.youtube.com/watch?v=JRPC7a_AcQo) Standard explanations can be complex and seemingly contrary to common sense, but are relatively straightforward when compared to explaining disk rotational behavior in the relativistic realm, the subject of what is known as the "Ehrenfest Paradox."

2. THE EHRENFEST PARADOX

"The Ehrenfest paradox [see Figure 1] concerns the rotation of a 'rigid' disc in the theory of relativity. … as presented by Paul Ehrenfest [in] 1909 in relation to the concept of Born rigidity within special relativity[1] … [I]t discusses an ideally rigid cylinder that is made to rotate about its axis of symmetry. The radius R as seen in the laboratory frame is always perpendicular to its motion and should therefore be equal to its value R_0 when stationary. However, the circumference ($2\pi R$) should appear Lorentz-contracted to a smaller value than at rest, by the usual factor γ. This leads to the contradiction that $R = R_0$ *and* $R < R_0$.

"The paradox has been deepened further by Albert Einstein, who showed that … the circumference … would … measure greater than $2\pi R$ [contrary to Ehrenfest]. This indicates that geometry is non-Euclidean for rotating observers, and was important for Einstein's development of general relativity. Any rigid object made from real materials that is rotating with a transverse velocity close to the speed of sound in the material must exceed the point of rupture due to centrifugal force, because centrifugal pressure cannot exceed the shear modulus of material ... Therefore, when considering velocities close to the speed of light, it is only a thought

[1] "Born rigidity is satisfied … if the length of the rigid body in momentary co-moving inertial frames measured by standard measuring rods (i.e., the proper length) is constant and is therefore subjected to Lorentz contraction in relatively moving frames. Born rigidity is a constraint on the motion of an extended body, achieved by careful application of forces to different parts of the body. A body rigid in itself would violate special relativity, as its speed of sound would be infinite." (https://en.wikipedia.org/wiki/Born_rigidity)

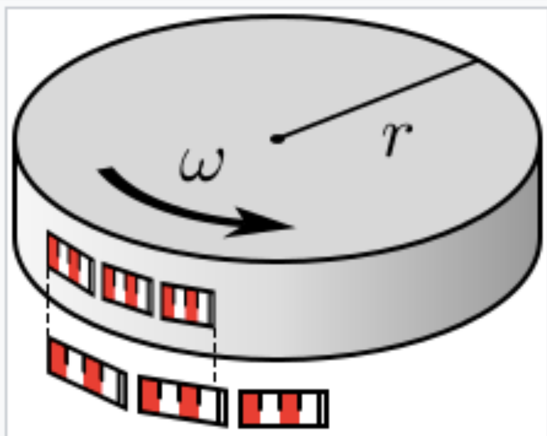
experiment. Neutron-degenerate matter allows velocities close to speed of light, because, e.g., the speed of neutron-star oscillations is relativistic; however; these bodies cannot strictly be said to be ‘rigid’ (per Born rigidity) …

“Ehrenfest considered an ideal Born-rigid cylinder that is made to rotate … [I]ts radius stays the same, [b]ut measuring rods laid out along the circumference should be Lorentz-contracted to a smaller value than at rest, by …[$\sqrt{1-\left(\frac{v}{c}\right)^2}$]. This leads to the paradox that the rigid measuring rods would have to separate from one another due to Lorentz contraction; the discrepancy noted by Ehrenfest seems to suggest that a rotated Born rigid disk should shatter … [and] that Born rigidity is not generally compatible with special relativity. According to special relativity an object cannot be *spun up* from a non-rotating state while maintaining Born rigidity, but once it has achieved a constant nonzero angular velocity it does maintain Born rigidity without violating special relativity, and then (as Einstein later showed) a disk-riding observer will measure a circumference [increased by $1\Big/\sqrt{1-\left(\frac{v}{c}\right)^2}$].

“The rotating disc and its connection with rigidity was also an important thought experiment for Albert Einstein in developing general relativity … [concluding] that the geometry of the disc becomes non-Euclidean for a co-rotating observer. Einstein wrote (1922):

> ‘Imagine a circle drawn about the origin in the x'y' plane of K' and a diameter of this circle. Imagine, further, that we have given a large number of rigid rods, all equal to each other ... laid in series along the periphery and the diameter of the circle, at rest relatively to K'. If U is the number of these rods along the periphery, D the number along the diameter, then, if K' does not rotate relatively to K, we shall have But if K' rotates … [w]ith respect to K all the rods upon the periphery experience the Lorentz contraction, but the rods upon the diameter do not experience this contraction (along their lengths!). It therefore follows that . It therefore follows that the laws of configuration of rigid bodies with respect to K' do not agree with the laws of configuration of rigid bodies that are in accordance with Euclidean geometry. If, further, we place two similar clocks (rotating with K'), one upon the periphery, and the other at the centre of the circle, then, judged from K, the clock on the periphery will go slower than the clock at the centre. The same thing must take place, judged from K' if we define time with respect to K' … Space and time, therefore, cannot be defined with respect to K' as they were in the special theory of relativity with respect to inertial systems ...’

“Grøn [in *Einstein's General Theory of Relativity,* Springer, p. 91 (2007), ISBN 978-0-387-69200-5] states that the resolution of the paradox stems from the impossibility of synchronizing clocks in a rotating reference frame ... The modern resolution can be briefly summarized as follows: … *Small distances measured by disk-riding observers ... is indeed well approximated (for small angular velocity) by the* ***geometry of the hyperbolic plane*** *... For physically reasonable materials, during the spin-up phase a real disk expands radially due to centrifugal forces, relativistic corrections partially counteract (but do not cancel) this Newtonian effect.*”
(https://en.wikipedia.org/wiki/Ehrenfest_paradox)



Ehrenfest paradox – Circumference of a rotating disk should contract but not the radius, as radius is perpendicular to the direction of motion.

Figure 1
(https://en.wikipedia.org/wiki/Ehrenfest_paradox)

In the chapter on "Relativistic Paradoxes" (*The Dynamics of Relativistic Length Contraction and the Ehrenfest Paradox*, 2007 [https://arxiv.org/pdf/0712.3891.pdf]), Fayngold tackles the Ehrenfest Paradox by "… analyz[ing] the Lorentz contraction in rotational motions. The analysis will demonstrate the physical mechanism of deformation of a spinning disk and, accordingly, the dynamical origin of its specific geometry." The analysis is too lengthy (and quite a bit too much for me to bite off and chew) for summary here, but he reaches a similar conclusion to the preceding:

"…[A] spinning disk is in a state of a complex deformation which renders its plane non-Euclidean. It is described by Lobachevsky's geometry with negative curvature; sometimes it is referred to as a **hyperbolic geometry** … [T]he congruence in the co-rotating frame is satisfied in a more subtle way taking account of the fact that time in rotating systems is not single-valued. Its most essential features are that the sum of the angles of a triangle is less than 2π, and the ratio of the length of a circle to its radius is greater than 2π …

"The relativistic kinematics of accelerated objects cannot be separated from the dynamics. A change in motion of particles constituting an object, changes the structure of their fields and thereby the shape of the object after acceleration … The size of an accelerated object cannot be uniquely determined … because the object generally cannot even be assigned a constant proper length. The concept of deformation in relativistic mechanics is more subtle than in classical physics. Its two intimately linked characteristics – geometric shape and physical structure – are not rigidly correlated.

"… A uniformly rotating ring, while retaining its circumference length L = 2πR, is physically deformed (circumferentially stretched at fixed R), which becomes evident in the co-rotating [reference frame] … [I]n a rotational boost, an object undergoes physical deformation lasting permanently after the boost and becoming one of the characteristics of its spinning state … [T]he dynamical aspect of the Lobachevsky's geometry in a rotating system is manifest in the increase of the system's rest mass. This is associated with an additional energy input necessary for boosting such a system, apart from the energy going into increase of relativistic mass of its constituting particles." (https://arxiv.org/pdf/0712.3891.pdf)

If all of this seems confusing, then welcome to the club. From these explanations, what I surmise is that, since the rotating disk experiences acceleration (centripetal), it cannot be considered as an inertial reference frame and subject to the laws of special relativity. The mysterious relativistic phenomena of length contraction and time dilation seem somehow to play off one another to maintain the shape of the rotating disk. The resolution of the paradox appears to require delving into the much more complex realm of general relativity (and hyperbolic geometry), although for a sufficiently large disk, perhaps the rotating periphery could at least approximately fit into the inertial reference frame requirement for special relativity, being the motion for at least a comparatively short distance should be close to linearly translational. However, although I will make no attempt to examine the interplay between relativistic length contraction and time dilation, I do note Fayngold's statement that "[T]he dynamical aspect of … a rotating system is manifest in the increase of the system's rest mass." It is this phenomenon, relativistic increase in mass of a rotating disk, and its effect upon angular momentum that I will endeavor upon which to shed at least a dim light.

3. RELATIVISTIC MASS AND ANGULAR MOMENTUM

Before diving into the relativistic angular momentum for a disk whose perimeter is rotating with a tangential speed (nearly) equal to that of light, I first examine a simpler case.

3.1 ROTATING ROD

Assume a thin rod of mass M and length R rotating about an axis through one of its ends such that the other end at R rotates with a tangential speed (nearly) equal to that of light. Assuming constant angular speed ω, the tangential speed at any position r ≤ R will just be $v = \omega r$, i.e., directly proportional to r. Therefore, over any incremental length dr from $r_- \ to \ r_+$, the non-relativistic mass is

$$\int_{r_-}^{r_+} \frac{M\,dr}{R} = \frac{M}{R}(r_+ - r_-)$$

where M/R represents the linear density.

For convenience, assume that M, R, ω and v can all be expressed in units such that each acquires a value of unity (e.g., c could be measured in units of light-sec/sec). Using the Lorentz factor

$^1\Big/\sqrt{1-\left(\frac{v}{c}\right)^2}$, which simplifies to $^1\!/\!\sqrt{1-r^2}$ with the unitized values ($c = 1$; $\omega = 1 \rightarrow v = r$), the relativistic mass for increment dr becomes (with the unitized values)

$$\int_{r_-}^{r_+} \frac{dr}{\sqrt{1-r^2}} = \sin^{-1} r_+ - \sin^{-1} r_-$$

via integration formula #201 from the CDC Standard Mathematical Tables, 27th Edition, W. Beyer, ed., CRC Press, Inc., Boca Raton, FL (1986):

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$$

The angular momentum $L = I\omega$, which reduces to L = I for $\omega = 1$, where I is the moment of inertia. For the incremental length dr, with our unitized values, both the moment of inertia and the angular momentum can be expressed as follows for the non-relativistically rotating rod:

$$L(r) = I(r) = \int_{r_-}^{r_+} r^2 dr = \frac{{r_+}^3 - {r_-}^3}{3}$$

For the relativistic angular momentum, the relativistic moment of inertia employs the formula for relativistic mass, yielding

$$L(r) = I(r) = \int_{r_-}^{r_+} \frac{r^2 dr}{\sqrt{1-r^2}}$$
$$= \frac{1}{2}\left(\left[\sin^{-1} r_+ - r_+\sqrt{1-{r_+}^2}\right] - \left[\sin^{-1} r_- - r_-\sqrt{1-{r_-}^2}\right]\right)$$

via integration formula #214 from the CDC Tables:

$$\int \frac{x^2 dx}{\sqrt{1-x^2}} = \frac{1}{2}\left(\sin^{-1} x - x\sqrt{1-x^2}\right)$$

Employing an EXCEL® spreadsheet to perform these integrations for radial increments of 0.01, results are obtained for both the non-relativistic and relativistic mass and angular momentum of the rod as shown in Figure 2. As expected, the non-relativistic mass increases linearly with radius, while the non-relativistic angular momentum rises more than linearly until reaching the value of 1/3 at r = 1.0. This corresponds exactly to the angular momentum for the rod, $L = \frac{MR^2}{3}\omega = \frac{1}{3}$ with the unitized values. For the rod rotating such that its end at R (nearly) attains light speed, the relativistic mass increases more than linearly with radius, reaching a maximum of 1.57 ($= ^{\pi}\!/_2$, the solution to the corresponding integral for $0 \le r \le 1$). This is 4.7 times greater than the non-relativistic mass. Correspondingly, the relativistic angular momentum also increases at an even greater more than linear rate with radius, reaching a maximum of 0.785 ($= ^{\pi}\!/_4$, the solution to the corresponding integral for $0 \le r \le 1$). This is 2.4 times greater than the non-relativistic angular momentum.

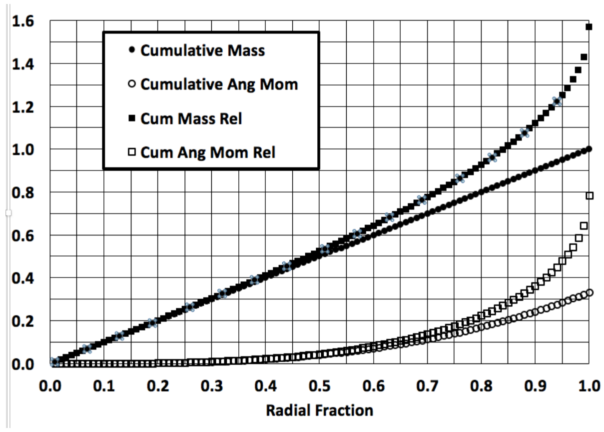


Figure 2. Mass and Angular Momentum for Rod

3.2. ROTATING DISK

Assume a thin disk of mass M and length R rotating about an axis through its center such that the periphery at R rotates with a tangential speed (nearly) equal to that of light. Assuming constant angular speed ω, the tangential speed at any position r ≤ R will just be $v = \omega r$, i.e., directly proportional to r. Therefore, over any incremental annulus dr from $r_-\ to\ r_+$, the non-relativistic mass is

$$\int_{r_-}^{r_+} \frac{M(2\pi r)dr}{\pi R^2} = \frac{M}{R^2}({r_+}^2 - {r_-}^2)$$

where $^M\!/_{\pi R^2}$ represents the areal density.

For convenience, assume once more that M, R, ω and v can all be expressed in units such that each acquires a value of unity. With the Lorentz factor $^1\!/\!\sqrt{1-r^2}$ via the unitized values, the relativistic mass for annular increment dr becomes (with the unitized values)

$$\int_{r_-}^{r_+} \frac{2r\ dr}{\sqrt{1-r^2}} = 2(\sqrt{1-{r_-}^2} - \sqrt{1-{r_+}^2})$$

via integration formula #204 from the CDC Tables:

$$\int \frac{x\ dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2}$$

Angular momentum $L = I\omega$, which reduces to L = I for $\omega = 1$, where I is the moment of inertia. For the annular increment dr, with our unitized values, both the moment of inertia and the angular momentum can be expressed as follows for the non-relativistically rotating disk:[2]

$$L(r) = I(r) = \int_{r_-}^{r_+} (2r)r^2 dr$$
$$= 2\int_{r_-}^{r_+} r^3 dr = \frac{{r_+}^4 - {r_-}^4}{2}$$

For the relativistic angular momentum, the relativistic moment of inertia employs the formula for relativistic mass, yielding

$$L(r) = I(r) = \int_{r_-}^{r_+} \frac{2r^3 dr}{\sqrt{1-r^2}}$$
$$= \frac{2}{3}\left(\left[\sqrt{1-{r_-}^2}\,({r_-}^2+2)\right] - \left[\sqrt{1-{r_+}^2}\,({r_+}^2+2)\right]\right)$$

via integration formula #220 from the CDC Tables:

$$\int \frac{x^3 dx}{\sqrt{1-x^2}} = \frac{-1}{3}\sqrt{1-x^2}\,(x^2+2)$$

Employing again an EXCEL® spreadsheet to perform these integrations for radial increments of 0.01, results are obtained for both the non-relativistic and relativistic mass and angular momentum of the disk as shown in Figure 3. As expected, the non-relativistic mass increases more than linearly with radius, while the non-relativistic angular momentum rises more than linearly until reaching the value of 1/2 at r = 1.0. This corresponds exactly to the angular momentum for the disk, $L = \frac{MR^2}{2}\omega = \frac{1}{2}$ with the unitized values. For the disk rotating such that its periphery at R (nearly) attains light speed, the relativistic mass increases at an even greater more than linear rate with radius, reaching a maximum of 2.00 (the solution to the corresponding integral for $0 \le r \le 1$). This is 4.0 times greater than the non-relativistic mass. Correspondingly, the relativistic angular momentum also increases at an even greater more than linear rate with radius, reaching a maximum of $\frac{4}{3} = 1.33$ (the solution to the corresponding integral for $0 \le r \le 1$). This is 2.7 times greater than the non-relativistic angular momentum.

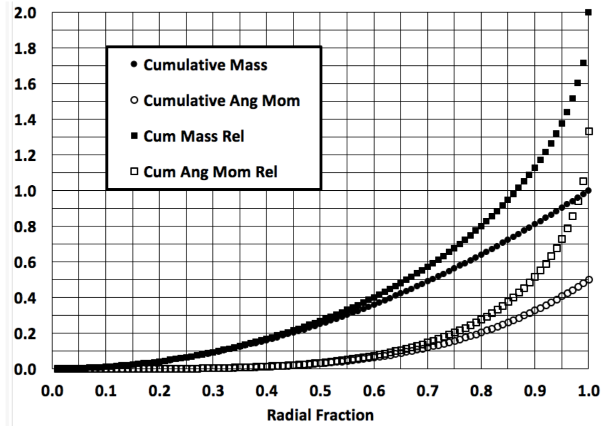


Figure 3. Mass and Angular Momentum for Disk

4. OBSERVATIONS

While I feel unable to draw any conclusions from my examination, I can at least make some observations in hope that they will serve as food for thought for the reader. The behavior of a relativistically rotating disk as discussed in the Ehrenfest Paradox, at least in terms of changes in size, geometry and maybe even speed, is beyond my area of expertise. I refer the reader to the summaries provided in Section 2 and their references. What I did attempt to tackle is the effect on mass as the disk attains (nearly) the speed of light tangentially at its periphery, assuming the formulas for relativistic mass increase can be applied to what is not strictly an inertial reference frame (although it may be approximately so for perhaps the rotating periphery since the motion for at least a comparatively short distance should be close to linearly translational). As a precursor to the rotating disk, I first examined a simple rotating rod, pivoted at one end. The results, shown in Figure 2, indicate that the relativistic mass, compared to the non-relativistic mass, increases by as much as a factor of 4.7 at the periphery. The corresponding relativistic angular momentum increases by a factor of 2.4 at the periphery.

For the rotating disk, the relativistic mass at the periphery is 4.0 times that of the non-relativistic mass, while the relativistic angular momentum is 2.7 times that of its non-relativistic counterpart. For non-relativistic rotation, the ratio of the angular momenta between the disk and the rod is

[2] With unitizing, the density = $\frac{1}{\pi}$ and the annular area = $2\pi r dr$, whose product is $2rdr$.

1.5, derivable from their moments of inertia given equal mass, radii and rotational speeds:

$$\frac{L_{disk}}{L_{rod}} = \frac{M\omega R^2/2}{M\omega R^2/3} = \frac{3}{2}$$

For relativistic motion, the ratio is greater:

$$\frac{L_{disk}}{L_{rod}} = \frac{1.33}{\pi/4} = 1.7$$

This ratio of relativistic masses is:

$$\frac{M_{disk}}{M_{rod}} = \frac{2.00}{\pi/2} = 1.3$$

Thus, the disk, when rotated with a tangential speed (nearly) that of light at its periphery, shows both a greater mass increase and greater angular momentum increase vs. the rod, rotated about one end such that the other end attains (nearly) tangential light speed.

Does all this mean that mass increase at relativistic speed is an actual phenomenon that would occur in the relativistically rotating disk? Or could it masquerade for some other effect that manifests itself as if there was a mass increase? Would the mass increase, if actual, occur by increasing the density of the rod or disk as one moves farther along the radii, assuming they preserve their original shapes and geometry? Or could it be that the density remains the same and the original shapes and geometry change to accommodate more mass? Or might it be some combination of both effects? Clearly answering questions about the behavior of a rotating disk with (nearly) tangential light speed at its periphery is very difficult, and even experts have to resort to fairly complex theories (hyperbolic geometry) and general relativity to fit this into the relativistic world of Einstein.