Evaluating the Alignment of the Polarized Starlight from 893 Stars in a Region on the Disk of the Milky Way

Richard Shurtleff *

Abstract

Detecting polarized starlight projects an intriguing pattern of polarization directions on the Galaxy. Polarized starlight is a well known tracer of Galactic Magnetic fields and is a tool for studying the dust that contaminates the view of more distant objects. The alignment of the polarization directions of a sample of stars on the Galactic Disk is investigated with a recently devised test. The Hub Test offers numerical metrics based on the geometry of spherical geodesics, *i.e.* great circles, to judge alignment. The test always compares the directions of two vectors at a single point, a process that avoids comparing the directions of two vectors at distinct points; no parallel transport needed. The sample of 893 stars, located from longitude 90° to 160° and latitude -15° to +15°, is among the best aligned regions on the Disk. The alignment function provides a full-sphere depiction of the collective alignment. The metrics include the likelihood that random polarization directions would produce equal or better alignments. For the 893 star sample considered here, the alignment occurs at the 54σ level. The alignment function has minima along an equator, which, for this sample, coincides with the Galactic Disk, and the function has maxima at poles, here coincident with the Galactic Poles. The source of the polarization data is the Heiles 2000 agglomeration catalog. This article is a Mathematica notebook which can be accessed and run via a link in the References.

Keywords: Polarized Starlight; Alignment; Computer Program; Uncertainties; Hub Test; Galactic Structure; Galactic Magnetic Field

*Department of Sciences, Wentworth Institute of Technology, 550 Huntington Avenue, Boston, MA, USA, 02115, orcid.org/0000-0001-5920-759X, e-mail addresses: shurtleffr@wit.edu, momentummatrix@yahoo.com

```
In[1]:= Print["The date and time that this statement was evaluated: ", Now]
```

```
The date and time that this statement was evaluated: Mon 29 Mar 2021 13:50:05 GMT−4.
```

0. Preface

The pdf version of this notebook is available online from the viXra archive. For the ready-to-run notebook follow the link in Ref. 1.

Notes:

(1) The large amount of catalog data needed for the ready-to-run notebook is not shown in the pdf version. One can use the record number list in the Appendix to generate the needed data, if the ready-to-run notebook is not available.

(2) The pdf version of this notebook reflects 503 uncertainty runs. That is a large number, consuming considerable computer time. The ready-to run notebook is set up to generate fewer uncertainty runs. [The “Uncertainty runs” follow the process of alignment evaluation but with data varying from the best values consistent with uncertainties in measurement. Experimental uncertainties produce uncertainties in the results.]

(3) The numerical values quoted in the pdf version are associated with the particular settings and uncertainty runs that were current when the pdf version was created. Other sets of uncertainty runs should alter those numerical values only slightly.

(4) A template for performing calculations similar to those in this notebook, but with other data, can be found online, Ref. 2.

(5) These notebooks were created using Wolfram Mathematica, Version Number: 12.1, Ref. 3.

(6) The formulas for creating Aitoff plots were found on Wikipedia, Ref. 4.

The Hub Test

One motivation for constructing this notebook is to present an application of the Hub Test, which is discussed more fully in Ref. 5.

Polarization directions are well-aligned with each other when they are well-aligned with some point on the Celestial Sphere. Consider the well-known alignment of the direction from Merak to Dubhe with Polaris. Guided by Fig. 1, let the source S be Merak, take the interval from Merak to Dubhe for the direction of polarization $\hat{v}_\psi$, and let Polaris be the point H. Then the alignment of the Merak to Dubhe direction $\hat{v}_\psi$ with Polaris, the point H, illustrates the concept of alignment in the Hub Test. With Merak as S, Merak-Dubhe as $\hat{v}_\psi$, and Polaris as H, the angle η would be about $\eta = 3.47°$. In that case, the blue great circle and the purple great circle in Fig. 1 would almost coincide.

Out[ ]=

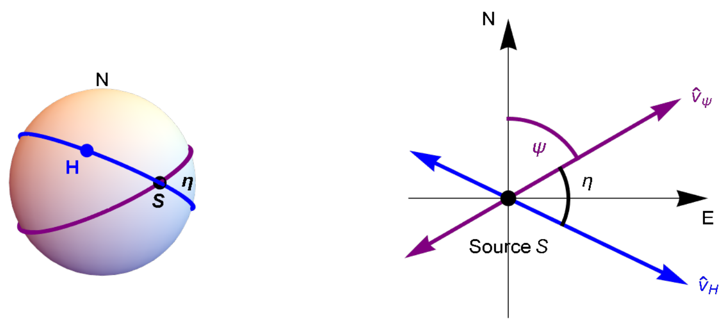

Figure 1: The Celestial sphere is pictured on the left and on the right is the plane tangent to the sphere at the source S. The linear polarization direction $\hat{v}_\psi$ lies in the tangent plane and determines the purple great circle on the sphere. A point H on the sphere and the point S determine a second great circle, the blue circle drawn on the sphere at the left. Clearly, H and S must be distinct. Choose the acute angle η between great circles, $0° \leq \eta \leq 90°$. The "alignment angle" η measures the alignment of the polarization direction $\hat{v}_\psi$ with the point H. Perfect alignment occurs when $\eta = 0°$ and the two great circles overlap. Perpendicular great circles, $\eta = 90°$, indicates maximum "avoidance" of the polarization direction $\hat{v}_\psi$ with the point H on the sphere.

With N sources S_i, $i = 1, ..., N$, there are N alignment angles η_{iH} for the point H and an average alignment angle at H,

$$\overline{\eta}(\mathrm{H}) = \frac{1}{N} \sum_{i=1}^{N} \eta_{\mathrm{iH}} \,. \qquad (1)$$

The alignment angle $\overline{\eta}(\mathrm{H})$ is a function of position H on the sphere. It is symmetric across diameters, $\overline{\eta}(\mathrm{H}) = \overline{\eta}(\mathrm{-H})$, because great circles are symmetric across diameters. The function $\overline{\eta}(\mathrm{H})$ measures convergence and divergence of the great circles determined by the polarization directions. For random polarization directions, the average $\overline{\eta}(\mathrm{H})$ should be near 45°, since each alignment angle η_{iH} is acute, $0° \leq \eta_{\mathrm{iH}} \leq 90°$. Points H where the alignment angle $\overline{\eta}(\mathrm{H})$ is smaller than 45°, the great circles converge, where $\overline{\eta}(\mathrm{H})$ is larger than 45°, the great circles diverge.

Thus the basic concept includes "avoidance", as well as alignment. Avoidance is high when the two directions $\hat{v}_\psi$ and $\hat{v}_H$ differ by a large angle, $\eta \to 90°$. Perpendicular great circles at S , $\eta = 90°$, would indicate the maximum avoidance of the polarization direction and the point on the sphere. The N sources' polarization directions most avoid the points $H_{\max}$ and $-H_{\max}$ where the function $\overline{\eta}(\mathrm{H})$ takes its maximum value $\overline{\eta}_{\max}$. The locations of the most extreme divergence are called "avoidance hubs".

The N sources' polarization directions are best aligned with the points $H_{\min}$ and $-H_{\min}$ where the alignment angle is a minimum $\overline{\eta}_{\min}$. The locations $H_{\min}$ and $-H_{\min}$ of their most extreme convergence are called "alignment hubs". Alignment and avoidance are equally viable, complementary concepts with the Hub Test.

The Hub test provides many calculated results to describe the collective behavior of the polarization directions in a sample. The alignment angle function $\overline{\eta}$(H), Eq. (1), can be mapped on the Celestial Sphere to give a visual display. The smallest alignment angle $\overline{\eta}_{\text{min}}$ and the largest avoidance angle $\overline{\eta}_{\text{max}}$ quantify the agreement of the directions. Known formulas, see Sec. 4 below, are available to calculate the significance of the alignment, *i.e.* the likelihood that random polarization directions would yield better results. The locations of the convergence hubs H_{min} and the divergence hubs H_{max} provide geometric clues to magnetic field direction and such quantities.

CONTENTS

1. Introduction

For those interested in the structure of the Milky Way, polarized starlight infers the direction of the Galactic magnetic field, see for example, Refs. 6 & 7. For those interested in deep space objects on the far side of the Galaxy, polarized starlight helps uncover the physics of the contaminating dust that obscures the objects of interest, see, for example, Refs. 8 & 9.

The Hub Test, described above in the Preface, supplies several quantitative measures that may be helpful in understanding the implications of the polarization directions of a given sample.

This work looks at a very significantly aligned sample of 893 stars occupying a region on the Galactic Disk. The stars' polarization directions are aligned at the 56σ level, with the chance that the alignment is random being nil. Since the alignment is quite well known and not surprising, analyzing this sample advertizes the Hub Test's supply of numerical metrics of the collective polarization behavior.

Certainly, alignment is an important characteristic. However, one aspect of collective behavior that is often overlooked is the concept of avoidance. Consider the great circles drawn by extending polarization directions outward from the sources. Suppose these great circles converge on a nearby point, while the sources themselves occupy an extended region. Their perpendicular directions fan out in many directions and the maximum alignment angle is relatively small, well away from 90°. There is no region of significant divergence. On the other extreme, for parallel polarization directions from a tight source region, the alignment directions form a kind of equator with poles that are significantly avoided. The sample here illustrates the latter extreme by showing alignment with the Galactic Disk, with avoidance hubs located just a few degrees from the Galactic poles. See Fig. 4. And, for some purposes, the direction perpendicular to the polarization directions is important and, in those cases, avoidance may be of more interest than alignment.

Some preliminary formulas and the construction of the grid are presented in Sec. 2. The grid is a 2°x2° mesh of 10518 grid points that is adjusted by latitude to maintain equal spacings. The needed data from the Heiles 2000 catalog, Ref. 10 & 11, is introduced in Sec. 3. Even a list of 893 four digit integers is fairly long, so, to save space, the needed position and polarization data is not displayed, only the record numbers in the catalog are displayed. From the record numbers and the catalog, the file of data that is used can be recreated. The probability and significance formulas in Sec. 4 depend, in part, on the average angular extent of the sample and on the number of stars in the sample.

Sec. 5 presents the analysis of the "best" polarization directions, meaning the values listed in the catalog for the polarization direction. One finds values for the smallest alignment angle $\overline{\eta}_{\min}$, the largest avoidance angle $\overline{\eta}_{\max}$, and the locations hubs on the sphere where these extreme alignment angles are found. The uncertainty in the statistics formulas give the significance of these results some uncertainty.

The inevitable, but important, uncertainty in measured values leads to uncertainty in results. Uncertainty in the measured polarization directions is data provided along with the polarization directions in the catalog. The effect of experimental uncertainty occupies the focus in Sec. 6. Sec. 7 finishes the article with some concluding remarks.

2. Coordinates, grid, and sundry basic formulas

2a. Coordinates

Consider a sphere in 3 dimensional Euclidean space. See Fig. 1 in the Introduction. The sphere is called the "Celestial Sphere" or simply the "sphere" or sometimes "the sky". The center of the sphere is the origin of a 3D Cartesian coordinate system with coordinates (x, y, z). The direction of the positive z -axis is associated with "North". Galactic longitude,gLON and latitude, gLAT, are measured as usual with the direction of the positive x-axis along (gLON,gLAT) = (0°, 0°). The viewpoint is generally from inside the sphere, say from the origin to be specific. Then the direction of increasing gLON, *i.e.* local East, is to the left with up toward North. Latitude gLAT = 90° indicates the North Galactic pole, the direction from the origin (0,0,0) to (0,0,1) . We do not use the conventional *UVW* notation.

Somewhat contrarily, from a point-of-view located outside the sphere, as in the left-hand sketch in Fig. 1, one pictures a source S plotted on the sphere and, in the 2D tangent plane at S, local North is upward and local East is to the right. See the right-hand sketch in Fig. 1. A "position angle" at the point S on the sphere, such as the angle ψ in Fig. 1, is measured in the 2D plane tangent to the sphere at S. The position angle ψ is measured clockwise from local North with East to the right.

It is important to note that from a point of view inside the sphere, position angles are measured counterclockwise from North, since increasing gLON, *i.e.* East, is to the left when viewed from inside the sphere. But it is much easier to draw a sphere from the outside viewpoint, hence Fig. 1.

Definitions:

er, eN, eE are unit vectors in a 3D Cartesian coordinate system
(gLON,gLAT) = galactic longitude and latitude
er(gLON,gLAT) = radial unit vectors from Origin
eN(gLON,gLAT) = local North at a point on the Celestial Sphere
eE(gLON,gLAT) = local East at a point on the Celestial Sphere
gLONFROMr(er) = gLON determined by radial unit vector er
gLATFROMr(er) = gLAT determined by radial unit vector er

Aitoff Plot Functions
αH(gLON,gLAT) , xH(gLON,gLAT) , yH(gLON,gLAT), where xH is centered on gLON = 0 and gLON increases from left-to-right.
xH180(gLON,gLAT) , yH180(gLON,gLAT), where xH is centered on gLON = 180° and gLON increases from left-to-right.
xHGal(gLON,gLAT) , yHGal(gLON,gLAT), where xH is centered on gLON = 0 and gLON increases from right-to-left, so gLON = +180° is on the left and gLON = -180° is to the right.

```
In[2]:= (* For a Source at (gLON,gLAT) = (gLON,gLAT): er, eN,
eE are unit vectors from Origin to Source, local North, local East, resp.  *)
er[gLON_, gLAT_] := er[gLON, gLAT] = {Cos[gLON] Cos[gLAT], Sin[gLON] Cos[gLAT], Sin[gLAT]}
eN[gLON_, gLAT_] := eN[gLON, gLAT] = {-Cos[gLON] Sin[gLAT], -Sin[gLON] Sin[gLAT], Cos[gLAT]}
eE[gLON_, gLAT_] := eE[gLON, gLAT] = {-Sin[gLON], Cos[gLON], 0}
{"Check er.er = 1, er.eN = 0, er.eE = 0, eN.eN
   = 1, eN.eE = 0,eE.eE = 1, erXeE = eN, eEXeN = er, eNXer = eE: ",
 {0} == Union[Flatten[Simplify[{er[gLON, gLAT].er[gLON, gLAT] - 1, er[gLON, gLAT].eN[gLON, gLAT],
      er[gLON, gLAT].eE[gLON, gLAT], eN[gLON, gLAT].eN[gLON, gLAT] - 1, eN[gLON, gLAT].
       eE[gLON, gLAT], eE[gLON, gLAT].eE[gLON, gLAT] - 1, Cross[er[gLON, gLAT], eE[gLON, gLAT]] -
       eN[gLON, gLAT], Cross[eE[gLON, gLAT], eN[gLON, gLAT]] - er[gLON, gLAT],
      Cross[eN[gLON, gLAT], er[gLON, gLAT]] - eE[gLON, gLAT]}]]]}

Out[5]= {Check er.er = 1, er.eN = 0, er.eE = 0, eN.eN = 1,
   eN.eE = 0,eE.eE = 1, erXeE = eN, eEXeN = er, eNXer = eE: , True}
```

Get (gLON,gLAT) in radians from a radial vector r:

```
In[6]:= gLONFROMr[r_] := N[ArcTan[Abs[r[[2]] / r[[1]]]]] /; (r[[2]] ≥ 0 && r[[1]] > 0)
gLONFROMr[r_] := N[π - ArcTan[Abs[r[[2]] / r[[1]]]]] /; (r[[2]] ≥ 0 && r[[1]] < 0)
gLONFROMr[r_] := N[-π + ArcTan[Abs[r[[2]] / r[[1]]]]] /; (r[[2]] < 0 && r[[1]] < 0)
gLONFROMr[r_] := N[-ArcTan[Abs[r[[2]] / r[[1]]]]] /; (r[[2]] < 0 && r[[1]] > 0)
gLONFROMr[r_] := π / 2. /; (r[[2]] ≥ 0 && r[[1]] == 0)
gLONFROMr[r_] := -π / 2. /; (r[[2]] < 0 && r[[1]] == 0)

In[12]:= gLATFROMr[r_] := N[ArcTan[r[[3]] / (√ (r[[1]]^2 + r[[2]]^2))]] /; (√ (r[[1]]^2 + r[[2]]^2) > 0)
gLATFROMr[r_] := Sign[r[[3]]] (π / 2.) /; (√ (r[[1]]^2 + r[[2]]^2) == 0)
```

The following Aitoff Plot formulas can be found in Wikipedia, Ref. 4.
For these formulas the angles gLON and gLAT should be in degrees.
They give an Aitoff Plot that is centered on (0°,0°)

In[14]:=
```
αH[gLON_, gLAT_] := αH[gLON, gLAT] = ArcCos[Cos[((2. π) / 360.) gLAT] Cos[((2. π) / 360.) gLON / 2.]]
xH[gLON_, gLAT_] :=
 xH[gLON, gLAT] = (2. Cos[((2. π) / 360.) gLAT] Sin[((2. π) / 360.) gLON / 2.]) / Sinc[αH[gLON, gLAT]]
yH[gLON_, gLAT_] := yH[gLON, gLAT] = Sin[((2. π) / 360.) gLAT] / Sinc[αH[gLON, gLAT]]
```

Using the following functions produces an Aitoff Plot that is centered on (180°,0°)

In[17]:=
```
xH180[gLON_, gLAT_] := xH180[gLON, gLAT] =
  (2. Cos[((2. π) / 360.) gLAT] Sin[((2. π) / 360.) (gLON - 180.) / 2.]) / Sinc[αH[(gLON - 180.), gLAT]]
yH180[gLON_, gLAT_] := yH180[gLON, gLAT] = Sin[((2. π) / 360.) gLAT] / Sinc[αH[(gLON - 180.), gLAT]]
```

For Galactic Coordinates, the following functions produces an Aitoff Plot that is centered on gLON = 0° and the gLON axis runs backwards from +180° on the left to 0° at the center to -180° on the right. The viewpoint is inside the Celestial Sphere, looking out.

In[19]:=
```
(*The plots of the sky in Galactic coordinates have the gLON axis running from +
 180° on the left to -180° on the right. Angles gLON and gLAT are in degrees*)
xHGal[gLON_, gLAT_] := xHGal[gLON, gLAT] =
  (2. Cos[((2. π) / 360.) gLAT] Sin[-(2. π / 360.) gLON / 2.]) / Sinc[αH[-gLON, gLAT]]
yHGal[gLON_, gLAT_] := yHGal[gLON, gLAT] = Sin[((2. π) / 360.) gLAT] / Sinc[αH[-gLON, gLAT]]
```

2b. Grid

We avoid bunching at the poles by taking into account the diminishing radii of constant latitude circles as the latitude approaches the poles. Successive grid points along any latitude or along any longitude make an arc that subtends the same central angle $d\theta$.
We grid one hemisphere at a time, then the grids are combined.

Definitions:

gridSpacing	separation in degrees between grid points on a constant latitude circle and separation of constant latitude circles.
$d\theta 1$	grid spacing in radians
idN, ai, ji	dummy indices, ID #s for grid points, longitude, latitude
gLONpointH,gLATpointH	gLON and gLAT of the grid points H_j
grid, gridN, gridS	tables data associated with grid points, listings are below
nGrid	number of grid points
gLONGrid	longitudes at the grid points ($-\pi \leq$ gLON $\leq +\pi$)
gLATGrid	latitudes at the grid points ($-\pi/2 \leq$ gLON $\leq \pi/2$)
rGrid	radial unit vectors from origin to grid points, in 3D Cartesian coordinates

Tables: ***grid, gridN*** **and** ***gridS***
1. sequential point # 2. gLON index 3. gLAT index 4. gLON (rad) 5. gLAT (rad) 6. Cartesian coordinates of the grid point

In[21]:=
```
gridSpacing = 2.(*, in degrees.*);
```

In[22]:=
```
(*KEEP this cell - DO NOT DELETE*)
(*The Northern Grid "gridN". *)
dθ1 = ((2. π) / 360.) gridSpacing;
(*Convert gridSpacing to radians*) gridN = {};
idN = 1;
For[gLATj = 0., gLATj < π / (2. dθ1), gLATj++, gLATpointH = gLATj dθ1;
 For[ ai = 0., ai < Ceiling[((2. π) / dθ1) (Cos[gLATpointH] + 0.01)],
  ai++, gLONpointH = ai dθ1 / (Cos[gLATpointH] + 0.01);
  AppendTo[gridN, {idN, ai, gLATj, gLONpointH, gLATpointH, er[gLONpointH, gLATpointH]}];
  idN = idN + 1
 ]]
```

In[24]:=
```
(*KEEP this cell - DO NOT DELETE*)
(*The Southern Grid "gridS". *)
dθ1 = ((2. π) / 360.) gridSpacing; (*Convert gridSpacing to radians*)
gridS = {}; idN = 1;
For[gLATj = 1., gLATj < π / (2. dθ1), gLATj++, gLATpointH = -gLATj dθ1;
 For[ ai = 0., ai < Ceiling[((2. π) / dθ1) (Cos[gLATpointH] + 0.01)],
  ai++, gLONpointH = ai dθ1 / (Cos[gLATpointH] + 0.01);
  AppendTo[gridS, {idN, ai, gLATj, gLONpointH, gLATpointH, er[gLONpointH, gLATpointH]}];
  idN = idN + 1
 ]]
```

In[27]:=
```
(*KEEP this cell - DO NOT DELETE*)
grid = {}; j = 1;
For[jN = 1, jN ≤ Length[gridN], jN++, AppendTo[grid, {j, gridN[[jN, 2]], gridN[[jN, 3]],
   gLONFROMr[gridN[[jN, 6]] ], gLATFROMr[gridN[[jN, 6]] ], gridN[[jN, 6]]}];
 j = j + 1]
For[jS = 1, jS ≤ Length[gridS], jS++, AppendTo[grid, {j, gridS[[jS, 2]], gridS[[jS, 3]],
   gLONFROMr[gridS[[jS, 6]] ], gLATFROMr[gridS[[jS, 6]] ], gridS[[jS, 6]]}];
 j = j + 1]
```

In[30]:=
```
nGrid = Length[grid];
```

In[31]:=
```
gLONGrid = Table[grid[[j, 4]] , {j, nGrid}];
gLATGrid = Table[grid[[j, 5]] , {j, nGrid}];
rGrid  = Table[grid[[j, 6]] , {j, nGrid}];
```

2c. The mean and standard deviation are convenient functions. Set directories for getting and putting data.

Definitions

mean — the arithmetic average of a set of numbers, $\frac{1}{N}\sum_{i=1}^{N} n_i$

stanDev — the standard deviation. Given a set of *N* numbers n_i with mean value *m*, the standard deviation is $\left(\frac{1}{N}\sum_{i=1}^{N}(n_i - m)^2\right)^{1/2}$, the square root of the average of the squares of the differences of the numbers with the mean. Note that we divide by *N* to get the average of the deviations squared.

catalogDirectory — directory containing the catalog files

homeDirectory — directory containing the notebook and data files

In[34]:=
```
mean[data_] := (1/Length[data]) Sum[data[[i4]], {i4, Length[data]}];
(* arithmetic average *)
stanDev[data_] :=
 ((1/Length[data]) Sum[(data[[i5]] - mean[data])^2, {i5, Length[data]}])^(1/2)
 (*standard deviation*)
```

In[36]:=
```
catalogDirectory =
  "C:\\Users\\shurt\\Dropbox\\HOME_DESKTOP-0MRE5OJ\\SendXXX_CJP_CEJPetc\\SendViXra\\
    20210221StellarPolarization\\20210221Catalog";
 (* location of the catalog data file on my computer*)
```

In[37]:=
```
homeDirectory =
  "C:\\Users\\shurt\\Dropbox\\HOME_DESKTOP-0MRE5OJ\\SendXXX_CJP_CEJPetc\\SendViXra\\
    20210221StellarPolarization\\20210221Notebooks\\20210228GalacticCoordsNotebooks\\
    20210310Lon100PlusOnDisk";
 (*The notebook file and data files for this notebook are put in this directory. *)
```

Section Summary

In[38]:=
```
Print["The grid points are separated by gridSpacing = ",
 gridSpacing, "° arcs along latitude and longitude."]
Print["The number of grid points is ", nGrid, " ."]
```

```
The grid points are separated by gridSpacing = 2.° arcs along latitude and longitude.
The number of grid points is 10518 .
```

3. Polarization and Position Data

Definitions:

cat the catalog data, Heiles 2000 Ref. 10.
allClumpsofStarsIDsInCatalog record numbers of the stars in the catalog for all clumps
clumpOfStarsIDinCatalog record numbers of the sample’s stars in the catalog (we treat this clump)
nSrc number of stars
gLONSrc galactic longitude (radians)
gLATSrc galactic latitude (radians)
ψn PPA, polarization position angle: counterclockwise from North with East to the left, as seen from inside the Celestial Sphere.
σψn uncertainty in PPA
percentPol percentage of linear polarization
rSrc unit vector from Origin to Sources on Celestial Sphere
eNSrc Local North at the ith Source
eESrc Local East at the ith Source
sourceCenter unit radial vector to the arithmetic center of the sources
angleSourceToCenter arc from Source to Center
showClump1 map of significance for alignments in the catalog, needed to discuss sample selection

Catalog data

The file contains the original unaltered catalog entries, except that the declination and Right Ascension have been separated. The object's record number is appended.

----

1. Declination (deg) 2 RA (hr) 3. HD number 4. Bonner DM number 5. Cordoba DM number 6. Cape DM number 7. Percentage polarization (%) 8. rms uncertainty on Pol (%) 9. Position angle, equatorial (deg.) 10. rms uncertainty on PA (deg.) 11. Position angle, Galactic (deg.) 12. Galactic longitude (deg.) 13. Galactic latitude (deg.) 14. Reddening (mag.) 15. Discrepancy between PA and PAgal (deg.) 16. Primary stellar database 17. Visual magnitude (mag.) 18. Distance (pc) 19. Spectral type 20. Polarization catalog numbers 21. Distance catalog 22. Object # in the catalog

----

See the ReadMe file in Ref. 11 for details.

In[40]:=
```
(*Get the catalog*)
SetDirectory[catalogDirectory];
cat = Get["20210321originalCatalog1.dat"];
```

In[42]:=
```
(*Get the IDs of the stars in the sample.*)
SetDirectory[homeDirectory];
clumpOfStarsIDinCatalog = Get["20210329clumpOfStarsIDinCatalog.dat"];
nSrc = Length[clumpOfStarsIDinCatalog]
```

Out[44]= 893

In[45]:=
```
gLONSrc = Table[cat[[i, 12]] (2. π/360.), {i, clumpOfStarsIDinCatalog}];
(*galactic longitude in radians*)
gLATSrc = Table[cat[[i, 13]] (2. π/360.), {i, clumpOfStarsIDinCatalog}];
(*galactic latitude in radians*)
ψn = Table[cat[[i, 11]] (2. π/360.), {i, clumpOfStarsIDinCatalog}];
(* galactic position angle in radians*)
σψn = Table[cat[[i, 10]] (2. π/360.), {i, clumpOfStarsIDinCatalog}];
(*uncertainty in ψ in radians*)
percentPol = Table[cat[[i, 7]], {i, clumpOfStarsIDinCatalog}]; (* % polarization*)
```

In[50]:=
```
rSrc = Table[er[ gLONSrc[[i]], gLATSrc[[i]] ],  {i, nSrc}];(*calculated from Input.*)
eNSrc = Table[eN[ gLONSrc[[i]], gLATSrc[[i]] ], {i, nSrc}];(*calculated from Input.*)
eESrc = Table[eE[  gLONSrc[[i]], gLATSrc[[i]]  ], {i, nSrc}];(*calculated from Input.*)
```

In[53]:=
```
sourceCenter0 = 1/nSrc Sum[rSrc[[i]], {i, nSrc}];
sourceCenter = sourceCenter0 / (sourceCenter0.sourceCenter0)^(1/2);
(*unit radial vector to the arithmetic center of the sources.*)
angleSourceToCenter = Table[ArcCos[rSrc[[i]].sourceCenter], {i, nSrc}];
```

Section Summary:

In[56]:=
```
Print["Catalog:"]
Print["The number of stars in the catalog is ", Length[cat], ". "]
Print["The first record: ", cat[[1]], "."]
Print["The last record: ", cat[[-1]], "."]
Print["The catalog data is filtered for %
   polarization p, p ≥ 0.6%, and PPA ψ uncertainty σψ, σψ ≤  14°."]
Print["After filtering for % polarization and experimental
   uncertainty σψ, there are 3719 stars remaining."]
```

```
Catalog:

The number of stars in the catalog is 9286.

The first record: {-85.6632, 8.94497, 79837., -999.9, -999.9, -85.0018, 0.04, 0.035,
  107.3, 23.6, 53., 298.851, -24.8156, 0., -0.1, 1, 5.4, 70., F0III, 1000000000, 120, 1}.

The last record: {89.2641, 2.52974, 8890., 88.0008, -999.9, -999.9, 0.171,
  0.12, 119.7, 19.3, 108., 123.28, 26.4614, 0., 0., 1, 2., 208.9, F8I, 11000, 4, 9286}.

The catalog data is filtered for %
  polarization p, p ≥ 0.6%, and PPA ψ uncertainty σψ, σψ ≤  14°.

After filtering for % polarization and
  experimental uncertainty σψ, there are 3719 stars remaining.
```

The Selection Process:

The stars in the catalog are filtered for % polarization and experimental uncertainty $\sigma\psi$. Then 5° radius regions are constructed on the 10518 grid points. There were 1829 regions with seven or more stars, $N \geq 7$, the minimum required for the statistics formulas in Sec. 4 to be valid. Of these, 1497 had very significant alignment, meaning at most one in a hundred, sig $\leq$ 1% $= 1 \times 10^{-2}$, samples with randomly directed polarization directions would be equally well aligned. See Fig. 2 for a plot of the 1497 very significantly aligned 5° radius regions. At each region's center point, the negative log of the significance is plotted for convenience, so the minimum value is $-\log_{10}(1 \times 10^{-2}) = +2$, corresponding to a significance of 1%.

The stars selected for the sample studied are all the stars in all the 5° radius regions that (*i*) have 7 or more stars, (*ii*) have longitude 95° $\leq$ gLON $\leq$ 155°, (*iii*) have latitude -15° $\leq$ gLAT $\leq$ 10°, and (*iv*) whose stars have polarization directions aligned with a significance less than a billionth, sig $\leq 10^{-9}$. There are 190 such regions containing 893 stars. The sample, shaded green in Fig. 2, includes the highest peak and the lower peak nearby.

Out[238]=

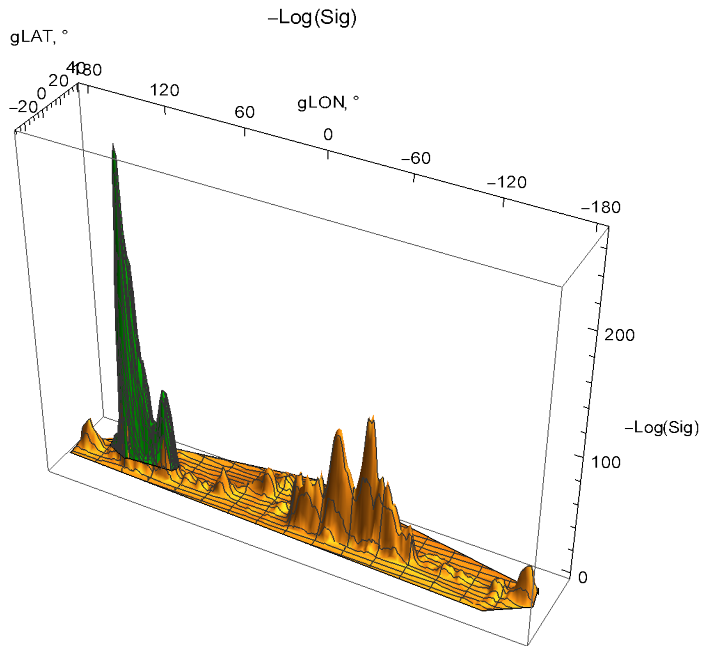

Figure 2. The significance of 5° radius regions. For convenience, the negative logarithm is plotted. The top of the green peak has a value of about 250, meaning that fewer than one in 10^{250} randomly directed regions would have better aligned polarization directions.

In[63]:=
```
Print["Sample:"]
Print[
 "Check that the smallest % polarization p in the sample is 0.6% or more. Smallest: ",
 Sort[percentPol][[1]], "% ."]
Print["Check that the largest PPA ψ uncertainty σψ is less than 14°. Largest: ",
 Sort[σψn][[-1]] (360./(2. π)), "° ."]
```

```
Sample:

Check that the smallest % polarization p in the sample is 0.6% or more. Smallest: 0.605% .

Check that the largest PPA ψ uncertainty σψ is less than 14°. Largest: 8.7° .
```

In[66]:=
```
Print["There are ", nSrc, " stars in the sample. Their record
   numbers in the Heiles 2000 catalog can be found in the Appendix."]
Print["For example, the Heiles 2000 catalog listing for the
   first star in the sample, star number ",
 clumpOfStarsIDinCatalog[[1]] , " : ", cat[[ clumpOfStarsIDinCatalog[[1]] ]], "."]
```

```
There are 893 stars in the sample. Their record
   numbers in the Heiles 2000 catalog can be found in the Appendix.

For example, the Heiles 2000 catalog listing for the first star in the sample, star number
 7802 : {44.2782, 4.08551, 25517., 43.0886, -999.9, -999.9, 2.53, 0.2, 160.,
  2.3, 118.3, 155.516, -5.9991, 0.5, 0., 1, 8.9, 1823., B1V, 10000, 120, 7802}.
```

```
In[68]:= ListPlot[Table[{-gLONSrc[[j]], gLATSrc[[j]]} (360./(2. π)), {j, nSrc}],
  PlotRange → {{-180, 180}, {-90, 90}},
  Ticks → {Table[{i, -i}, {i, -180, 180, 60}], Table[{j, j}, {j, -90, 90, 30}]},
  PlotLabel → "Sources", AxesLabel → {"°gLON", "°gLAT"}, PlotStyle → Green]
Print["Figure 3. The locations of the ", nSrc, " stars in the sample. "]
Print[
 "Sample Size: The angular separation of the furthest star from the region center is ",
 Sort[angleSourceToCenter][[-1]] (360./(2. π)), "°."]
```

Out[68]=

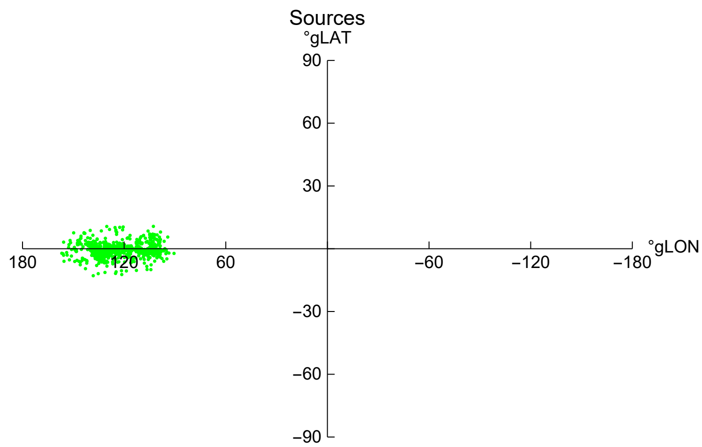

Figure 3. The locations of the 893 stars in the sample.

Sample Size: The angular separation of the furthest star from the region center is 33.8849°.

4. Probability Distributions and Significance Formulas

The problem of “significance” is to determine the likelihood that random polarizations directions would have better alignment or avoidance than the observed polarization directions. To determine the probability distributions and related formulas, we made many runs with random data and fit the results.

Definitions:

norm	a constant used to normalize the distribution so the integral of probability is 1.
probMIN0, probMAX0	probability distributions for alignment (MIN) and avoidance (MAX), functions of η, η_0, σ
ρciaiMIN,MAX	constants used in the formulas to mean η_0 and uncertainty σ
$\sigma\rho$ciaiMIN,MAX	uncertainty σ in the constants used in the formulas to mean η_0 and uncertainty σ
regionRadiusChoices	radii used in random runs performed elsewhere, not in this notebook
regionChoice	determines the best choice for the current sample
rgnRadius	assumed radius of the region for the purpose of selecting the statistics constants c_i and a_i
iρ	dummy variable used to select region radius
ciMIN,MAX and aiMIN,MAX	parameters for statistics formulas for η_0 and σ
η0MIN, MAX	function to estimate mean η_0
σMIN, MAX	function to estimate uncertainty σ

probMIN, probMAX probability distributions using estimated values of η_0, σ
signiMIN0, signiMAX0 significance as a function of (η, η_0, σ)
signiMIN, signiMAX significance of η using estimated values of η_0, σ

In[71]:=
```
(* y = ((η - η0)/σ); dy = dη/σ *)
(* The normalization factor "norm" is needed for the probability density *)
```

$$\text{norm} = \left(\frac{1}{(2\pi)^{1/2}}\,\text{NIntegrate}\left[\left(1 + e^{4\,(y-1)}\right)^{-1} e^{-\frac{y^2}{2}},\ \{y, -\infty, \infty\}\right]\right)^{-1};$$

```
norm;(*Constant needed for Eq. (10) and (11) in Ref. 5.*)
```

In[73]:=

$$\text{probMIN0}[\eta\_,\ \eta0\_,\ \sigma\_] := \left(\frac{\text{norm}}{\sigma\,(2\pi)^{1/2}}\right)\left(1 + e^{4\,\frac{(\eta-\eta0-\sigma)}{\sigma}}\right)^{-1} e^{-\frac{1}{2}\left(\frac{\eta-\eta0}{\sigma}\right)^2}$$

$$\text{signiMIN0}[\eta\_,\ \eta0\_,\ \sigma\_] := \text{NIntegrate}[\text{probMIN0}[\eta1,\ \eta0,\ \sigma],\ \{\eta1, -\infty, \eta\}]$$

In[75]:=

$$\text{probMAX0}[\eta\_,\ \eta0\_,\ \sigma\_] := \left(\frac{\text{norm}}{\sigma\,(2\pi)^{1/2}}\right)\left(1 + e^{-4\,\frac{(\eta-\eta0+\sigma)}{\sigma}}\right)^{-1} e^{-\frac{1}{2}\left(\frac{\eta-\eta0}{\sigma}\right)^2}$$

$$\text{signiMAX0}[\eta\_,\ \eta0\_,\ \sigma\_] := \text{NIntegrate}[\text{probMAX0}[\eta1,\ \eta0,\ \sigma],\ \{\eta1,\ \eta,\ \infty\}]$$

The significance signiMIN0[η, η0, σ] is the Integral of probMIN0, i.e. signiMIN0 = $\int_{-\infty}^{\eta} P_{MIN}(\eta)\, d\eta$.

The significance signiMAX0[η, η0, σ] is the Integral of probMAX0, i.e. signiMAX0 = $\int_{\eta}^{\infty} P_{MAX}(\eta)\, d\eta$.

The formulas for mean $\eta_0 = \frac{\pi}{4} \pm \frac{c1}{N^{a1}}$ and half-width $\sigma = \frac{c2}{4\,N^{a2}}$ estimate η_0 and σ by functions of the number N of sources. These formulas depend on the size of the region (radius ρ) by the choice of parameters c_i and a_i, i = 1,2. The following values for the parameters c_i and a_i are based on random runs. For each combination of N = {8,16,32,64,128,181,256,512} and ρ = {0°,5°,12°,24°,48°,90°}, there were 2000 random runs completed.

A notation conflict between this notebook and the article, Ref. 5, should be noted. We doubled the exponent “a” so $N^{a/2}$ appears in the article, whereas in the formulas here we see N^a. Thus $a \approx 1/2$ here, but the paper has $a_{\text{Article}} \approx 1$. That explains the “/2” in the following arrays.

In[77]:= ρciaiMIN = (matrix below);

"ρ"	"c1"	"a1"	"c2"	"a2"
90	0.9423	1.0046/2	1.061	0.954/2
48	0.9505	1.0156/2	1.166	0.9956/2
24	0.9235	1.0069/2	1.127	0.964/2
12	0.8912	1.0054/2	1.238	1.021/2
5	0.8363	1.0088/2	1.076	0.940/2
0	0.5031	1.0153/2	1.522	1.053/2

In[78]:= ρciaiMAX =

"ρ"	"c1"	"a1"	"c2"	"a2"
90	0.9441	1.0055 / 2	1.000	0.931 / 2
48	0.9572	1.0165 / 2	1.090	0.958 / 2
24	0.927	1.0068 / 2	1.101	0.964 / 2
12	0.9049	1.0090 / 2	1.228	1.018 / 2
5	0.8424	1.0062 / 2	1.168	0.992 / 2
0	0.4982	1.0093 / 2	1.543	1.060 / 2

;

In[79]:= ρΔciaiMIN =

"ρ"	"c1"	"a1"	"c2"	"a2"
90	0.0050	0.0036 / 2	0.026	0.016 / 2
48	0.0079	0.0057 / 2	0.016	0.0095 / 2
24	0.0024	0.0018 / 2	0.022	0.013 / 2
12	0.0034	0.0026 / 2	0.039	0.021 / 2
5	0.0035	0.0028 / 2	0.030	0.019 / 2
0	0.0059	0.0080 / 2	0.052	0.024 / 2

;

In[80]:= ρΔciaiMAX =

"ρ"	"c1"	"a1"	"c2"	"a2"
90	0.0061	0.0044 / 2	0.038	0.025 / 2
48	0.0063	0.0045 / 2	0.026	0.016 / 2
24	0.011	0.0079 / 2	0.019	0.011 / 2
12	0.0069	0.0052 / 2	0.039	0.022 / 2
5	0.0038	0.0031 / 2	0.022	0.013 / 2
0	0.0058	0.0080 / 2	0.057	0.025 / 2

;

In[81]:=
```
(*The region radius controls the constants ci and ai for statistics in Sec. 4.*)
regionRadiusChoices = {90, 48, 24, 12, 5, 0};(*Do not change this statement*)
regionChoice = 3; (*This is a setting. The choice 24° is 3rd in the list. *)
rgnRadius = regionRadiusChoices[[regionChoice]];
Print["The region radius ρ is set at ", rgnRadius, "°."]
```

The region radius ρ is set at 24°.

In[85]:=
```
iρ = regionChoice + 1; (* Parameters ci, ai, i = 1,2. *)
Print["These constants are for sources confined to regions with radii ρ = ",
 ρciaiMIN[[iρ, 1]], "°."]
{c1MIN, a1MIN, c2MIN, a2MIN} = Table[ρciaiMIN[[iρ, j]], {j, 2, 5}]
{c1MAX, a1MAX, c2MAX, a2MAX} = Table[ρciaiMAX[[iρ, j]], {j, 2, 5}]
```

These constants are for sources confined to regions with radii ρ = 24°.

Out[87]= {0.9235, 0.50345, 1.127, 0.482}

Out[88]= {0.927, 0.5034, 1.101, 0.482}

In[89]:=
```
iρ = regionChoice + 1; (* ± uncertainty for the Parameters ci and ai, i = 1,2. *)
Print["These uncertainties are for sources confined to regions with radii ρ = ",
 ρciaiMAX[[iρ, 1]], "°."]
{c1MINplusMinus, a1MINplusMinus, c2MINplusMinus, a2MINplusMinus} =
 Table[ρΔciaiMIN[[iρ, j]], {j, 2, 5}]
{c1MAXplusMinus, a1MAXplusMinus, c2MAXplusMinus, a2MAXplusMinus} =
 Table[ρΔciaiMAX[[iρ, j]], {j, 2, 5}]
```

These uncertainties are for sources confined to regions with radii ρ = 24°.

Out[91]= {0.0024, 0.0009, 0.022, 0.0065}

Out[92]= {0.011, 0.00395, 0.019, 0.0055}

In[93]:=
```
η0MIN[nSrc_, c1_, a1_] := π/4 - c1/nSrc^a1
σMIN[nSrc_, c2_, a2_] := c2/(4 nSrc^a2)
```

In[95]:=
```
η0MAX[nSrc_, c1_, a1_] := π/4 + c1/nSrc^a1
σMAX[nSrc_, c2_, a2_] := c2/(4 nSrc^a2)
```

The following probability distributions and significances make use of the above formulas for mean η_0 and half-width σ. They are functions of the alignment angle η and the number of sources N.

In[97]:=
```
probMIN[η_, nSrc_] := probMIN0[ η, η0MIN[nSrc, c1MIN, a1MIN], σMIN[nSrc, c2MIN, a2MIN] ]
```

In[98]:=
```
signiMIN[η_, nSrc_] := signiMIN0[η, η0MIN[nSrc, c1MIN, a1MIN], σMIN[nSrc, c2MIN, a2MIN]]
```

In[99]:=
```
probMAX[η_, nSrc_] := probMAX0[ η, η0MAX[nSrc, c1MAX, a1MAX], σMAX[nSrc, c2MAX, a2MAX] ]
signiMAX[η_, nSrc_] := signiMAX0[η, η0MAX[nSrc, c1MAX, a1MAX], σMAX[nSrc, c2MAX, a2MAX]]
```

Section Summary

In[101]:=
```
Print["The angular separation of the furthest star from the region center is ",
 Sort[angleSourceToCenter][[-1]] (360./(2. π)), "°.",
 " We choose the statistics constants ai and ci, i = 1,2, for
   sources confined to regions with radii ρ = ", ρciaiMIN[[iρ, 1]], "°."]
Print["The formulas also depend on the number of sources, nSrc = ", nSrc, "."]
Print["For this sample, but with observed replaced by random polarization
   directions, the expected smallest alignment angle η̄min  is  η̄min^(Random ψ) = ",
 η0MIN[nSrc, c1MIN, a1MIN] (360./(2. π)), "° ± ", σMIN[nSrc, c2MIN, a2MIN] (360./(2. π)),
 "°. (Random ψ)"]
Print["For this sample, but with observed replaced by random polarization
   directions, the expected largest avoidance angle η̄max is  η̄max^(Random ψ) = ",
 η0MAX[nSrc, c1MAX, a1MAX] (360./(2. π)), "° ± ", σMAX[nSrc, c2MAX, a2MAX] (360./(2. π)),
 "°. (Random ψ)"]
```

The angular separation of the furthest star from the region center is 33.8849°. We choose the statistics constants a_i and c_i, i = 1,2, for sources confined to regions with radii ρ = 24°.

The formulas also depend on the number of sources, nSrc = 893.

For this sample, but with observed replaced by random polarization directions, the expected smallest alignment angle $\overline{\eta}_{min}$ is $\overline{\eta}_{min}{}^{Random\,\psi}$ = 43.2704° ± 0.610487°. (Random ψ)

For this sample, but with observed replaced by random polarization directions, the expected largest avoidance angle $\overline{\eta}_{max}$ is $\overline{\eta}_{max}{}^{Random\,\psi}$ = 46.7368° ± 0.596403°. (Random ψ)

5. Results using the Best Values ψn of the Polarization Directions

"Best" means we use the ψn that were listed in the catalog. We calculate the alignment function $\overline{\eta}$(H) at the grid points H . Given the alignment function $\overline{\eta}$(H) , one can find the smallest alignment angle $\overline{\eta}_{min}$ and the largest avoidance angle $\overline{\eta}_{max}$ and determine the significances for the alignment and avoidance of the polarization directions.

Note that, in Sec. 6 below, we consider other values, those that are consistent with uncertainty $\sigma\psi$ in the measured values.

5a. The alignment function $\overline{\eta}$(H) .

Definitions:

vψSrc unit vectors along the polarization directions in the tangent planes of the sources
eN local unit vectors along local North
eE local unit vectors along local East
jηBarHj {j,$\overline{\eta}$(H)}, where j is the index for grid point H_j and $\overline{\eta}$(H) is the average alignment angle at H_j. See Eq. (1) in the Introduction.
sortjηBarHj {j,$\overline{\eta}$(H)}, sorted, with smallest angles $\overline{\eta}$(H) first.
jηBarMin {j,$\overline{\eta}$(H)}, the j and $\overline{\eta}$ for the smallest value of $\overline{\eta}$(H) , best alignment
ηBarMin the smallest value of $\overline{\eta}$(H) , measures alignment of the polarization directions
jηBarMax {j,$\overline{\eta}$(H)}, the j and $\overline{\eta}$ for the largest value of $\overline{\eta}$(H) , most avoided
ηBarMax the largest value of $\overline{\eta}$(H) , measures avoidance

nSxψn unit vector, $S_i \times \psi_i$, cross product of the radial vector to the source with the vector in the direction of the polarization
nSxHnj unit vector, $S_i \times H_j$, cross product of the radial vector to the source with the radial vector to the grid point H_j
ηnHj alignment angle between source and grid point H_j, see Fig. 1
ηBarHj alignment angle $\overline{\eta}(H_j)$ between source and grid point H_j, avegLONged over all sources
jηBarHj {j, $\overline{\eta}(H_j)$ }, the j and $\overline{\eta}$ for grid point H_j

sigηBarMin significance of the smallest alignment angle
sigrangeηBarMin get the range of sigs using the plus/minus values on the parameters c_i, a_i
sigSmallηBarMin the smallest of the values in sigrangeηBarMin
sigBigηBarMin the largest of the values in sigrangeηBarMin
sigηBarMax significance of the largest alignment angle (i.e. avoidance)
sigrangeηBarMax get the range if sigs using the plus/minus values on the parameters c_i, a_i
sigSmallηBarMax the smallest of the values in sigrangeηBarMax

sigBigηBarMax the largest of the values in sigrangeηBarMax
gLONHminDegrees gLON of the point H_{min} where $\overline{\eta}$(H) is the smallest
gLATHminDegrees gLAT of the point H_{min} where $\overline{\eta}$(H) is the smallest
gLONHmaxDegrees gLON of the point H_{max} where $\overline{\eta}$(H) is the largest
gLATHmaxDegrees gLAT of the point H_{max} where $\overline{\eta}$(H) is the largest

In[105]:=

```
(* vψ, eN, eE unit vectors in the tangent plane of each source Si,
pointing along the polarization direction, local North,
and local East, respectively. See Fig. 1.*)
vψSrc = Table[Cos[ ψn[[i]] ] eN[ gLONSrc[[i]], gLATSrc[[i]] ] +
    Sin[ ψn[[i]] ] eE[ gLONSrc[[i]], gLATSrc[[i]] ], {i, nSrc}];
```

In[106]:=

```
(* Analysis using Eq (5) in Ref. 5 to get first ηiH,
cos(η) = |v̂H.v̂ψ|, and then η̄(Hj). *)
jηBarHj =
  Table[{j, (1/nSrc) Sum[ArcCos[ Abs[ rGrid[[j]].vψSrc[[i]] / ((rGrid[[j]] - (rGrid[[j]].
             rSrc[[i]]) rSrc[[i]]).(rGrid[[j]] - (rGrid[[j]].rSrc[[i]])
           rSrc[[i]]))^(1/2) ] - 0.000001 ] , {i, nSrc}]}, {j, nGrid}];
 sortjηBarHj = Sort[jηBarHj, #1[[2]] < #2[[2]] &];
jηBarMin = sortjηBarHj[[1]]; (*  {j,η̄(Hj)} for smallest η̄(Hj) *)
ηBarMin = jηBarMin[[2]];
jηBarMax = sortjηBarHj[[-1]]; (*  {j,η̄(Hj)} for largest η̄(Hj) *)
ηBarMax = jηBarMax[[2]] ;
```

In[112]:=

```
(*Significance of the smallest alignment angle η̄min .*)
sigηBarMin = signiMIN[ηBarMin, nSrc];
sigrangeηBarMin = Sort[Partition[Flatten[Table[
      {signiMIN0[ηBarMin, η0MIN[nSrc, c1MIN + γ1 c1MINplusMinus, a1MIN + α1 a1MINplusMinus],
        σMIN[nSrc, c2MIN + γ2 c2MINplusMinus, a2MIN + α2 a2MINplusMinus]], γ1, α1, γ2, α2},
      {γ1, -1, 1}, {α1, -1, 1}, {γ2, -1, 1}, {α2, -1, 1}] ], 5 ]   ];
{sigrangeηBarMin[[1]], sigrangeηBarMin[[-1]]};
sigSmallηBarMin = sigrangeηBarMin[[1, 1]];
sigBigηBarMin = sigrangeηBarMin[[-1, 1]];
```

In[117]:=

```
(*Significance of the largest avoidance angle η̄max .*)
sigηBarMax = signiMAX[ηBarMax, nSrc];
sigrangeηBarMax = Sort[Partition[Flatten[Table[
      {signiMAX0[ηBarMax, η0MAX[nSrc, c1MAX + γ1 c1MAXplusMinus, a1MAX + α1 a1MAXplusMinus],
        σMAX[nSrc, c2MAX + γ2 c2MAXplusMinus, a2MAX + α2 a2MAXplusMinus]], γ1, α1, γ2, α2},
      {γ1, -1, 1}, {α1, -1, 1}, {γ2, -1, 1}, {α2, -1, 1}] ], 5 ]   ];
{sigrangeηBarMax[[1]], sigrangeηBarMax[[-1]]};
sigSmallηBarMax = sigrangeηBarMax[[1, 1]];
sigBigηBarMax = sigrangeηBarMax[[-1, 1]];
```

In[122]:=
```
(* Galactic coordinates (gLON,gLAT) for the hubs Hmin and Hmax .*)
gLONHminDegrees = gLONGrid[[ jηBarMin[[1]] ]] (360 / (2 π)); (*Hmin*)
gLATHminDegrees = gLATGrid[[ jηBarMin[[1]] ]] (360 / (2 π));

gLONHmaxDegrees = gLONGrid[[ jηBarMax[[1]] ]] (360 / (2 π)); (*Hmax*)
gLATHmaxDegrees = gLATGrid[[ jηBarMax[[1]] ]] (360 / (2 π));
```

In[126]:=
```
(*The names "jηBarMin", "jηBarMax" are similar to quantities below,
so save the current values labeled by "Best".*)
(* jηBar entries: 1. grid point # , 2. alignment angle .*)
{jηBarMinBest, jηBarMaxBest} = {jηBarMin, jηBarMax} ;
```

In[127]:=
```
Print["The min alignment angle is ηmin = ", jηBarMinBest[[2]] * (360. / (2. π)),
 "° , which has a significance of sig. = ", sigηBarMin, ",  plus/minus = + ",
 sigBigηBarMin - sigηBarMin, " and - ", sigηBarMin - sigSmallηBarMin,
 " , giving a range from sig. = ", sigSmallηBarMin, " to ", sigBigηBarMin, " ."]
Print["The max avoidance angle is ηmax = ", jηBarMaxBest[[2]] * (360. / (2. π)),
 "° , which has a significance of sig. = ", sigηBarMax, ",  plus/minus = + ",
 sigBigηBarMax - sigηBarMax, " and - ", sigηBarMax - sigSmallηBarMax,
 " , giving a range from sig. = ", sigSmallηBarMax, " to ", sigBigηBarMax, " ."]
Print["These uncertainties are due to the uncertainties in the constants ci, ai."]
```

```
The min alignment angle is ηmin = 10.3974° , which has a significance of sig. =
 0.,  plus/minus = + 0. and - 0. , giving a range from sig. = 0. to 0. .

The max avoidance angle is ηmax = 79.9885° , which has a significance of sig. =
 0.,  plus/minus = + 0. and - 0. , giving a range from sig. = 0. to 0. .

These uncertainties are due to the uncertainties in the constants ci, ai.
```

5b. Plot of the Alignment Angle Function $\overline{\eta}$(H)

Definitions

gLONjgLATjηBarHjTable	{gLON$_j$, gLAT$_j$, $\overline{\eta}$(H)} at each grid point $H = H_j$, in degrees
ηBarHjSmooth	interpolation of gLONjgLATjηBarHjTable yields $\overline{\eta}$(H) as a smooth function of the (gLON,gLAT) of H
xyηBarAitoffTable	{x, y, $\overline{\eta}$(x,y)} , where x,y are Aitoff coordinates and $\overline{\eta}$(x,y) is the alignment angle
xyAitoffSources	{x,y} Aitoff coordinates for the sources' locations on the sphere
dηContourPlot	separation of successive contour lines, in degrees
listCP	list contour plot of $\overline{\eta}$(H) from xyηBarAitoffTable
mapOfηBar	contour plot of the alignment angle $\overline{\eta}$(H) , adorned with source locations and labels

In[130]:=
```
(*The following table gLONjgLATjηBarHjTable is created to be interpolated below,
yielding a smooth function ηBarHjSmooth of the alignment angle η̅(H) over the sphere.*)
(* Table gLONjgLATjηBarHjTable
 entries: 1.  gLON  2. gLAT  3. alignment angle ηBarRgnkj at grid point (all in degrees)*)
gLONjgLATjηBarHjTable = (gLONjgLATjηBarHjTable0 = {};
   For[j = 1, j ≤ Length[jηBarHj], j++, AppendTo[ gLONjgLATjηBarHjTable0,
     {gLONGrid[[j]] * (360. / (2. π)), gLATGrid[[j]] * (360. / (2. π)),
      jηBarHj[[j, 2]] * (360. / (2. π))}]   ; If[ 180 ≥ gLONGrid[[j]] * (360. / (2. π)) > 174.,
     AppendTo[ gLONjgLATjηBarHjTable0,  {gLONGrid[[j]] * (360. / (2. π)) - 360.,
       gLATGrid[[j]] * (360. / (2. π)), jηBarHj[[j, 2]] * (360. / (2. π))}]   ] ;
    If[ -174. > gLONGrid[[j]] * (360. / (2. π)) ≥ -180., AppendTo[ gLONjgLATjηBarHjTable0,
      {gLONGrid[[j]] * (360. / (2. π)) + 360, gLATGrid[[j]] * (360. / (2. π)),
       jηBarHj[[j, 2]] * (360. / (2. π))}]   ]         ];
   gLONjgLATjηBarHjTable0);
```

In[131]:=
```
ηBarHjSmooth = Interpolation[gLONjgLATjηBarHjTable] (*The smooth alignment angle function η̅(H).*)
```

••• Interpolation: Interpolation on unstructured grids is currently only supported for InterpolationOrder->1 or InterpolationOrder->All. Order will be reduced to 1.

Out[131]=
```
InterpolatingFunction[ Domain: {{-186., 186.}, {-88., 88.}}
                       Output: scalar ]
```

In[132]:=
```
(*Transcribe the alignment function  η̅(H), the location of the sources,
and the Celestial Equator onto an Aitoff plot.*)
xyηBarAitoffTable =
  Partition[Flatten[Table[{xHGal[gLON, gLAT], yHGal[gLON, gLAT], ηBarHjSmooth[gLON, gLAT]},
     {gLON, -178., 178., 2.}, {gLAT, -88., 88., 2.}]], 3];
(* The smooth alignment angle function η̅(H)  = ηBarHjSmooth mapped
  onto a 2D Aitoff projection of the sphere. *)

xyAitoffSources = Table[{xHGal[ gLONSrc[[n]] (360 / (2 π)), gLATSrc[[n]] (360 / (2 π)) ],
    yHGal[ gLONSrc[[n]] (360 / (2 π)), gLATSrc[[n]] (360 / (2 π)) ]}, {n, nSrc}];
 (*The Aitoff coordinates for the sources' locations.*)
```

In[134]:=
```
(* Contour plot of the alignment function ηBarHjSmooth. *)
dηContourPlot = 10 ;
(*, in degrees. *)listCP = ListContourPlot[Union[xyηBarAitoffTable(*,{{xHGal[gLONHminDegrees,
       gLATHminDegrees],yHGal[gLONHminDegrees,gLATHminDegrees],ηBarMin*(360./(2.π))-1.0}},
    {{xHGal[gLONHmaxDegrees,gLATHmaxDegrees],yHGal[gLONHmaxDegrees,gLATHmaxDegrees],
      ηBarMax*(360./(2.π))+1.0}}*)], AspectRatio → 1 / 2, Contours → Table[η,
     {η, Floor[jηBarMin[[2]] * (360. / (2. π))] + 1, Ceiling[jηBarMax[[2]] * (360. / (2. π))] - 1,
      dηContourPlot}], ColorFunction → "TemperatureMap", PlotRange → {{-7, 7}, {-3, 3}},
  Axes -> False, Frame → False, PlotLabel → "The alignment function η̅(H)", PlotLegends → Automatic] ;
```

In[135]:=
```
(*Construct the map of η̄(H).*)
mapOfηBar =
  Show[{listCP, Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]},
      {gLAT, -90, 90}, PlotStyle → {Black, Thickness[0.002]}, (*Mesh→{11,5,0}
      (*{23,11,0}*),MeshStyle→Thick,*)PlotPoints → 60], {gLON, -180, 180, 30}],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]}, {gLON, -180, 180},
      PlotStyle → {Black, Thickness[0.002]}, (*Mesh→{11,5,0}(*{23,11,0}*),
      MeshStyle→Thick,*)PlotPoints → 60], {gLAT, -60, 60, 30}],
    Graphics[{PointSize[0.004], Text[StyleForm["N", FontSize -> 10, FontWeight -> "Plain"],
       {0, 1.85}], (*Sources S:*)Green, Point[ xyAitoffSources ],
       Black, Text[StyleForm["Hmax", FontSize → 8, FontWeight -> "Bold"], {-3.3, -1.0}],
      {Arrow[BezierCurve[{{-3.3, -1.2}, {-2.3, -2.0}, {xHGal[gLONHmaxDegrees, gLATHmaxDegrees],
           yHGal[gLONHmaxDegrees, gLATHmaxDegrees]}}]]},
      Text[StyleForm["Hmin", FontSize → 8, FontWeight -> "Bold"], {3.3, -1.0}],
      {Arrow[BezierCurve[{{3.3, -1.2}, {2.3, -2.0}, {xHGal[gLONHminDegrees - 180,
            -gLATHminDegrees], yHGal[gLONHminDegrees - 180, -gLATHminDegrees]}}]]},
      Text[StyleForm["Hmin", FontSize → 8, FontWeight -> "Bold"], {-3.3, 1.0}],
      {Arrow[BezierCurve[{{-3.3, 1.2}, {-2.3, 2.0}, {xHGal[gLONHminDegrees, gLATHminDegrees],
           yHGal[gLONHminDegrees, gLATHminDegrees]}}]]},
      Text[StyleForm["Hmax", FontSize → 8, FontWeight -> "Bold"], {3.3, 1.0}] ,
      {Arrow[BezierCurve[{{3.3, 1.2}, {2.3, 2.0}, {xHGal[gLONHmaxDegrees - 180, -gLATHmaxDegrees],
           yHGal[gLONHmaxDegrees - 180, -gLATHmaxDegrees]}}]]}      }]}, ImageSize → 2 × 432];
```

Section Summary

In[136]:=
```
mapOfηBar
Print[
 "Figure 4: The alignment function η̄(H), Eq. (1). The map is centered on (gLON,gLAT)=(0°,0°)."]
Print["The sources are located at the dots, shaded ", Green, " ."]
Print["The smallest alignment angle is η̄min = ", jηBarMinBest[[2]] (360. / (2. π)),
 "°, located at the hubs Hmin in the most aligned areas shaded ", Blue,
 " . The alignment hubs Hmin and -Hmin are located at (gLON,gLAT) = ",
 {gLONHminDegrees, gLATHminDegrees }, " and ",
 {gLONHminDegrees - 180, -gLATHminDegrees }, " , in degrees."]
Print["The largest avoidance angle is η̄max = ", jηBarMaxBest[[2]] (360. / (2. π)),
 "°, located at the hubs Hmax in the most avoided areas shaded ", Red,
 " . The avoidance hubs Hmax and -Hmax are located at (gLON,gLAT) = ",
 {gLONHmaxDegrees - 180, -gLATHmaxDegrees }, " and at ",
 {gLONHmaxDegrees, gLATHmaxDegrees }, " , in degrees."]
Print["Notes: Although somewhat obscured by the distortion needed to plot a
   sphere on a flat surface, the function η̄(H) is symmetric across diameters.
   Diametrically opposite points -H and H have the same alignment angle η̄(H)."]
```

Out[136]=

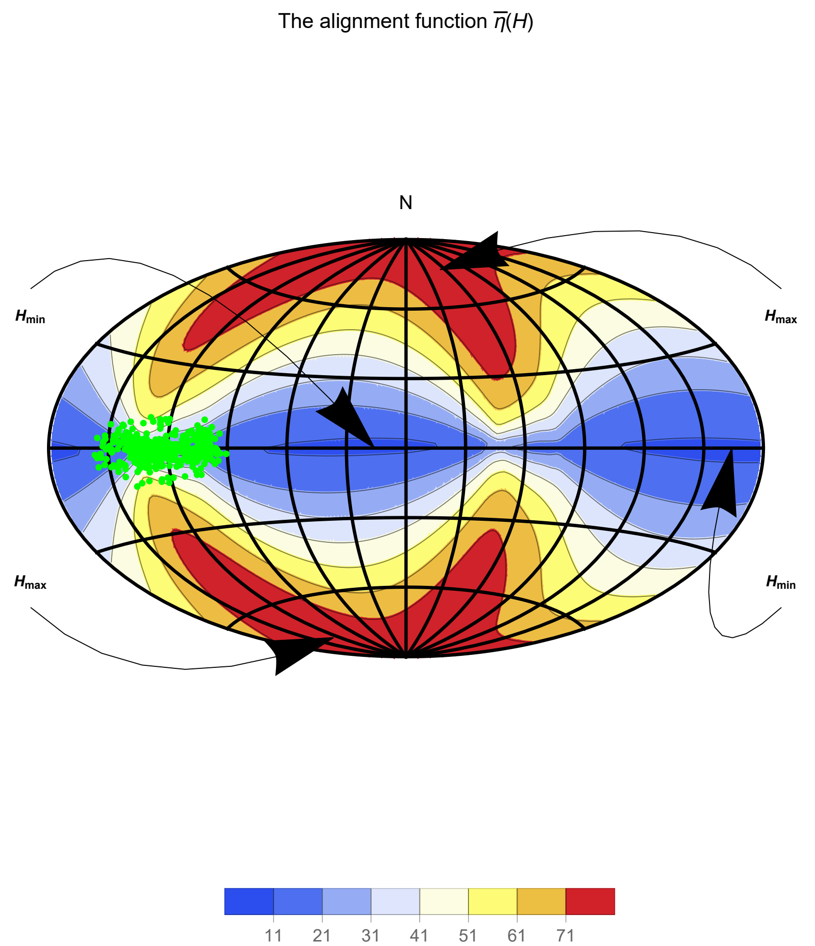

Figure 4: The alignment function $\overline{\eta}(H)$, Eq. (1). The map is centered on (gLON,gLAT) = (0°,0°).

The sources are located at the dots, shaded ■ .

The smallest alignment angle is $\overline{\eta}_{min}$ = 10.3974 °, located at the hubs H_{min} in the most aligned areas shaded ■ . The alignment hubs H_{min} and $-H_{min}$ are located at (gLON,gLAT) = {15.8416, 0.} and {-164.158, 0.} , in degrees.

The largest avoidance angle is $\overline{\eta}_{max}$ = 79.9885 °, located at the hubs H_{max} in the most avoided areas shaded ■ . The avoidance hubs H_{max} and $-H_{max}$ are located at (gLON,gLAT) = {-52.9765, 76.} and at {127.023, -76.} , in degrees.

Notes: Although somewhat obscured by the distortion needed to plot a sphere on a flat surface, the function $\overline{\eta}(H)$ is symmetric across diameters. Diametrically opposite points -H and H have the same alignment angle $\overline{\eta}(H)$.

In[142]:=
```
(*Statistics*)
Print["Statistics of the Alignment Function η̄(H) :"]
Print[" "]
Print["The number of sources: N = ", nSrc]
Print["The min alignment angle, ηmin = ", jηBarMinBest[[2]] * (360. / (2. π)),
 "°, is ", (η0MIN[nSrc, c1MIN, a1MIN] - jηBarMinBest[[2]]) * (360. / (2. π)),
 "° below the most likely value, ", η0MIN[nSrc, c1MIN, a1MIN] * (360. / (2. π)),
 "°, for random runs. Since the uncertainty σ is ",
 σMIN[nSrc, c2MIN, a2MIN] * (360. / (2. π)), "°, the difference ",
 (η0MIN[nSrc, c1MIN, a1MIN] - jηBarMinBest[[2]]) * (360. / (2. π)), "° is ",
 (η0MIN[nSrc, c1MIN, a1MIN] - jηBarMinBest[[2]]) / σMIN[nSrc, c2MIN, a2MIN],
 "σs from the most likely random run value."]
Print["Thus, the smallest alignment angle η̄min is " ,
 (η0MIN[nSrc, c1MIN, a1MIN] - jηBarMinBest[[2]]) / σMIN[nSrc, c2MIN, a2MIN],
 "σs below the most likely random run value."]
Print[""]
Print["The largest avoidance angle, ηmax = ", jηBarMaxBest[[2]] * (360. / (2. π)),
 "°, is ", - (η0MAX[nSrc, c1MAX, a1MAX] - jηBarMaxBest[[2]]) * (360. / (2. π)),
 "° above the most likely value, ", η0MAX[nSrc, c1MAX, a1MAX] * (360. / (2. π)),
 "°, for random runs. Since the uncertainty σ is ",
 σMAX[nSrc, c2MAX, a2MAX] * (360. / (2. π)), "°, the difference ",
 - (η0MAX[nSrc, c1MAX, a1MAX] - jηBarMaxBest[[2]]) * (360. / (2. π)), "° is ",
 - ((η0MAX[nSrc, c1MAX, a1MAX] - jηBarMaxBest[[2]]) / σMAX[nSrc, c2MAX, a2MAX]),
 "σs from the most likely random run value." ]
Print["Thus, the largest avoidance angle η̄max is " ,
 (jηBarMaxBest[[2]] - η0MAX[nSrc, c1MAX, a1MAX]) / σMAX[nSrc, c2MAX, a2MAX],
 "σs above the most likely random run value."]
```

Statistics of the Alignment Function $\overline{\eta}(H)$:

The number of sources: N = 893

The min alignment angle, ηmin = 10.3974°, is 32.873° below the most likely value, 43.2704°, for random runs. Since the uncertainty σ is 0.610487 °, the difference 32.873° is 53.8472σs from the most likely random run value.

Thus, the smallest alignment angle $\overline{\eta}_{min}$ is 53.8472σs below the most likely random run value.

The largest avoidance angle, ηmax = 79.9885°, is 33.2517 ° above the most likely value, 46.7368°, for random runs. Since the uncertainty σ is 0.596403°, the difference 33.2517° is 55.7537σs from the most likely random run value.

Thus, the largest avoidance angle $\overline{\eta}_{max}$ is 55.7537σs above the most likely random run value.

6. Uncertainty Runs

6a. Creating and Storing Uncertainty Runs

For each "uncertainty run", let the polarization direction ψ for each source be allowed to differ from the best value ψn by an amount $\delta\psi$ chosen according to a Gaussian distribution with mean (best) value ψn and half-width $\sigma\psi$, $\psi = \psi n + \delta\psi$. Both values ψn and $\sigma\psi$ were taken from the catalog.

Definitions:

rSrcxrGrid unit vector $S_i \times H_j$ in the direction of the cross product of the radial vector S_i to a source with the radial vector H_j to a grid point
μ the mean value μ of the measurement Gaussian for ψ
σ the uncertainty of the measured polarization position angle ψ
ψData polarization directions $\psi = \psi n + \delta\psi$
runData collection of data to save from the uncertainty runs, see below for content list
nRunPrint dummy index controlling when current TimeUsed and MemoryInUse are printed
ψSrc the polarization direction ψ for the run.
rSrcxψSrc unit vector, $S_i \times \psi_i$, cross product of the radial vector S_i to the source with the vector $\hat{v}_\psi$ in the direction of the polarization
jηBarToGrid {j, $\overline{\eta}(H_j)$}, where j is the index for the grid point H_j and $\overline{\eta}(H_j)$ is the alignment angle function, (1), at H_j
sortjηBarToGrid sort {j, $\overline{\eta}(H_j)$}, with the smaller angle $\overline{\eta}$(H) first.
jηBarMin1 {j,$\overline{\eta}$(H)} for the smallest value of $\overline{\eta}$(H) , best alignment
jηBarMax1 {j,$\overline{\eta}$(H)}, for the largest value of $\overline{\eta}$(H) , most avoided

ηBarMinData values of $\overline{\eta}_{\min}$ from uncertainty runs, alignment
ηBarMaxData values of $\overline{\eta}_{\max}$ from uncertainty runs, avoidance
HmingLONData values of gLON = gLON for hub $H_{\min}$ from uncertainty runs, alignment
HmingLATData values of gLAT = gLAT for hub $H_{\min}$ from uncertainty runs, alignment
HmaxgLONData values of gLON = gLON for hub $H_{\max}$ from uncertainty runs, avoidance
HmaxgLATData values of gLAT = gLAT for hub $H_{\max}$ from uncertainty runs, avoidance

Tables:
ψData entries: 1. Run # 2. ψSrc, list of polarization position angles ψ
runData entries: 1. Run # 2. {$\overline{\eta}_{\min}$, {gLON,gLAT} at $H_{\min}$} 3. {$\overline{\eta}_{\max}$, {gLON,gLAT} at $H_{\max}$}

To create Uncertainty Runs, you need to calculate "rSrcxrGrid" and evaluate the "For" statement in the following cells. Be sure to save the results with the "Put[]" statements. Then comment out the "rSrcxrGrid" and "For" statements by enclosing each in (*comment brackets*).

In[150]:=
```
(*
rSrcxrGrid1=Table[ Cross[  rSrc[[i]],rGrid[[j]]  ] ,    {i,nSrc},{j,nGrid}];
(*first step: gLONw cross product, not unit vectors*)
rSrcxrGrid=Table[ rSrcxrGrid1[[i,j]]/
   (rSrcxrGrid1[[i,j]].rSrcxrGrid1[[i,j]]+ 0.000001)^1/2. ,    {i,nSrc},{j,nGrid}];
Clear[rSrcxrGrid1];
*)
(*rSrcxrGrid: table of the unit vectors perpendicular to the plane
  of the great circle containing the source Si and the grid point Hj*)
```

In[151]:=
```
(*
nR=500;
(*number of runs with the PPA ψ allowed by measurement uncertainty. *)
μ=ψn;σ=σψn;runData={};ψData={};nRunPrint=0;
For[nRun=1,nRun≤nR,nRun++,
 If[nRun>nRunPrint,Print["At the start of run ",nRun,", the time is ",
   TimeUsed[]," seconds and the memory in use is ",MemoryInUse[]," bytes."];
  nRunPrint=nRunPrint+200];
   ψSrc=Table[RandomVariate[NormalDistribution[μ[[i]],σ[[i]]]],{i,nSrc}];
 (*table of PPA angles ψ for the sources in region j0, in radians*)
 rSrcxψSrc = Table[ Sin[ψSrc[[i]]]eNSrc[[i]]-Cos[ψSrc[[i]]] eESrc[[i]],    {i,nSrc}];
 (*table of the cross product of rSrc and vector in direction of ψSrc,
 a unit vector*)jηBarToGrid = Table[{j,(1/nSrc)Sum[ArcCos[
       Abs[  rSrcxψSrc[[i]].rSrcxrGrid[[i,j]]  ] - 0.000001  ],{i,nSrc}]},{j,nGrid}];
 (*
 {grid point #, value of the alignment angle ηnHj[j] averaged over all sources,
  in radians}*) sortjηBarToGrid=Sort[jηBarToGrid,#1[[2]]<#2[[2]]&];
 (*jηBarToGrid, {j,ηj}, but sorted with the smallest alignment angles first
 *)
 jηBarMin1=sortjηBarToGrid[[1]]; (*  {j,ηj}, at the grid point Hj with minimum η̄*)
 jηBarMax1=sortjηBarToGrid[[-1]]; (*  {j,ηj},
 at the grid point Hj with maximum η̄*)AppendTo[ψData,{nRun,ψSrc}];
 AppendTo[runData,{nRun,{ jηBarMin1[[2]],{gLONGrid [ [ jηBarMin1[[1]]  ]],
     gLATGrid [[  jηBarMin1[[1]]  ]]}},{ jηBarMax1[[2]],{gLONGrid [[
      jηBarMax1[[1]]  ]],gLATGrid [[  jηBarMax1[[1]]  ]]}}} ](*collect data*)    ]
*)
```

Hint: You can save memory if you do not get the "ψData". The table ψData can be used to reconstruct the runData table, but it is not needed in any following calculation.

In[152]:=
```
SetDirectory[homeDirectory];(*Save memory space; ψData is not used below.*)
(*Put[ψData,"20210320PsiDataAllStarsClump1Lon100PlusOnDisk.dat"  ] *)
(*Save a new "ψData"*)
(*ψData=Get["20210322PsiDataAllStarsClump1Lon100PlusOnDisk.dat"]; *)
(*Get an old "ψData"*)
```

Hint: Saving “runData” to a file avoids the time it takes to complete the “For” statement. Make the “For” statement into a remark so that it doesn’t evaluate.

In[153]:=
```
SetDirectory[homeDirectory];
(*Put[runData,"20210320runDataAllStarsClump1Lon100PlusOnDisk.dat"  ]*)
(*Save a new "runData".*)
runData = Get["20210322runDataAllStarsClump1Lon100PlusOnDisk.dat"];
(*Get an old "runData".*)
```

In[155]:=
```
ηBarMinData = Table[runData[[i1, 2, 1]] , {i1, Length[runData]}];
ηBarMaxData = Table[runData[[i1, 3, 1]] , {i1, Length[runData]}];
HmingLONData = Table[ runData[[i1, 2, 2, 1]] , {i1, Length[runData]}];
HmingLATData = Table[runData[[i1, 2, 2, 2]], {i1, Length[runData]}];
HmaxgLONData = Table[ runData[[i1, 3, 2, 1]] , {i1, Length[runData]}];
HmaxgLATData = Table[runData[[i1, 3, 2, 2]], {i1, Length[runData]}];
```

In[161]:=
```
Print["The number of uncertainty runs is ", Length[runData], "."]
```

```
The number of uncertainty runs is 503.
```

6b. The Effects of Uncertainty on the Smallest Alignment Angle $\overline{\eta}_{\min}$

This section fits a Gaussian distribution to the $\overline{\eta}_{\min}$ from the uncertainty runs.

Definitions

sortηBarMin	sort the list of $\overline{\eta}_{\min}$ from the uncertainty runs
η0B	estimated mean of the Gaussian fit
σB	estimated half-width of the Gaussian fit
histogramrange	{min η, max η, Δη} for the histogram
hl0, hl	histogram {η, bin height} tables needed to set up the NonlinearModelFit
nlmB	non-linear model fit of a Gaussian to the $\overline{\eta}_{\min}$ histogram
showNLMB	plot of Gaussian and histogram
ParametersNLMB	amplitude, half-width, and mean of the Gaussian fit
pTableNLMB	table of parameter attributes, including standard error

In[162]:=
```
sortηBarMin = Sort[ηBarMinData];
η0B = mean[ηBarMinData ]; (*Guess the mean for the Gaussian. *)
σB = stanDev[ηBarMinData ];(*Guess the half-width.*)
histogramrange = {η0B - 5 σB, η0B + 5 σB, 0.4 σB};
hl0 = HistogramList[sortηBarMin, histogramrange];
hl =
  Table[{(1/2) (hl0[[1, i1]] + hl0[[1, i1 + 1]]), hl0[[2, i1]]}, {i1, Length[ hl0[[2]] ]}];
nlmB = NonlinearModelFit[hl, a Exp[- (1/2.) ((x - x0) / b)^2],
   {{a, Length[sortηBarMin / 6]}, {b, σB}, {x0, η0B}}, x];(*x is ηBarMin*)
```

In[168]:=
```
showNLMB = Show[{Histogram[sortηBarMin, histogramrange,
    PlotLabel → "η̄min ", AxesLabel → {"η̄min, radians", "ΔR"}],
   Plot[Normal[nlmB], {x, η0B - 5 σB, η0B + 5 σB}, PlotLabel → "η̄min"],
   ListPlot[hl, PlotLabel → "η̄min"] }]
Print["Figure 5: The Gaussian fit to the alignment angle η̄min histogram, where
   the height is the number of runs ΔR in each bin of width Δη̄min  =  ", 0.4 σB,
 " radians. The total number of runs is R = Σ(ΔR) = ", Length[runData], "."]
```

Out[168]=

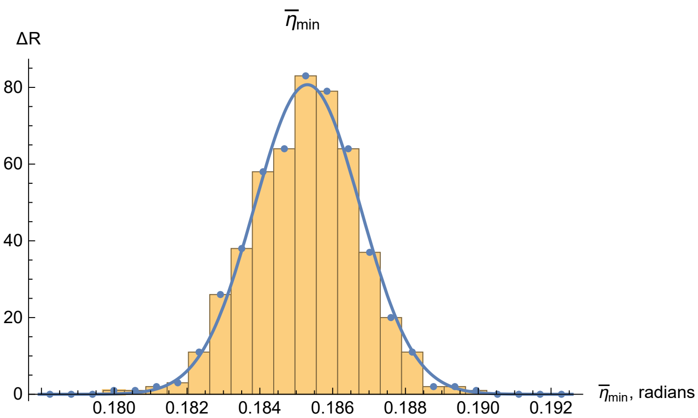

Figure 5: The Gaussian fit to the alignment angle $\overline{\eta}_{min}$ histogram, where the height is the number of runs ΔR in each bin of width $\Delta\overline{\eta}_{min}$ = 0.00058555 radians. The total number of runs is R = Σ(ΔR) = 503.

In[170]:=
```
ParametersNLMB = {a, b, x0} /. nlmB["BestFitParameters"];
pTableNLMB = nlmB["ParameterTable"]
{σηBarMinFit, ηBarMinFit} = {ParametersNLMB[[2]] , ParametersNLMB[[3]] };(*radians*)
```

Out[171]=

	Estimate	Standard Error	t-Statistic	P-Value
a	80.6937	1.69664	47.561	1.11843×10^{-23}
b	0.00145383	0.0000352965	41.189	2.56012×10^{-22}
x0	0.185305	0.0000352965	5249.97	1.40925×10^{-68}

6c. The Effects of Uncertainty on the Largest Avoidance Angle $\overline{\eta}_{max}$

This section fits a Gaussian distribution to the $\overline{\eta}_{max}$ returned by the uncertainty runs.

Definitions: See the list of Definitions in Sec. 7b. Trade avoidance (Max) here for alignment (Min) there.

In[173]:=
```
sortηBarMax = Sort[ηBarMaxData];
η0MaxB = mean[ηBarMaxData ]; (*Guess the mean for the Gaussian. *)
σMaxB  = stanDev[ηBarMaxData ];(*Guess the half-width.*)
histogramrangeMAX = {η0MaxB - 5 σMaxB, η0MaxB + 5 σMaxB, 0.4 σMaxB};
hl0Max = HistogramList[sortηBarMax, histogramrangeMAX];
hlMax = Table[{(1/2) (hl0Max[[1, i1]] + hl0Max[[1, i1 + 1]]), hl0Max[[2, i1]]},
  {i1, Length[ hl0Max[[2]] ]}];
nlmMaxB = NonlinearModelFit[hlMax, a Exp[-(1/2.) ((x - x0) / b)^2],
  {{a, 300.}, {b, σMaxB}, {x0, η0MaxB}}, x];(*x is ηBarMax *)
```

In[179]:=
```
showNLMMaxB = Show[{Histogram[sortηBarMax,
    histogramrangeMAX, PlotLabel → "η̄max", AxesLabel → {"η̄max, radians", "ΔR"}],
   Plot[Normal[nlmMaxB], {x, η0MaxB - 5 σMaxB, η0MaxB + 5 σMaxB}, PlotLabel → "η̄max"],
   ListPlot[hlMax, PlotLabel → "η̄max"]}]
Print["Figure 6: The Gaussian fit to the avoidance angle η̄max
   histogram. The bins have a width Δη̄max  =  ", 0.4 σMaxB,
 " radians and have a height equal to the number of runs ΔR in the bin.
   The total number of runs is R = Σ(ΔR) = ", Length[runData], "."]
```

Out[179]=

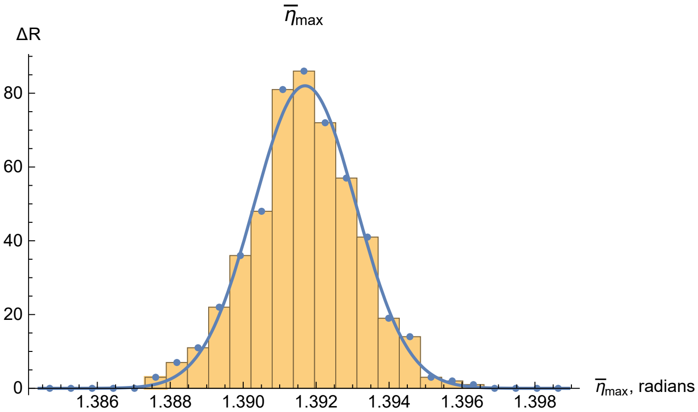

Figure 6: The Gaussian fit to the avoidance angle $\bar{\eta}_{max}$ histogram. The bins have a width $\Delta\bar{\eta}_{max}$ = 0.000580852 radians and have a height equal to the number of runs ΔR in the bin. The total number of runs is R = Σ(ΔR) = 503.

In[181]:=
```
ParametersNLMMaxB = {a, b, x0} /. nlmMaxB["BestFitParameters"];
pTableNLMMaxB = nlmMaxB["ParameterTable"]
{σηBarMaxFit, ηBarMaxFit} = {ParametersNLMMaxB[[2]], ParametersNLMMaxB[[3]]};
(*radians*)
```

Out[182]=

	Estimate	Standard Error	t-Statistic	P-Value
a	82.0186	1.84385	44.4822	4.80532×10^{-23}
b	0.00139776	0.000036284	38.5227	1.09471×10^{-21}
x0	1.3917	0.000036284	38355.7	1.40669×10^{-87}

6d. The Effects of Uncertainty on the Locations (gLON,gLAT) of the Alignment Hubs H_{min}

Each uncertainty run returns an alignment hub H_{min}. In this section, we calculate the mean and standard deviation to approximate the distribution of the locations the Alignment Hubs H_{min}.

In any one run, the analysis produces an alignment angle $\bar{\eta}$ at each grid point. There can be just one minimum alignment angle $\bar{\eta}_{min}$, but there are two hubs, H_{min} and $-H_{min}$, by the symmetry across a diameter. So we collect all the hubs together by moving the $-H_{min}$ hubs across a diameter to join the H_{min} hubs.

Definitions

HmingLON	gLON in radians for H_{min}
HmingLAT	gLAT in radians for H_{min}
σgLONMinFit1	half-width for gLON uncertainty runs
gLONMinFit1	mean for gLON uncertainty runs

σgLATMinFit1	half-width for gLAT uncertainty runs
gLATMinFit1	mean for gLAT uncertainty runs
HmingLONAVE	average over all uncertainty runs of gLON for H_{min}
HmingLONgLAT	(gLON,gLAT) table for ListPlot
lpHmin	plot Hmin hubs from uncertainty runs
gLON1,2Min1	values needed for framing the most likely hubs
gLAT1,2Min1	ditto for latitude

In[184]:=
```
(* Gather the hubs. Move the hubs across diameters,
ΔgLON = π, or around a complete circle, ΔgLON = 360°,
if necessary, so that all hubs satisfy 0° ≤  gLON  <  180° .*)
HmingLON0 = HmingLONData;
HmingLAT0 = HmingLATData;
HmingLONBy180n = Round[HmingLON0 / π];
HmingLON1 = Table[HmingLON0[[i1]] - HmingLONBy180n[[i1]] π , {i1, Length[HmingLON0]}];
HmingLAT1 = Table[(-1)^HmingLONBy180n[[i1]] HmingLAT0[[i1]] , {i1, Length[HmingLAT0]}];
HmingLON  = Table[If[HmingLON1[[i1]] < 0, HmingLON1[[i1]] + π, HmingLON1[[i1]], "huh?"] ,
   {i1, Length[HmingLON1]}];
HmingLAT  = Table[If[HmingLON1[[i1]] < 0, -HmingLAT1[[i1]], HmingLAT1[[i1]], "huh?"] ,
   {i1, Length[HmingLAT1]}];
```

In[190]:=
```
(*Check that 0° ≤  gLON  <  180°  and -90° ≤  gLAT  <  90° *)
(*ListPlot[{Sort[HmingLON],Sort[HmingLAT]},
 PlotLabel→"gLON and gLAT for Hmin, radians",AxesLabel→{"Run #","gLON,gLAT"}]*)
```

In[191]:=
```
{σgLONMinFit1, gLONMinFit1} = {stanDev[HmingLON], mean[HmingLON]};(*radians*)
{σgLATMinFit1, gLATMinFit1} = {stanDev[HmingLAT], mean[HmingLAT]};(*radians*)
```

In[193]:=
```
(*Define quantities for the plot of the Hmin from the uncertainty runs. *)
HmingLONgLAT = Sort[Table[{-HmingLON[[i5]], HmingLAT[[i5]]}, {i5, Length[HmingLON]}]];
{HmingLONgLAT[[1]], HmingLONgLAT[[-1]]} ;(*radians*)
{HmingLONgLAT[[1]], HmingLONgLAT[[-1]]} (360. / (2. π)) ;(*degrees*)
lpHmin = ListPlot[HmingLONgLAT (360. / (2. π)), PlotRange → {{-180, 180}, {-90, 90}},
   PlotMarkers → Automatic, AxesLabel → {"-gLON, degrees", "gLAT, degrees"},
   PlotLabel → "(-gLON,gLAT) for the Hmin hubs",
   Ticks → {Table[{t, -t}, {t, -180, 180, 45}], Automatic}];
gLON1Min1 = (gLONMinFit1 - σgLONMinFit1) (360. / (2. π));
gLON2Min1 = (gLONMinFit1 + σgLONMinFit1) (360. / (2. π));
gLAT1Min1 = (gLATMinFit1 - σgLATMinFit1) (360. / (2. π));
gLAT2Min1 = (gLATMinFit1 + σgLATMinFit1) (360. / (2. π));
```

6e. The Effects of Uncertainty on the Locations (gLON,gLAT) of the Avoidance Hubs H_{max}.

Each uncertainty run returns an alignment hub H_{max}. In this section, we simply calculate the mean and standard deviation to approximate the distribution of the locations of the Avoidance Hubs H_{max}.

Definitions: Explore the definitions for H_{min} at the start of Sec. 7d. Find the similarly named quantity by interchanging Max for Min. Adjust the definition to the present context.

In[201]:=
```
(* Move hubs, if necessary, so that 0°  ≤  gLON  <  360° *)
HmaxgLON0 = HmaxgLONData;
HmaxgLAT0 = HmaxgLATData;
HmaxgLONBy180n = Round[HmaxgLON0 / π];
HmaxgLON1 = Table[HmaxgLON0[[i1]] - HmaxgLONBy180n[[i1]] π , {i1, Length[HmaxgLON0]}];
HmaxgLAT1 = Table[(-1)^HmaxgLONBy180n[[i1]] HmaxgLAT0[[i1]] , {i1, Length[HmaxgLAT0]}];
HmaxgLON  = Table[If[0 > HmaxgLON1[[i1]], HmaxgLON1[[i1]] + π, HmaxgLON1[[i1]], "huh?"] ,
    {i1, Length[HmaxgLON1]}];
HmaxgLAT  = Table[If[0 > HmaxgLON1[[i1]], -HmaxgLAT1[[i1]], HmaxgLAT1[[i1]], "ah"] ,
    {i1, Length[HmaxgLAT1]}];
```

In[207]:=
```
(*Check that 0°  ≤  gLON  <  180°  and -90°  ≤  gLAT  <  90° *)
(*ListPlot[{Sort[HmaxgLON],Sort[HmaxgLAT]},PlotRange→{-2π,2π},
 AxesLabel→{"Run #","gLON,gLAT radians"},PlotLabel→"gLONs, gLATs for Hmax"]*)
```

In[208]:=
```
{σgLONMaxFit, gLONMaxFit}  =  {stanDev[HmaxgLON], mean[HmaxgLON]};(*radians*)
{σgLATMaxFit, gLATMaxFit}  =  {stanDev[HmaxgLAT], mean[HmaxgLAT]};(*radians*)
```

In[210]:=
```
(* Define quantities for the plot of the
 locations of the Hmax from the uncertainty runs. *)
HmaxgLONgLAT  = Table[{-HmaxgLON[[i8]], HmaxgLAT[[i8]]}, {i8, Length[HmaxgLAT ]}];
{HmaxgLONgLAT[[1]], HmaxgLONgLAT[[-1]]} ;(*radians*)
{HmaxgLONgLAT[[1]], HmaxgLONgLAT[[-1]]} (360. / (2. π)) ;(*degrees*)
lpHmax1 = ListPlot[HmaxgLONgLAT (360. / (2. π)), PlotRange → {{-180, +180}, {-90, 90}},
    PlotMarkers → Automatic, AxesLabel → {"-gLON, degrees", "gLAT, degrees"},
    PlotLabel → "Hmax hubs with the most likely region indicated ",
    Ticks → {Table[{t, -t}, {t, -180, 180, 45}], Automatic}];
gLON1Max = (gLONMaxFit - σgLONMaxFit) (360. / (2. π));
gLON2Max = (gLONMaxFit + σgLONMaxFit) (360. / (2. π));
gLAT1Max = (gLATMaxFit - σgLATMaxFit) (360. / (2. π));
gLAT2Max = (gLATMaxFit + σgLATMaxFit) (360. / (2. π));
```

6f. Map of the Hubs for the Uncertainty Runs

In this subsection, we map the locations of the alignment hubs $H_{\min}$ and the locations of the avoidance hubs $H_{\max}$, one set for each uncertainty run.

Definitions:

xyAitoffHmin	Aitoff coordinates for the alignment hubs $H_{\min}$ from the uncertainty runs
xyAitoffHmax	Aitoff coordinates for the avoidance hubs $H_{\max}$ from the uncertainty runs
xyAitoffOppositeHmin	Aitoff coordinates for the $-H_{\min}$
xyAitoffOppositeHmax	Aitoff coordinates for the $-H_{\max}$
mapOfσψHminHmax	plot of the alignment and avoidance hubs $H_{\min}$, $-H_{\min}$, $H_{\max}$, and $-H_{\max}$

```
In[218]:= (*The Aitoff coordinates for the hubs Hmin locations.*)xyAitoffHmin =
  Table[{xHGal[ HmingLON[[n]] (360/(2 π)), HmingLAT[[n]] (360/(2 π)) ], yHGal[
     HmingLON[[n]] (360/(2 π)), HmingLAT[[n]] (360/(2 π)) ]}, {n, Length[HmingLAT]}];
 (*The Aitoff coordinates for the hubs Hmax locations.*)xyAitoffHmax =
  Table[{xHGal[ HmaxgLON[[n]] (360/(2 π)), HmaxgLAT[[n]] (360/(2 π)) ], yHGal[
     HmaxgLON[[n]] (360/(2 π)), HmaxgLAT[[n]] (360/(2 π)) ]}, {n, Length[HmingLAT]}];
 (*The Aitoff coordinates for the hubs -Hmin locations.*)
 xyAitoffOppositeHmin = Table[{xHGal[ If[0 ≤ HmingLON[[n]] (360/(2 π)) < +180,
       HmingLON[[n]] (360/(2 π)) - 180, If[0 > HmingLON[[n]] (360/(2 π)) > -180,
        HmingLON[[n]] (360/(2 π)) + 180]], -HmingLAT[[n]] (360/(2 π)) ],
     yHGal[ If[0 ≤ HmingLON[[n]] (360/(2 π)) < +180, HmingLON[[n]] (360/(2 π)) - 180,
       If[0 > HmingLON[[n]] (360/(2 π)) > -180, HmingLON[[n]] (360/(2 π)) + 180]],
      -HmingLAT[[n]] (360/(2 π)) ]}, {n, Length[HmingLAT]}];
 (*The Aitoff coordinates for the hubs -Hmax locations.*)
 xyAitoffOppositeHmax =
  Table[{xHGal[ If[0 ≤ HmaxgLON[[n]] (360/(2 π)) < +180, HmaxgLON[[n]] (360/(2 π)) - 180,
       If[0 > HmaxgLON[[n]] (360/(2 π)) > -180, HmaxgLON[[n]] (360/(2 π)) + 180]],
      -HmaxgLAT[[n]] (360/(2 π)) ],
     yHGal[ If[0 ≤ HmaxgLON[[n]] (360/(2 π)) < +180, HmaxgLON[[n]] (360/(2 π)) - 180,
       If[0 > HmaxgLON[[n]] (360/(2 π)) > -180, HmaxgLON[[n]] (360/(2 π)) + 180]],
      -HmaxgLAT[[n]] (360/(2 π)) ]}, {n, Length[HmaxgLAT]}];
```

In[222]:=
```
(*Construct the map of uncertainty run Hmin and Hmax hubs with ± regions indicated.*)
mapOfσψHminHmax =
  Show[{Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]},
      {gLAT, -90, 90}, PlotStyle → {Black, Thickness[0.002]}, PlotPoints → 60,
      PlotRange → {{-7, 7}, {-3, 3}}, Axes → False], {gLON, -180, 180, 30}],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]}, {gLON, -180, 180},
      PlotStyle → {Black, Thickness[0.002]}, PlotPoints → 60], {gLAT, -60, 60, 30}],
    Graphics[{PointSize[0.007], Text[StyleForm["N", FontSize -> 10, FontWeight -> "Plain"],
       {0, 1.85}], LightBlue, (*Hmin:*)Point[ xyAitoffHmin ],
      (*-Hmin:*)Point[ xyAitoffOppositeHmin ], LightRed, (*Hmax:*)
      Point[ xyAitoffHmax ], (*-Hmax:*)Point[ xyAitoffOppositeHmax ]    }],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]}, {gLAT, gLAT1Max, gLAT2Max},
      PlotStyle → {Purple, Thickness[0.002]}, PlotPoints → 60],
     {gLON, gLON1Max, gLON2Max, gLON2Max - gLON1Max}],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]}, {gLON, gLON1Max, gLON2Max},
      PlotStyle → {Purple, Thickness[0.002]}, PlotPoints → 60],
     {gLAT, gLAT1Max, gLAT2Max, gLAT2Max - gLAT1Max}],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]}, {gLAT, -gLAT2Max, -gLAT1Max},
      PlotStyle → {Purple, Thickness[0.002]}, PlotPoints → 60],
     {gLON, gLON1Max - 180, gLON2Max - 180, gLON2Max - gLON1Max}],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]},
      {gLON, gLON1Max - 180, gLON2Max - 180}, PlotStyle → {Purple, Thickness[0.002]},
      PlotPoints → 60], {gLAT, -gLAT2Max, -gLAT1Max, gLAT2Max - gLAT1Max}],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]}, {gLAT, -gLAT2Min1, -gLAT1Min1},
      PlotStyle → {Purple, Thickness[0.002]}, PlotPoints → 60],
     {gLON, gLON1Min1 - 180, gLON2Min1 - 180, gLON2Min1 - gLON1Min1}],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]},
      {gLON, gLON1Min1 - 180, gLON2Min1 - 180}, PlotStyle → {Purple, Thickness[0.002]},
      PlotPoints → 60], {gLAT, -gLAT2Min1, -gLAT1Min1, gLAT2Min1 - gLAT1Min1}],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]}, {gLAT, gLAT1Min1, gLAT2Min1},
      PlotStyle → {Purple, Thickness[0.002]}, PlotPoints → 60],
     {gLON, gLON1Min1, gLON2Min1, gLON2Min1 - gLON1Min1}],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]}, {gLON, gLON1Min1, gLON2Min1},
      PlotStyle → {Purple, Thickness[0.002]}, PlotPoints → 60],
     {gLAT, gLAT1Min1, gLAT2Min1, gLAT2Min1 - gLAT1Min1}]},
   ImageSize → 2 × 432, PlotLabel → "The Hubs Found from the Uncertainty Runs"];
```

Section Summary

In[223]:=
```
Print[
 "To estimate the effects of experimental uncertainty, there were uncertainty runs."]
Print["Uncertainty runs have polarization directions ψ = ψn + δψ, ",
 "where δψ is chosen with a normal
   distribution of half-width σψ about the best value ψn."]
Print["The number of uncertainty runs: ", Length[runData], "."]
Print["The uncertainty runs determine the smallest alignment angle to be  η̄min = ",
 ηBarMinFit (360. / (2. π)), "° ± ", σηBarMinFit (360. / (2. π)), "°."]
Print["The uncertainty runs determine the largest avoidance angle to be  η̄max = ",
 ηBarMaxFit (360. / (2. π)), "° ± ", σηBarMaxFit (360. / (2. π)), "°."]
Print["The uncertainty runs give the location
   for the alignment hub Hmin as  (gLON, gLAT)  = ",
 {gLONMinFit1 (360. / (2. π)), gLATMinFit1 (360. / (2. π))}, " ± ",
 {σgLONMinFit1 (360. / (2. π)), σgLATMinFit1 (360. / (2. π))}, ", in degrees."]
Print["The uncertainty runs give the location of the avoidance hub Hmax as
   (gLON, gLAT)  = ", {gLONMaxFit (360. / (2. π)), gLATMaxFit (360. / (2. π))},
 " ± ", {σgLONMaxFit (360. / (2. π)), σgLATMaxFit (360. / (2. π))}, ", in degrees."]
```

To estimate the effects of experimental uncertainty, there were uncertainty runs.

Uncertainty runs have polarization directions $\psi = \psi n + \delta\psi$,
where $\delta\psi$ is chosen with a normal distribution of half-width $\sigma\psi$ about the best value ψn.

The number of uncertainty runs: 503.

The uncertainty runs determine the smallest alignment angle to be $\bar{\eta}_{min}$ = 10.6172° ± 0.0832982°.

The uncertainty runs determine the largest avoidance angle to be $\bar{\eta}_{max}$ = 79.7383° ± 0.0800856°.

The uncertainty runs give the location for the alignment hub H_{min} as (gLON, gLAT) =
{15.0518, 0.12326} ± {2.78686, 0.480965}, in degrees.

The uncertainty runs give the location of the avoidance hub H_{max} as (gLON, gLAT) =
{125.814, -76.4334} ± {1.76182, 1.37394}, in degrees.

In[230]:=
```
mapOfσψHminHmax
Print["Figure 7: The ", Length[runData],
 " hubs found for the uncertainty runs.", (*" The arrows point to the
   hubs found with the best values of the polarization directions ψn.   ",*)
 "The alignment hubs Hmin and -Hmin are plotted as light blue dots, ", LightBlue,
 ". ", " The avoidance hubs Hmax and -Hmax are plotted as pink dots, ", LightRed,
 ".", "The most likely locations of the hubs are outlined in purple, ", Purple, "."]
```

The Hubs Found from the Uncertainty Runs

Out[230]=

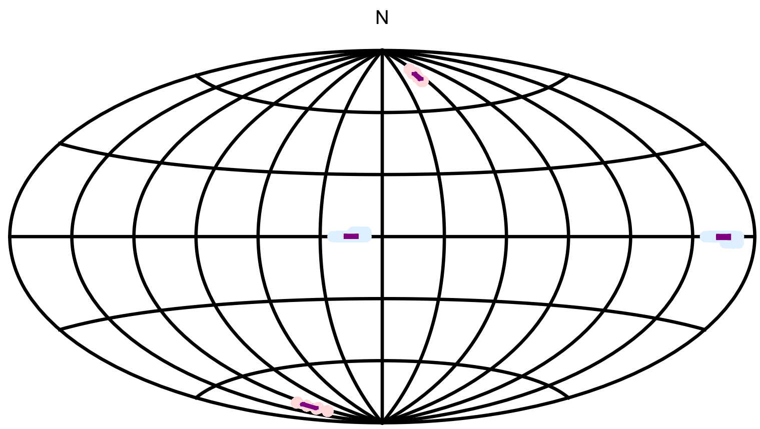

Figure 7: The 503 hubs found for the uncertainty runs.
The alignment hubs H_{min} and $-H_{min}$ are plotted as light blue dots,
. The avoidance hubs H_{max} and $-H_{max}$ are plotted as pink dots,
.The most likely locations of the hubs are outlined in purple, .

As a final image, we superimpose the map of the uncertainty run hubs $H_{\min}$, $-H_{\min}$, $H_{\max}$, and $-H_{\max}$ in Fig. 7 on the graph of the alignment angle function $\overline{\eta}(H)$, Fig. 4.

In[232]:=

```
Show[{mapOfηBar, mapOfσψHminHmax}]
Print[
 "Figure 8: Overlay Fig. 7, Uncertainty Run Hubs, onto Fig. 4, Alignment Function η̅(H)
   using Best Values ψn.  Note that the light blue alignment hubs from the uncertainty
   runs closely follow the areas of convergence (blue) for the best values
   ψn. And the pink avoidance hubs follow the areas of extreme divergence
   (red). One sees that shifting the polarization directions slightly due
   to experimental uncertainty, shifts the locations of the hubs slightly.
   The shifted hubs favor areas, in blue and red, that are close to the
   extremes for η̅(H) using the best values ψn listed in the catalog."]
```

Out[232]=

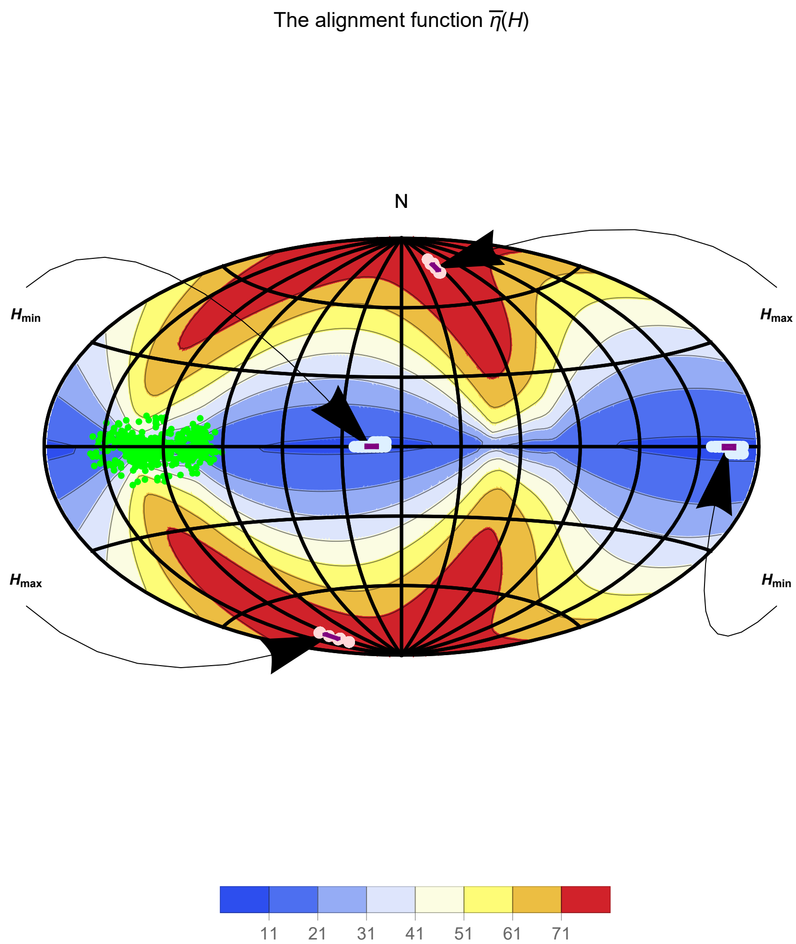

Figure 8: Overlay Fig. 7, Uncertainty Run Hubs, onto Fig. 4, Alignment Function $\bar{\eta}(H)$ using Best Values ψn. Note that the light blue alignment hubs from the uncertainty runs closely follow the areas of convergence (blue) for the best values ψn. And the pink avoidance hubs follow the areas of extreme divergence (red). One sees that shifting the polarization directions slightly due to experimental uncertainty, shifts the locations of the hubs slightly. The shifted hubs favor areas, in blue and red, that are close to the extremes for $\bar{\eta}(H)$ using the best values ψn listed in the catalog.

7. Concluding Remarks

The polarization of starlight is a well-known phenomenon that has been important in understanding the structure of the magnetic field of the Milky Way Galaxy. So, it is not surprising to find that the stars in a region of the Galaxy are well aligned.

The 893 stars in the sample studied here are well-known to be polarized in the direction of the Galactic Disk. Thus the application of the Hub Test to these stars offers a new way to view the alignment and, this is perhaps a new concept, the avoidance of the polarization directions with points on the Celestial Sphere.

One can summarize the alignment metrics. Randomly directed sources would likely have a smallest alignment angle around 43° and a largest avoidance angle around 47°, neither far from a 45° average. The observed polarization directions put the smallest angle $\overline{\eta}_{\text{min}}$ at 10°. That is 33° and 54σs below the random polarization value. For avoidance, the largest angle, $\overline{\eta}_{\text{max}}$, is 80°. That is 33° and 56σs above the random polarization value. One may conclude that the alignment and avoidance are not due to chance.

The alignment and avoidance patterns of the alignment function $\overline{\eta}(H)$ coincide with the Galactic Structure. A glance at Figs. 4 and 8 shows that the alignment hubs H_{min} and $-H_{\text{min}}$ are centered along the Disk and the avoidance hubs H_{max} and $-H_{\text{max}}$ approach the Galactic Poles.

The explanation of the alignment and avoidance patterns is thought to involve magnetic fields that align grains of interstellar dust along the line of sight and, by selective extinction, polarize the light from these stars, Refs. 12,13. There are ambitious projects underway to both locate stars, Ref. 14, and measure their polarization directions, Ref. 15. All that data will need considerable analysis. One hopes that the metrics and descriptions of alignment and avoidance with the Hub Test can assist with the study of Galactic Magnetic fields and the Interstellar Medium.

References

1. R. Shurtleff, the ready-to-run Mathematica version of this notebook is available at the following URL:
   https://www.dropbox.com/s/31dgjiby8j85oqr/20210323AllStarsClump1Lon100PlusOnDisk3.nb?dl=0
2. R. Shurtleff, Evaluating the Alignment of Astronomical Linear Polarization Data, Intermediate Level Software, viXra:2101.0144 (2021).
3. Wolfram Research, Inc., Mathematica, Version 12.1, Champaign, IL (2020).
4. Wikipedia contributors. "Aitoff projection." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 25 May. 2017. Web. (3 Jan. 2018).
5. R. Shurtleff, "Indirect polarization alignment with points on the sky, the Hub Test" , https://vixra.org/abs/2011.0026 (2020).
6. Boulanger, F., Ensslin, T., Fletcher, A., et al., JCAP, 08, 049 (2018).
7. Ian W. Stephens et al, The Galactic Magnetic Field's Effect in Star-forming Regions, ApJ 728 99 (2011), Fig. 4.
8. Planck Collaboration, Planck 2018 results, XII. Galactic astrophysics using polarized dust emission, Astronomy & Astrophysics 641, A12 (2020).
9. A. Lazarian and Thiem Hoang, Alignment and Rotational Disruption of Dust, ApJ 908 12 (2021).
10. Heiles, C., An agglomeration of stellar polarization catalogs, Astron. J. 119, 923 (2000).
11. The catalog, ReadMe, and other files in Ref. 10 are also available online at VizieR; https://cdsarc.unistra.fr/viz-bin/cat/II/226 (2000).
12. Lazarian, A., Andersson, B.-G., & Hoang, T. 2015, in Polarimetry of stars and planetary systems, ed. L. Kolokolova, J. Hough, & A.-C. Levasseur-Regourd ((New York: Cambridge Univ. Press)), 81-113
13. Andersson, B.-G., Lazarian, A., & Vaillancourt, J. E. 2015, ARA&A, 53, 501
14. *Gaia* Collaboration, The *Gaia* Mission, A&A, 595, A1, DOI: https://doi.org/10.1051/0004-6361/201629272 (2016).
15. K. Tassis *et al.*, PASIPHAE: A high-Galactic-latitude, high-accuracy optopolarimetric survey , arXiv:1810.05652v1 [astro-

ph.IM] (12 Oct 2018).

Appendix: List of the Record Numbers in the Heiles 2000 Catalog for the Stars in the Sample

In[234]:=

```
Print["There are ", nSrc, " stars in the sample.
   Their record numbers in the Heiles 2000 catalog are listed here."]
Print["The catalog listing for the last star in the sample, star number ",
 clumpOfStarsIDinCatalog[[-1]] , ", : ", cat[[ clumpOfStarsIDinCatalog[[-1]] ]], "."]
clumpOfStarsIDinCatalog
```

```
There are 893
  stars in the sample. Their record numbers in the Heiles 2000 catalog are listed here.
The catalog listing for the last star in the sample, star number 9147, :
 {73.1226, 0.321706, 1467., 72.0017, -999.9, -999.9, 0.82, 0.05, 102.7, 1.7, 110.,
  120.568, 10.3986, -99.9, -999.9, 1, -999.9, 240., , 1000000000000000000, 100, 9147}.
```

Out[236]=

```
{7802, 7878, 7881, 7892, 7905, 7912, 7920, 7929, 7944, 7950, 7952, 7956, 7959, 7963, 7964, 7968,
 7972, 7981, 7983, 7990, 7991, 7992, 7995, 8003, 8006, 8009, 8012, 8014, 8015, 8020, 8021,
 8023, 8024, 8027, 8031, 8032, 8033, 8036, 8037, 8043, 8045, 8047, 8048, 8049, 8050, 8053,
 8054, 8055, 8056, 8058, 8060, 8061, 8062, 8063, 8064, 8065, 8066, 8068, 8069, 8070, 8071,
 8072, 8073, 8077, 8079, 8081, 8082, 8083, 8084, 8085, 8087, 8090, 8091, 8093, 8094, 8097,
 8099, 8102, 8103, 8107, 8108, 8110, 8111, 8112, 8113, 8114, 8116, 8117, 8118, 8120, 8122,
 8127, 8128, 8129, 8134, 8135, 8136, 8137, 8139, 8140, 8141, 8143, 8145, 8146, 8147, 8148,
 8149, 8151, 8152, 8153, 8154, 8155, 8156, 8157, 8158, 8159, 8161, 8163, 8164, 8165,
 8166, 8167, 8171, 8172, 8173, 8174, 8176, 8178, 8179, 8180, 8182, 8183, 8184, 8186,
 8187, 8188, 8189, 8190, 8191, 8192, 8193, 8194, 8197, 8200, 8201, 8202, 8203, 8205,
 8206, 8208, 8210, 8211, 8213, 8214, 8216, 8217, 8218, 8219, 8220, 8221, 8222, 8223, 8224,
 8228, 8229, 8230, 8231, 8232, 8233, 8234, 8235, 8236, 8239, 8240, 8241, 8243, 8244,
 8246, 8247, 8248, 8249, 8250, 8251, 8252, 8253, 8254, 8255, 8256, 8257, 8258, 8260,
 8261, 8262, 8263, 8264, 8265, 8266, 8267, 8268, 8269, 8270, 8271, 8273, 8274, 8275,
 8276, 8277, 8278, 8279, 8280, 8283, 8285, 8286, 8287, 8288, 8289, 8290, 8291, 8292, 8293,
 8294, 8296, 8297, 8298, 8299, 8300, 8301, 8304, 8307, 8308, 8309, 8310, 8311, 8312,
 8313, 8315, 8316, 8317, 8319, 8320, 8321, 8322, 8323, 8324, 8326, 8327, 8328, 8329,
 8330, 8332, 8333, 8334, 8335, 8336, 8337, 8338, 8339, 8340, 8341, 8342, 8343, 8344,
 8345, 8346, 8347, 8348, 8349, 8350, 8351, 8352, 8354, 8355, 8356, 8357, 8358, 8359, 8360,
 8361, 8362, 8363, 8364, 8365, 8366, 8367, 8368, 8370, 8371, 8373, 8374, 8375, 8376,
 8377, 8378, 8380, 8381, 8382, 8383, 8384, 8385, 8386, 8387, 8388, 8389, 8390, 8391,
 8392, 8393, 8394, 8395, 8396, 8399, 8400, 8401, 8402, 8403, 8404, 8405, 8406, 8407,
 8408, 8410, 8411, 8413, 8415, 8416, 8417, 8419, 8420, 8421, 8422, 8424, 8425, 8426, 8427,
 8429, 8430, 8431, 8432, 8433, 8434, 8435, 8436, 8437, 8438, 8439, 8440, 8441, 8443,
 8444, 8445, 8446, 8447, 8448, 8451, 8452, 8453, 8454, 8455, 8456, 8457, 8458, 8459,
 8461, 8462, 8463, 8465, 8466, 8468, 8469, 8470, 8471, 8473, 8475, 8476, 8477, 8478,
 8479, 8480, 8481, 8482, 8483, 8484, 8488, 8489, 8490, 8491, 8492, 8493, 8494, 8495, 8496,
 8497, 8498, 8501, 8502, 8503, 8505, 8506, 8508, 8509, 8510, 8511, 8512, 8513, 8514,
 8515, 8517, 8518, 8519, 8520, 8521, 8522, 8523, 8524, 8525, 8526, 8527, 8528, 8529,
 8530, 8531, 8532, 8533, 8534, 8535, 8536, 8537, 8538, 8539, 8540, 8541, 8542, 8543,
 8545, 8546, 8547, 8548, 8549, 8550, 8551, 8552, 8553, 8554, 8555, 8556, 8557, 8558,
 8561, 8562, 8563, 8564, 8565, 8566, 8567, 8568, 8569, 8570, 8572, 8573, 8575, 8576, 8578,
 8579, 8580, 8581, 8584, 8585, 8586, 8587, 8588, 8590, 8591, 8592, 8593, 8594, 8597,
 8598, 8599, 8601, 8604, 8606, 8607, 8609, 8611, 8612, 8614, 8616, 8617, 8618, 8619, 8620,
```

8625, 8626, 8627, 8629, 8630, 8631, 8632, 8634, 8640, 8641, 8642, 8643, 8645, 8646, 8647,
8648, 8649, 8650, 8652, 8653, 8654, 8656, 8657, 8659, 8660, 8661, 8663, 8665, 8666, 8667,
8668, 8669, 8671, 8672, 8673, 8674, 8675, 8676, 8677, 8678, 8679, 8680, 8682, 8683, 8684,
8685, 8686, 8687, 8690, 8691, 8692, 8693, 8694, 8695, 8696, 8697, 8698, 8700, 8701, 8702,
8703, 8704, 8705, 8706, 8707, 8709, 8710, 8711, 8712, 8713, 8714, 8716, 8717, 8718, 8719,
8720, 8721, 8722, 8723, 8725, 8726, 8727, 8729, 8730, 8731, 8732, 8733, 8734, 8735, 8736,
8737, 8738, 8739, 8740, 8741, 8742, 8743, 8744, 8746, 8747, 8748, 8749, 8750, 8751, 8752,
8753, 8754, 8755, 8756, 8757, 8758, 8759, 8760, 8761, 8762, 8763, 8764, 8765, 8766, 8767,
8768, 8769, 8770, 8771, 8772, 8773, 8774, 8775, 8776, 8777, 8778, 8779, 8782, 8783, 8784,
8785, 8786, 8787, 8789, 8791, 8792, 8793, 8794, 8795, 8797, 8799, 8800, 8801, 8803, 8805,
8806, 8807, 8808, 8809, 8810, 8811, 8814, 8815, 8816, 8818, 8819, 8820, 8821, 8822,
8823, 8824, 8825, 8826, 8827, 8828, 8830, 8831, 8832, 8833, 8834, 8835, 8836, 8837,
8839, 8840, 8841, 8842, 8843, 8844, 8845, 8846, 8847, 8848, 8849, 8850, 8852, 8853,
8854, 8855, 8856, 8857, 8858, 8860, 8861, 8862, 8863, 8864, 8865, 8866, 8867, 8868,
8869, 8870, 8871, 8872, 8873, 8874, 8875, 8876, 8877, 8878, 8879, 8880, 8881, 8882,
8883, 8884, 8885, 8886, 8887, 8888, 8889, 8890, 8891, 8893, 8894, 8895, 8896, 8897,
8898, 8899, 8900, 8901, 8902, 8903, 8904, 8905, 8906, 8907, 8908, 8909, 8910, 8911,
8913, 8914, 8915, 8917, 8918, 8919, 8920, 8921, 8922, 8923, 8924, 8925, 8926, 8928,
8929, 8930, 8931, 8932, 8933, 8934, 8935, 8936, 8938, 8940, 8942, 8943, 8944, 8945,
8946, 8947, 8948, 8949, 8950, 8951, 8952, 8953, 8954, 8955, 8956, 8957, 8958, 8959,
8960, 8961, 8962, 8963, 8964, 8965, 8966, 8967, 8969, 8970, 8971, 8972, 8973, 8974,
8975, 8976, 8977, 8979, 8980, 8981, 8982, 8983, 8984, 8985, 8986, 8987, 8988, 8989,
8990, 8991, 8993, 8994, 8995, 8996, 8997, 8998, 8999, 9000, 9001, 9003, 9006, 9007,
9008, 9009, 9010, 9011, 9012, 9013, 9015, 9016, 9017, 9018, 9019, 9020, 9022, 9023,
9025, 9026, 9027, 9028, 9029, 9030, 9031, 9032, 9033, 9034, 9035, 9036, 9038, 9039,
9041, 9042, 9043, 9044, 9045, 9047, 9048, 9049, 9050, 9051, 9053, 9054, 9055, 9057,
9058, 9060, 9061, 9062, 9063, 9064, 9065, 9066, 9067, 9068, 9074, 9080, 9083, 9086,
9090, 9094, 9100, 9104, 9113, 9122, 9124, 9127, 9131, 9132, 9135, 9142, 9146, 9147}