Neutrino Masses

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Abstract: *It is possible to predict the neutrino masses within existing knowledge. The elastic weak current neutrino-nucleon interactions* $(\nu_i, n) \to (l, n' \sim (\nu_e, n) \to (e, p)$, *are considered, and the vector* $\mathbf{m} = |m_e, m_\mu, m_\tau\rangle = |1.35 \times 10^{-8}, 5.81 \times 10^{-4}, 1.64 \times 10^{-1}\rangle$ eV/c^2 *of the increasing neutrino masses is predicted.*

Introduction

A physical particle $\mathbf{P}$ is an object characterized by its internal variables, $\{m, e, \sigma, \mu, \cdots\}$ motion state $(E, \vec{p})$ and its state function $\mathbf{\Psi}$. Fundamental particle state is its rest state, all internal particle variables are associated with the particle rest state, and all other particle states are the evolution of the fundamental particle under interaction transformation. Simply, the particle is all of this $\mathbf{P} = \{m, e, \sigma, \mu, \cdots, (E, \vec{p}), \mathbf{\Psi}\}$.

Here we consider the neutrino-nucleon interaction $(\nu_i, n) \longrightarrow (l, \bar{n}')$. Neutrino interacts on a nucleon creating a lepton and transforming nucleon into nucleon of opposite charge. The detailed process is the neutrino-quark or the quasi-elastic neutrino-quark scattering.

In particular we look at the the $(\nu_e, n) \longrightarrow (p, e)$ interaction presented by the following mapping diagram

$$\hat{\nu} : n \longrightarrow (p, e) \longrightarrow (p, e)^*.$$

Initially, electron neutrino is the free state particle, acting as an operator $\hat{\nu}$ transforming neutron to a proton, creating an electron and bringing the pair (p, e) in an excited state. The electron neutrino undergoes complete decay transformation in the course of the process. Neutrino and electron are in (1:1) correspondence related by the amount of the neutrino energy to create the electron in an energy state. Consequently, there is a function measuring the level of neutrino decay by the "level" of electron creation. The construction of such a function bases on the following sequence of thoughts.

Free neutrino decays in a complex interaction potential, its state function defines the probability of the neutrino decay and consequently the probability of the electron creation in terms of the interaction potential. The theoretical cross-section of the neutrino-electron interaction depends on the electron energy[1] and the interaction probability. Consequently, the cross-section relates the interaction potential and electron energy. Finally, the interaction potential is coupled minimally to the matter in the Kline-Gordon-Dirac equations and the endpoint of the neutrino decay relates the neutrino and electron rest energies.

Interaction Probability

The neutrino decay starts with its sinking in the constant complex potential Γ at $t = 0$, and continues together with complementary process of the electron creation. The neutrino state decay function

$$\Psi \quad = A'\psi(\vec{r}, t)e^{-\frac{\Gamma}{\hbar}t}, \quad A \in \mathbb{C}, \;\; \psi \in \mathrm{L}^2(\mathrm{R}, \mathrm{R}),$$

defines the neutrino decay probability density function

$$\rho(\vec{r}, t) = \frac{dP}{d\vec{r}^3 dt} \quad = |A'|^2 |\psi(\vec{r}, t, *)|^2 e^{-\varpi t}, \quad \varpi = 2\Gamma/\hbar.$$

We will assume that the integration of $|\psi(\vec{r}, t, *)|^2$ reduces to one so that

$$\dot{\mathsf{P}} \quad = |A'|^2 e^{-\varpi t}.$$

Consequently the probability that the neutrino exists at a $t \geq 0$ is

$$\mathsf{P}(t) \quad = \int P(t')dt' = |A'|^2 \int e^{-\varpi t'}dt' + B = -\frac{|A'|^2}{\varpi}\, e^{-\varpi t} + B = Ae^{-\varpi t} + B.$$

Boundary conditions are set to be $P(0) = 1$ and $P(\infty) = 0$ which implies that $B = 0$ and $A = 1$. Hence the neutrino probability is

$$\mathsf{P}(t) \quad = e^{-\varpi t}.$$

Since electron creation and neutrino destruction are complementary events, the probability of electron creation is

$$\mathsf{P}^c \quad = 1 - \mathsf{P} = 1 - e^{-\varpi t}.$$

At equal probabilities point

$$\mathsf{P} = \mathsf{P}^c \quad \rightarrow \quad e^{-\varpi \tau} = 1/2 \quad \Rightarrow \quad \varpi\tau = \ln 2$$

the decay variable has value $\xi_{1/2} = \varpi\tau = \ln 2$, and the $\varpi\tau > \ln 2$ is the electron a creation condition, which eventually imposes some conditions on the neutrino decay potential.

Minimal Uncertainty

Interaction probability, thus the electron creation probability, explicitly depends on the neutrino decay potential. The energy-time uncertainty principle states that the uncertainty $\mathcal{E}\mathrm{T}$ to create an electron of an energy $\mathcal{E}$ in a time interval T cannot be smaller than its minimum $\hbar/2$. This means that all possible realizations of the created electrons are defined by the uncertainty parameter $\theta \geq 1$ and in terms of the minimal uncertainty

$$\mathtt{A} = \mathcal{E}\textsc{t} = \frac{\theta\hbar}{2} \geq \frac{\hbar}{2} \quad \Rightarrow \quad \textsc{t} = \frac{\mathtt{A}}{\mathcal{E}} = \frac{\theta\hbar}{2\mathcal{E}} \geq \frac{\hbar}{2\mathcal{E}} = \textsc{t}_{\textsc{o}}.$$

The time $\textsc{t}_{\textsc{o}}$ is the minimal uncertainty creation or the minimal creation time. For each uncertainty $\mathtt{A}(\theta)$ or each $\theta \geq 1$ an electron of an energy $\mathcal{E}$ is created in the time interval T with the probability $\mathsf{P}^c(T) = \beta$. Consequently the decay variable

$$\varpi T \;\; = \frac{\varpi\mathtt{A}}{\mathcal{E}} = \frac{2\Gamma}{\hbar\mathcal{E}}\frac{\theta\hbar}{2} = \frac{\theta\Gamma}{\mathcal{E}} \geq \;\; \frac{\Gamma}{\mathcal{E}}$$

$$\therefore \qquad e^{-\varpi T} \leq e^{-\Gamma/\mathcal{E}} \;\Leftrightarrow\; \varpi T \geq \Gamma/\mathcal{E}.$$

The value $\xi_{\textsc{o}} = \Gamma/\mathcal{E}$ is the minimum uncertainty decay variable, the $\alpha_{\textsc{o}} = e^{-\Gamma/\mathcal{E}}$ and $\beta_{\textsc{o}} = 1 - \alpha_{\textsc{o}}$ are the minimal uncertainty neutrino existence and electron creation probabilities. Either the minimum uncertainty decay variable is greater or smaller than the equal probability decay value. When the minimum uncertainty decay variable is greater than its half decay probability value

$$\begin{aligned} &\beta \geq e^{-\Gamma/\mathcal{E}} + (1 - 2e^{-\Gamma/\mathcal{E}}) \geq e^{-\Gamma/\mathcal{E}} \\ &\alpha \leq e^{-\Gamma/\mathcal{E}} \leq \beta. \end{aligned}$$

Consequently, for $\xi_{\textsc{o}} \geq \xi_{1/2}$ the decay variable ξ is bounded by the interaction probabilities and for all θ

$$\ln a \geq \frac{\Gamma}{\mathcal{E}} \geq \ln b, \quad a = 1/\alpha, \; b = 1/\beta.$$

The ordering is valid for $\alpha < 1/2 \Leftrightarrow a > 2$ and $\beta > 1/2 \Leftrightarrow b < 2$. Consequently

$$\xi(\theta) \geq \xi(\theta = 1) = \xi_{\textsc{o}} = \frac{\Gamma_{\textsc{o}}}{\mathcal{E}_{\textsc{o}}} \geq \ln b = \; (\ln 2).$$

Definition: *The neutrino state decay interaction potential is measured by the created electron energy and*

$$\Gamma: \;\; \frac{\Gamma}{\mathcal{E}} = g(\mathcal{E}) \geq \ln b, \quad \big(= g(\mathcal{E}) \geq \ln 2 \big).$$

Cross Section

All knowledge about the neutrino decay potential is in the theoretical, experimentally well-known, the cross-section of the decay processes at low energies. If $(\mathcal{E}, \vec{p}, m)$ refers to the electron, $\mathcal{E}_{\textsc{o}} = m_e c^2$ and $f_{\tau/2}$ is the comparative half-life characteristic of the process, the cross section[1] is

$$\sigma \;\; = \frac{2\pi^2 c^3 \hbar^3 \, p\,\mathcal{E}}{\mathcal{E}_{\rm O}^5 f_{\tau/2}} = k_{\rm O}\frac{cp\,\mathcal{E}}{\mathcal{E}_{\rm O}^2}, \quad k_{\rm O} = \frac{2\pi^2 c^2 \hbar^3}{\mathcal{E}_{\rm O}^3 f_{\tau/2}} = 2.3\times 10^{-44}\, cm^2.$$

Since $cp = \sqrt{\mathcal{E}^2 - \mathcal{E}_{\rm O}^2} = \mathcal{E}\Phi$, $\Phi = \sqrt{1-(\mathcal{E}_{\rm O}/\mathcal{E})^2}$, the cross section is

$$\sigma \;\; = k_{\rm O}\frac{\mathcal{E}^2}{\mathcal{E}_{\rm O}^2}\Phi\Big(\frac{\mathcal{E}}{\mathcal{E}_{\rm O}}\Big).$$

Interacting neutrinos are in (1:1) correspondence with produced electrons, and the σ is also the electrons production cross-section. However, the cross-section holds implicitly the probability of the neutrino decay, $\sigma = \sigma(\mathsf{P}) = \sigma(\mathsf{P}(\Gamma))$, which implies $\Gamma = \Gamma(\sigma)$ relation. Further, we look for an explicit relation.

Let N be the total number of the neutrinos in the beam per unit time of the beam, an electron is target. The number of the neutrinos on the target is N_ν, the number of the neutrino electron interactions is $N_{e\cap\nu}$. The conditional probability of an electron when the neutrino event happen is

$$\mathsf{P}_{e/\nu} \;\; = \frac{N_{e\cap\nu}}{N_\nu} = \frac{N_{e\cap\nu}/N}{N_\nu/N} = \frac{\mathsf{P}(e\cap\nu)}{\mathsf{P}_\nu}.$$

The conditional probability is measured by the rate of the neutrino cross section and the electron geometric cross section $A_e \sim a_e^2$ and $a_e = e^2/m_e c^2$, and $\mathsf{P}_{e/\nu} = \sigma/A_e$. The probability $\mathsf{P}(e\cap\nu)$ is the probability P^c of all electros created by the neutrinos, and the probability P_ν of the neutrino interaction is proportional to the comparative half-time $t_{1/2}$ of the neutrino on the electron target in the unit time of the beam. Consequently $\mathsf{P}_\nu \sim 1/t_{1/2}$ and

$$\frac{\sigma}{A_e} \sim \frac{\mathsf{P}^c}{1/t_{1/2}} \quad\Rightarrow\quad \mathsf{P}^c = \; k_{\rm P}\sigma, \qquad k_{\rm P} = \frac{1}{t_{1/2}A_e}.$$

Finally the electron creation probability is

$$\mathsf{P}^c \;\; = k_{\rm P}k_{\rm O}\frac{\mathcal{E}^2}{\mathcal{E}_{\rm O}^2}\Phi\Big(\frac{\mathcal{E}}{\mathcal{E}_{\rm O}}\Big).$$

Remark: The cross section is zero when $\vec{p} = 0$ and production of the electrons of the rest energy is zero. However, at $t = T$ the probability of the electron production is $\beta = 1-\alpha > 0$ for all electron energies $\mathcal{E} \geq \mathcal{E}_{\rm O}$. Hence, the cross section cannot be zero. We notice that for all $\eta = \mathcal{E}_{\rm O}/\mathcal{E} < 1$

$$cp \;\; = \mathcal{E}\sqrt{1-\eta^2} = \mathcal{E}\Big(1 - \frac{\eta^2}{2} - \cdots\Big) \geq 0$$

When the electron energy is approaching the rest energy point from above, the momentum is approaching zero from the above. Hence, any truncated function Φ when $cp \to \mathcal{E}\Phi_*$ will give nonzero production of the electrons at the rest energy. As the first approximation, we take $\Phi_* = 1$.

Definition: *The electrons are produced at the rest energy, and the electron production cross section at an energy $\mathcal{E} \geq \mathcal{E}_{\rm O}$, is*

$$\sigma \quad = k_{\mathcal{E}} k_{\rm O} \begin{cases} \frac{\mathcal{E}^2}{\mathcal{E}_{\rm O}^2}, & \text{if } \mathcal{E} \geq \mathcal{E}_{\rm O}, \\ 0, & \text{if } \mathcal{E} < 0. \end{cases}$$

The electron energy coefficient $k_{\mathcal{E}}$ in the first approximation is one.

Corollary: The electron creation probability is

$$\mathsf{P}^c \quad = k_{\rm P}\, k_{\rm A} k_{\mathcal{E}} k_{\rm O} \frac{\mathcal{E}^2}{\mathcal{E}_{\rm O}^2} \xrightarrow{\mathcal{E} \to \mathcal{E}_{\rm O}} k_{\rm P}\, k_{\rm A} k_{\mathcal{E}} k_{\rm O}, \quad \mathcal{E} \geq \mathcal{E}_{\rm O}.$$

□ This follows directly from the $\mathsf{P}^c = k_{\rm P}\sigma$ after introducing an eventual correction coefficient $k_{\rm A}$ of the electron cross section area, for example $k_{\rm A} = 1/\pi$ or some more convenient factor. In the first approximation we will take $k_{\rm A} = 1 = k_{\mathcal{E}}$. □

Interaction Potential

The electron creation probability, explicitly defined by the decay interaction potential, is directly related to the theoretical electron creation cross section, a function of the created electron energy. Hence, $\mathsf{P}^c = 1 - e^{-\varpi\Gamma}$ so that

$$\begin{aligned} \varpi\Gamma \quad &= -\ln\Big(1 - \mathsf{P}^c\Big) = -\ln\Big(1 - k_{\rm P}\sigma\Big) = k_{\rm P}\sigma + \frac{(k_{\rm P}\sigma)^2}{2} \cdots \\ \Rightarrow \quad \varpi\Gamma \quad &= k_{\rm P}\, k_{\rm A} k_{\mathcal{E}} k_{\rm O} \frac{\mathcal{E}^2}{\mathcal{E}_{\rm O}^2} + \cdots . \end{aligned}$$

The variable $\varpi\Gamma = \Gamma/\mathcal{E}$ at the minimum uncertainty electron production reduces to the

$$\frac{\Gamma}{\mathcal{E}} = k_{\rm P}\sigma + \cdots = k_{\rm P}\, k_{\rm A} k_{\mathcal{E}} k_{\rm O} \frac{\mathcal{E}^2}{\mathcal{E}_{\rm O}^2} + \cdots .$$

The last defines the function $g(\mathcal{E})$ and consequently the decay interaction potential to be

$$\Gamma = k_{\rm P}\sigma\mathcal{E} = \; k_{\rm P}\, k_{\rm A} k_{\mathcal{E}} k_{\rm O} \frac{\mathcal{E}^2}{\mathcal{E}_{\rm O}^2}\, \mathcal{E} \; \xrightarrow{\mathcal{E} \to \mathcal{E}_{\rm O}} \; k_{\rm P}\, k_{\rm A} k_{\mathcal{E}} k_{\rm O}\, \mathcal{E}_{\rm O} = \; k_{\rm A} k_{\mathcal{E}} k_{\rm P} k_{\rm O}\, m_e c^2.$$

Kline-Gordon-Dirac Equations

Finally, an energy equation is necessary to complete the system of equations to find the neutrino mass. A natural equation is a Kline-Gordon equation extended by the neutrino decay interaction potential. An extension is made by the minimal coupling of the decay interaction to the free massive particle, see the Appendix. The particle is the vector $\mathsf{R} = |E, \vec{p}, m, u\rangle$ in the in the $\mathbb{R}^5_1 \otimes \mathbb{R}^2_1 = \mathbb{R}^4_1 \otimes \mathbb{R}^1 \otimes \mathbb{R}^2_1$ Minkowski space.

Definition: *Nonnegative energy function of free particle in the complex decay potential is the norm $|\mathcal{R}|^2$ and the particle energy conservation law is*

$$\mathcal{Z} = |\mathcal{R}|^2 = 0 \quad \Leftrightarrow \quad P^2 - z = \quad E^2 - \vec{p}^{\,2} - z = 0, \quad z = \mu^2 = |\, m^2 + u_{\rm o}^2 - u^2|.$$

A particle is real if its energy is real number, else it is an exotic particle, it is regular particle if its interaction rest energy μ is real.

With each particle is associated the collection of quantum operators residing in the operator space $\hat{\mathbf{R}} = \hat{\mathbf{R}}^4_1 \otimes \hat{\mathbf{R}}^1_1 \otimes \hat{\mathbf{R}}^2_1$, and inheriting all algebraic properties of the vector space $\mathbf{R}$. The quantization of the energy equation defines the particle quantum state equation.

Definition: *The particle state variables and its state decay function Ψ are entangled in the quantum state Kline-Gordon equation, (the light velocity $c = 1$, the particle interaction rest energy, i-rest energy or the i-rest mass is $\mu : \mu^2 = |m^2 - u^2|$)*

$$\hat{\mathcal{H}}^2\Psi \quad = (\widehat{P^2} - \widehat{\mu^2})\Psi = \ (\widehat{E^2} - \widehat{\vec{p}^{\,2}} - \widehat{\mu^{\,2}})\Psi = 0.$$

Corollary: The energy function operator $\hat{\mathcal{H}}^2 = \hat{P}^2 - \hat{\mu}^2 = (\hat{P} - \hat{\mu})(\hat{P} + \hat{\mu}) = \hat{\Pi}\hat{\Pi}^*$ and

$$\hat{\mathcal{H}}^2\boldsymbol{\Psi} \quad = (\hat{P}^2 - \hat{\mu}^2)\Psi = \ (\hat{E}^2 - \hat{\vec{p}}^{\,2} - \hat{\mu}^{\,2})\Psi.$$

In addition, the linear factorization or the Kline-Gordon equation, or the Kline-Gordon-Dirac equations for the free particle state decay holds and

$$\begin{aligned} \hat{\Pi}\psi \quad &= \ (\hat{P} - \mu)\psi = (E - \vec{p} - \mu)\psi = 0, \\ \hat{\Pi}^*\psi \quad &= \ (\hat{P} + \mu)\psi = (E + \vec{p} + \mu)\psi = 0. \end{aligned}$$

□ To shorten notation we use $\hat{P} = \hat{E} + \hat{\vec{p}}$, $\hat{\mu} = m + u_{\rm o} - u$. By definition $\hat{P}\Psi = P\Psi$, $\hat{\mu}\Psi = \mu\Psi$, and $\hat{P}^2\Psi = P^2\Psi$ and $\hat{\mu}^2\Psi = \mu^2\Psi$, so that

$$\begin{aligned} \hat{\mathcal{H}}^2\Psi \quad &= (\widehat{P^2} - \widehat{\mu^2})\Psi = (P^2 - \mu^2)\Psi = P^2\Psi - \mu^2\Psi \\ &= P\hat{P}\Psi - \mu\hat{\mu}\Psi = \hat{P}^2\Psi - \hat{\mu}^2\Psi = (\hat{P}^2 - \hat{\mu}^2)\Psi. \end{aligned}$$

However the operators inherit the scalar product of the space $\mathbb{R}^3_1 \otimes \mathbb{R}^1 \otimes \mathbb{R}^2_1$ and

$$(\hat{P}^2 - \hat{\mu}^2)\Psi = \langle \hat{P} - \hat{\mu}|\, \hat{P} + \hat{\mu}\rangle\Psi = \hat{\Pi}\hat{\Pi}^*\Psi.$$

Suppose that $\hat{\Pi}\Psi \neq 0$ and $\hat{\Pi} * \Psi \neq 0$. Then

$$0 \quad = \hat{\mathcal{H}}^2 = \Pi\Pi * \Psi = \Pi\hat{\Pi} * \Psi = \hat{\Pi} * \Pi\Psi = \hat{\Pi} * \hat{\Pi}\Psi \neq 0.$$

contradiction. Hence the linear state equations must hold. We notice that the Kline-Gordon-Dirac linear equations differ from the Dirac linear relativistic spinor equation. □

Definition: *The pair* $(\mathbf{P}, \mathbf{P}^*)$ *of the objects in the states described by the linear Kline-Gordon or Dirac equation are dual particles.*

Kline-Gordon Particles

We notice that the particle $\mathbf{P}$ is the evolution of the particle $\mathbf{P}_o$ in the imaginary decay potential $G = iu$, and consequently that is the particle of interaction rest mass, i-mass or the interaction i-rest energy $z = \mu^2$.
The particles that satisfy the Kline-Gordon equation may be characterized by their energy solutions $E^2 = \vec{p}^2 + z$ and i-rest energy function $z = m^2 - u^2$. Let $M = m - u$ and $M_* = m + u$. Then $M_* + M = 2m$, $M_* - M = 2u$ and i-rest, energy $M_* M$ is

$$z = \begin{cases} +M_* M, & E_o = \pm\sqrt{|M_* M|}, & \text{if } m > u, \\ 0, & E_o = 0, & \text{if } m = u, \\ -M_* M, & E_o = \pm i\sqrt{|M_* M|}, & \text{if } m < u. \end{cases}$$

Positive and negative energy solutions correspond to the dual particle pairs $(\mathbf{P}, \mathbf{P}^*)$. To get an understanding of the imaginary energy solution we will look at the particle decay in the sequence of the increasing constant interaction potential depths. Thus, we have pairs (m, u) with $m > u$, $m = u$ and $m < u$ or the $z > 0$, $z = 0$ and $z < 0$ i-rest energy cases.

1. The choice of (m, u) sets the zero i-rest energy point and places the particle into the class of the positive or negative i-rest energy function z. The particle is real as long as its energy $E^2 = \vec{p}^2 + z$ is nonnegative. This is true for all regular particles, $m \geq u$. All free particles start as the regular particles and hold their regularity in the depth increasing decay potential until the zero i-rest energy.

Definition: *The zero i-rest energy is the decay particle splitting point.*

2. When the depth of the decay potential reaches the particle rest energy, the particle i-rest energy becomes zero, and the Kline-Gordon equation splits into two equations

$$P^2 \quad = E^2 - \vec{p}^2 = 0, \quad z \quad = m^2 - u^2 = 0.$$

The real particle $\mathbf{P}$ enters the zero i-rest energy point with either $(E, \vec{p}) \equiv 0$ or $(E, \vec{p}) > 0$.

When $(E, \vec{p}) \equiv 0$ the particle split is $\mathbf{P}_{\mathrm{O}} \sim \mathbf{P} \wedge \mathbf{P}_{\emptyset}$ where $\mathbf{P}_{\emptyset} = (0,0,0,*)$ is the zero Kline-Gordon photon, and the $\mathbf{P}$ the particle of the mass $\mathrm{M} = m$, sitting in the rest mass field. Such particle is decoupled from the motion.

When $(E, \vec{p}) \neq 0$ the particle split is $\mathbf{P}_{\mathrm{O}} \sim \mathbf{P} \wedge \mathbf{P}_{\gamma}$ where $\mathbf{P}_{\gamma}$, is the Kline-Gordon photon. The original particle decouples into Kline-Gordon photon and the massive particle $\mathbf{P}$ of the mass $\mathrm{M} = m$, sitting in the rest mass field.

3. The decay potential may exceed the particle rest energy and, the i-rest particle energy function becomes negative. In that case

$$E^2 \quad = \vec{p}^{\,2} - \mu^2 \geq 0, \quad \forall\, \vec{p} \geq \vec{p}_m = \mu^2.$$

Real particles with $\vec{p}_m \neq 0$ are never at the rest, a such one is the photon. We notice that the equation is equivalent to the

$$\vec{p}^{\,2} \quad = E^{\,2} + \mu^2 \geq 0, \quad \mu^2 \geq 0.$$

Remark: We recall that upper and lower extremals differ in the sign of the four-momentum, and that all above minimal coupling extensions done to the lower extremal $-x_{\mathrm{O}}^2 + x^2$ would produce the energy function

$$\begin{aligned} \mathcal{Z} \quad &= -E^2 + \vec{p}^{\,2} - z, \quad z = \mu^2 = |\, m^2 - u^2|. \\ \Rightarrow \quad \mathcal{Z} \quad &= 0 \quad \Rightarrow \quad \vec{p}^{\,2} = \quad E^2 + \mu^2. \end{aligned}$$

For $z > 0$ the energy is the same as the energy on the upper extremal solution for the negative i-rest energy. Hence

$$E^2 \quad = \begin{cases} \vec{p}^{\,2} + \mu^2 & \geq 0, \quad \text{if } \; z > 0, \\ 0, & \qquad\quad\; \text{if } \; z = 0, \\ \vec{p}^{\,2} - \mu^2 & \geq 0, \quad \text{if } \; z < 0. \end{cases}$$

Dirac Particles

Dirac particles are the particle pair $(\mathbf{P}, \mathbf{P}^*)$ solution to the linear factorization equations of the Kline-Gordon equation and have the rest energies $M = m - u$ and $M_* = m + u$. If the particle has the i-rest energy M the i-rest energy of its $M_* = 2m - M$. At the splitting point, the i-rest mass is zero, the pair $(\mathbf{P}, \mathbf{P}^*)$ of the dual particles is the massless particle $\mathbf{P} \sim M = 0$ and massive dual particle $\mathbf{P}^* \sim M_* = 2m$.

Neutrino Masses

In the neutrino-nucleon interaction process minimal creation is the creation of an electron at its rest state. Consequently, the minimal requirement is that the neutrino state completely decays at the splitting point. The splitting point is the neutrino decay end point, there, $\mu = 0$, an electron at the the rest state followed the $\mathbf{P}_\emptyset$ Kline-Gordon particle is created. Thus

$$\mu^2 \; = 0 \;\; \Leftrightarrow \;\; m^2 - \Gamma^2 = 0 \;\; \Rightarrow \;\; m = k_{\rm P}\sigma \, m_e = k_{\rm A} k_{\mathcal{E}} k_{\rm P} k_{\rm O} \, m_e.$$

For the $(\nu_e, n) \longrightarrow (p, e)$ process $t_{1/2} = 1.1 \times 10^3$. Since $k_{\rm O} = 2.3 \times 10^{-44}\, s$ and the area $a_e^2 = 7.90 \times 10^{-34}\, cm^2$, $k_{\rm P} = 1/1.15 \cdot 10^{30}$ the electron neutrino mass in the first approximation $k_{\rm A} = 1 = k_{\mathcal{E}}$ is

$$\begin{aligned} m_{\nu_e} \; &= k_{\rm P} k_{\rm O} \, m_e = 1.15 \times 10^{30} \cdot 2.3 \times 10^{-44} \cdot 0.511 \\ &= 1.35 \times 10^{-14}\, \mathrm{MeV/c^2} = \; 1.35 \times 10^{-8}\, \mathrm{eV/c^2}. \end{aligned}$$

Electron neutrino elastic scattering is the prototype of the interactions of all neutrinos with nucleons through the charged week currents. Thus, the muon and the tau neutrinos follow the $(\nu_i, N) \longrightarrow (\bar{N}, l)$ process.
It is reasonable to assume that all cross-sections have the same structure and differ among themselves due to charged lepton mass entries, and it may be the characteristic half-time $t_{1/2}$. If m_i is a neutrino and m_i its lepton mass and e refers to the electron then

$$\mathsf{m}_i \;\; = k_{\rm P}\sigma^i \, m_i = k_{\rm P}\sigma^e m_e \, \frac{\sigma^i}{\sigma^e} \frac{m_i}{m_e} = \mathsf{m}_e \, \frac{k_{\rm O}^i}{k_{\rm O}^e} \frac{m_i}{m_e} = \mathsf{m}_e \, \frac{m_e^3}{m_i^3} \frac{t_{1/2}^e}{t_{1/2}^i} \frac{m_i}{m_e} = \quad F(m_i, m_e) \, \frac{m_i}{m_e}.$$

Table 1. Calculation of the Neutrino Masses

	e	μ	τ	$\sum$
m_i	5.11×10^{-1}	1.06×10^{-2}	1.78×10^{-3}	
$\frac{m_i}{m_e}$	1	$2.07 \times 10^{+2}$	$3.48 \times 10^{+3}$	
$\frac{m_e}{m_e}$	1	4.83×10^{-3}	2.87×10^{-4}	
$\mathsf{m}_i^{\rm I}$	1.35×10^{-8}	5.81×10^{-4}	1.64×10^{-1}	0.165
$\mathsf{m}_i^{\rm II}$	1.35×10^{-8}	3.15×10^{-13}	1.11×10^{-15}	1.35×10^{-8}
	$\Delta\mathsf{m}^2_{\lvert 12\rvert}$	$\Delta\mathsf{m}^2_{\lvert 23\rvert}$	$\Delta\mathsf{m}^2_{\lvert 31\rvert}$	
I	3.37×10^{-7}	2.70×10^{-2}	$2.7 = \times 10^{-2}$	eV^2/c^4
II	1.83×10^{-16}	9.94×10^{-26}	183×10^{-16}	eV^2/c^4

We assume that the half-life time does not vary bay the charged leptons masses, and consider the following cross-section cases

Case I: The cross section does not vary by the lepton generation/masses, $F = 1$,
Case II: The cross section does vary by the lepton generation/masses. $F \neq 1$.

The table above shows the calculation masses are in the $\mathrm{eV/c^2}$.

Conclusion

There are two generation chains of the neutrino masses, each starting with the electron neutrino mass. In the case that the cross-sections do not vary over the generations the neutrinos constitute the increasing mass chine ordered as $\nu_e \prec \nu_\mu \prec \nu_\tau$, and the decreasing mass chine ordered as $\nu_e \succ \nu_\mu \succ \nu_\tau$ when the cross-sections vary over the generations. We notice that the both chines satisfy the muss sum limit[2] $\Sigma \mathsf{m}_i < 0.17\,\mathrm{eV}$ with 95% confidence set by the Planck satellite data. However, counted oscillation square mass differences are consistent with the neutrino oscillation experiments only for the mass-increasing neutrino ordering. Thus, the mass-increasing neutrino ordering chain $|\mathsf{m}_e, \mathsf{m}_\mu, \mathsf{m}_\tau\rangle = |1.35 \times 10^{-8},\, 5.81 \times 10^{-4},\, 1.64 \times 10^{-1}\rangle$ eV/c^2 is predicted.

References

[1] Hans Frauenfelder, Ernest M Hennley, *Subatomic Physics,* Prentice Hall. /Englewood Cliffs, New Jersey 07632,

[2] Pablo F. de. Salas, et al. *Neutrino Mass Ordering from Oscillations and Beyond: 2018 status and Future prospects*, Front Astron. Space Sci., 09 Oct 2018

Appendix

Minimal Coupling

In this part $\mathbb{R}^{m+n}$ is an Euclidean space spanned by the base $(\hat{x}_\mathrm{O}, \hat{\mathsf{x}})$ with defined scalar product. The general vector $\mathrm{X} = \vec{x} + \vec{\mathsf{x}} = x_\mathrm{O}\hat{x} + \mathsf{x}\hat{\mathsf{x}}$ square

$$\mathrm{X}^2 \;= x_\mathrm{O}^2 + 2\vec{x}_\mathrm{O}\vec{\mathsf{x}} + \mathsf{x}^2 = x_\mathrm{O}^2 + 2x_\mathrm{O}\mathsf{x}\cos\theta + \mathsf{x}^2 = (x_\mathrm{O} + \mathsf{x}\cos\theta)^2 + \mathsf{x}^2\sin^2\theta > 0,$$

is the Euclidean norm $|\mathrm{X}|^2$ in $\mathbb{R}^{m+n}$. The vector square is the function $y(x_\mathrm{O}, \mathsf{x})$ of the magnitudes $(x_\mathrm{O}, \mathsf{x})$. The function $y(x_\mathrm{O}, \mathsf{x})$, upper paraboloid over $\mathbb{R}^{m+n}$ space, has minimum $y(0,0) = 0$, and does not have other extremes. Further, we look to find the extremes of the functions, $\mathrm{U}(x_\mathrm{O}) = y(x_\mathrm{O}, const)$ and $\mathrm{V}(\mathsf{x}) = y(const, \mathsf{x})$, restrictions of the square function by the $x = const$ and $x_\mathrm{O} = const$ cuts. Exactly,

$$\begin{aligned}
\partial_\mathsf{x}\mathrm{V} \;&= (2\vec{x}_\mathrm{O} + 2\vec{\mathsf{x}})\hat{\mathsf{x}} = 0 \;\Rightarrow\; (2\vec{x}_\mathrm{O}\vec{x} + 2\vec{\mathsf{x}}^2) = 0, \quad \partial_{\mathsf{x}\mathsf{x}}y \;= 2\hat{\vec{\mathsf{x}}}^2 \;= 2 > 0\\
\partial_{x_\mathrm{O}}\mathrm{U} \;&= (2\vec{x}_\mathrm{O} + 2\vec{\mathsf{x}})\hat{x}_\mathrm{O} = 0 \;\Rightarrow\; (2\vec{x}_\mathrm{O}^2 + 2\vec{x}_\mathrm{O}\vec{\mathsf{x}}) = 0, \;\; \partial_{x_\mathrm{O}x_\mathrm{O}}y \;= 2\hat{x}_\mathrm{O}^2 = 2 > 0,
\end{aligned}$$

and the extremals are the upper and lower hyperbolic saddles at the point $(\vec{0}, \vec{0})$ over $\mathrm{N} = n+m$ dimensional space

$$y = \begin{cases} Z = \{+x_{\rm o}^2 - x^2\} \subset \ \mathbb{R}_n^{m+n}, & \text{upper saddle at } x_{\rm o} \text{ fixed cut,} \\ Y = \{-x_{\rm o}^2 + x^2\} \subset \ \mathbb{R}_m^{m+n}, & \text{lover saddle at } \mathsf{x} \ \ \text{fixed cut.} \end{cases}$$

Definition: *The space $\mathbb{R}^n$ is minimally coupled to the space $\mathbb{R}^m$ into Minkowski space $\mathbb{R}_n^{m+n}$ with the hyperbolic norm function*

$$\mathcal{Z} \quad = |\mathrm{x}|^2 = \vec{x}^2 - \vec{\mathsf{x}}^2.$$

Scalar product induced by the hyperbolic norm is the function

$$\langle \mathrm{x}|\mathrm{x}\rangle \quad = \langle \vec{x}, -\vec{\mathsf{x}}\,|\, \vec{x}, \vec{\mathsf{x}}\rangle.$$

where $\langle \mathrm{x}| = \langle \vec{x}, -\vec{\mathsf{x}}\,|$ is conjugate of the vector $|\mathrm{x}\rangle = |\,\vec{x}, \vec{\mathsf{x}}\rangle$. The norm in the physical application is the energy square function $\mathcal{Z} = \mathcal{H}^2$.

Corollary: The space and time are minimally coupled into space-time four-vector $\mathrm{x} = (x_{\rm o}, \vec{\mathsf{x}})$ of the Minkowski space $\mathbb{R}_1^4$ with the norm $|\mathrm{x}|^2 = x_{\rm o}^2 - \vec{\mathsf{x}}^2$. Scalar product consistent with the norm is $\langle \mathrm{x}|\mathrm{x}\rangle = \langle x_{\rm o}, -\vec{\mathsf{x}}|\, x_{\rm o}, \vec{\mathsf{x}}\rangle$.

Corollary: The massless particle energy and momentum are coupled into energy-momentum four-vector $P = (E, c\vec{p}) \sim (E, \vec{p})$ of the Minkowski space $\mathbb{R}_1^4$. The massless particle $\mathbf{P} = (E, \vec{p})$ is the vector in the $\mathbb{R}_1^4$, and its energy function is

$$\mathcal{Z} \quad = E^2 - \vec{p}^{\,2}.$$

The free massless particles preserves its energy and $\mathcal{Z} = |P|^2 = E^2 - \vec{p}^{\,2} = 0$. Scalar product induced by the norm is $\langle P|P\rangle = \langle E, -\vec{p}|\, E, \vec{p}\rangle$.

$\square$ Euclidean norm $|P|$ of the vector $P = (E, \vec{p})$, with the speed of the light $c = 1$, implies

$$\begin{aligned} P^2 \quad &= E^2 + 2E\vec{p} + \vec{p}^{\,2} \Rightarrow \ \partial_p P^2 = 2E\hat{1} + 2p\hat{p}, \ \ \partial_{pp} P^2 = 2 > 0 \\ &\partial_p P^2 = 0 \ \Rightarrow \ 2(E\vec{p} + \vec{p}^{\,2}) = 0 \ \ \Rightarrow \quad \mathcal{Z} = E^2 - \vec{p}^{\,2}. \end{aligned}$$

The energy function $\mathcal{Z}$ is the energy-momentum for the physical particles. In particular for a free massless particle energy is preserved and $\mathcal{Z} = E^2 - \vec{p}^{\,2} = 0$. Under this condition $|P|^2 = \langle E, -\vec{p}|\, E, \vec{p}\rangle = \langle P|\, P\rangle$. $\square$

The evaluation $\mathcal{Z} = 0$ is the energy-momentum conservation law for the free massless particle. Else, $\mathcal{Z} = const > 0$ is the statement of the energy-momentum conservation law for massive particles. However, the massive particle may be introduced into energy-momentum function by the minimal coupling of the matter to space-time.

Corollary: The matter is minimally coupled to the space-time by the hyperbolic norm $|\mathcal{P}|^2 = P^2 - m^2$ into five-vector $(E, \vec{p}, m)$ in the Minkowski space $\mathbb{R}_{1+3+1}^5$. For either massive or massless particles the norm $|\mathcal{P}|^2 = 0$ is the free particle energy-momentum conservation law. Scalar product consistent with the norm is $|\mathcal{P}|^2 = \langle E, -\vec{p}, -m|\, E, \vec{p}, m\rangle = \langle \mathcal{P}|\, \mathcal{P}\rangle$.

□ Independently on that wether $P = (E, \vec{p})$ is minimally coupled into energy-momentum four vector, the vector $\mathcal{P} = (P, mc^2) \sim (P, m)$ is the vector in the five dimensional space and

$$\begin{aligned} \mathcal{P}^2 &= P^2 + 2Pm + m^2 \Rightarrow\ \partial_m \mathcal{P}^2 = 2P + 2m,\ \partial_{mm}\mathcal{P}^2 = 2 > 0 \\ &\partial_m \mathcal{P}^2 = 0\ \Rightarrow\ 2(Pm + m^2) = 0\ \ \Rightarrow\ \ \ \mathcal{Z} = P^2 - m^2. \\ \mathcal{Z} &= P^2 - m^2 = E^2 - \vec{p}^2 - m^2 \geq 0 \end{aligned}$$

The nonnegative energy function is the hyperbolic norm and represents the energy-momentum for a massive particle. The energy-momentum is conserved for a free massive particle. Accordingly, $|\mathcal{P}|^2 = \langle P, -m|\, P, m\rangle = \langle E, -\vec{p}, -m|\, E, \vec{p}, m\rangle = \langle \mathcal{P}|\, \mathcal{P}\rangle$. □

Further, a free particle $\mathbf{P}_{\rm O}$ is a physical object characterized by the five-vector $\mathcal{P} = (P, m) = (E, \vec{p}, m)$ in the six dimensional space $\mathbb{R}^4_1 \otimes \mathbb{R}^1$. However, an interacting particle must include the interaction potential G.

<u>**Definition:**</u> *An interaction potential is function $G = u_{\rm O} + iu$, minimally coupled to the particle.*

<u>**Corollary:**</u> The energy function of a particle in the complex interaction potential G is

$$\mathcal{Z} \ \ = |\mathcal{P}|^2 - G^2.$$

When the potential imaginary part is minimally coupled to its real part

$$\mathcal{Z} \ \ = |P|^2 - z = \ \ \ E^2 - \vec{p}^2 - z.$$

where $z = m^2 + u_{\rm O}^2 - u^2$ is the particle interaction rest mass, call it i-rest mass or i-rest energy. Interacting particle is the vector R in the $\mathbb{R}^5_1 \otimes \mathbb{R}^2_1 = \mathbb{R}^4_1 \otimes \mathbb{R}^1 \otimes \mathbb{R}^2_1$ vector space, and the scalar product consistent with the particle energy function is

$$\langle \mathsf{R}|\, \mathsf{R}\rangle \ \ = \langle E, -\vec{p}, -m, -u_o, u|\, E, \vec{p}, m, u_{\rm O}, u\rangle.$$

□ The interacting particle is the vector $\mathcal{R} = (\mathcal{P}, G)$ in the six dimensional space, interaction potential is minimally coupled to the particle and

$$\begin{aligned} \mathcal{R}^2 &= \mathcal{P}^2 + 2\mathcal{P}G + G^2 \ \ \Rightarrow\ \ \partial_{\rm G}\mathcal{P}^2 = 2(\mathcal{P} + G) = 0 \Rightarrow 2\mathcal{P}G + 2G^2 = 0 \\ \Rightarrow &\ \ \ \ \ \mathcal{Z} = \mathcal{R}^2 \ \ = |\mathcal{P}|^2 - G^2. \end{aligned}$$

In general the G^2 is the measure of the interaction with particle. However, the measure is also the interaction potential with minimally coupled components. Since particle mass couples minimally to the four-momentum we will couple the real part of the interaction to the four-momentum with the negative sign. Thus, the imaginary part of the interaction must minimally couple to the real part, and

$$G^2 = u_\text{O}^2 + 2iu_\text{O}\cdot u + u^2, \quad \partial_u G^2 = 2iu_\text{O}\hat{u} = 0 \;\Leftrightarrow\; 2iu_\text{O}u + 2u^2 = 0 \;\Rightarrow\; G^2 = u_\text{O}^2 - u^2.$$

However, $|\mathcal{P}^2| = P^2 - m^2 = E^2 - \vec{p}^2 - m^2$ so that

$$\mathcal{Z} = P^2 - m^2 = E^2 - \vec{p}^2 - m^2 - u_\text{O}^2 + u^2 = E^2 - \vec{p}^2 - z.$$

The function $z = m^2 + u_\text{O}^2 - u^2$ is the particle interaction rest energy function and may be positive, negative and zero. The particle vector $\mathbf{P} = |E, \vec{p}, m, u_\text{O}, u\rangle$ is in the $\mathbb{R}_1^5 \otimes \mathbb{R}_1^2 = \mathbb{R}_1^4 \otimes \mathbb{R}^1 \otimes \mathbb{R}_1^2$. Nonnegative energy function is the norm $|\mathsf{R}|^2$, and the scalar product induced by the norm

$$\langle \mathsf{R} | \, \mathsf{R} \rangle = \langle E, -\vec{p}, -m, -u_o, u | \, E, \vec{p}, m, u_\text{O}, u \rangle = |\mathsf{R}|^2.$$

□

The interaction mass or the interaction rest energy function for the particle in the complex interaction potential

$$z = \begin{cases} m^2 + u_\text{O}^2 - u^2, & \text{if } m^2 + u_\text{O}^2 - u^2 > 0, \\ 0, & \text{if } m^2 + u_\text{O}^2 - u^2 = 0, \\ u^2 - m^2 - u_\text{O}^2, & \text{if } m^2 + u_\text{O}^2 - u^2 < 0. \end{cases}$$