The Collatz Conjecture and the Quantum Mechanical Harmonic Oscillator

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Abstract

By establishing a dictionary between the QM harmonic oscillator and the Collatz process, it reveals very important clues as to why the Collatz conjecture most likely is true. The dictionary requires expanding any integer n into a binary basis (bits) $n = \sum a_{nl} 2^l$ (l ranges from 0 to $N-1$) that allows to find the correspondence between every integer n and the state $|\Psi_n\rangle$, obtained by a superposition of bit states $|l\rangle$, and which are related to the energy eigenstates of the QM harmonic oscillator. In doing so, one can then construct the one-to-one correspondence between the Collatz iterations of numbers $n \to \frac{n}{2}$ (n even); $n \to 3n+1$ (n odd) and the operators $\mathbf{L}_{\frac{n}{2}}; \mathbf{L}_{3n+1}$, which map Ψ_n to $\Psi_{\frac{n}{2}}$, or to Ψ_{3n+1}, respectively, and which are constructed explicitly in terms of the creation $\mathbf{a}^\dagger$, annihilation $\mathbf{a}$, and unit operator $\mathbf{1}$ of the QM harmonic oscillator. A rigorous analysis reveals that the Collatz conjecture is most likely true, if the composition of a chain of $\mathbf{L}_{\frac{n}{2}}; \mathbf{L}_{3n+1}$ operators (written as L_* in condensed notation) leads to the null-eigenfunction conditions $(\mathbf{L}_*\mathbf{L}_* \ldots \mathbf{L}_* - \mathcal{P})\Psi_n = 0$, where $\mathcal{P}$ is the operator that *projects* any state Ψ_n into the ground state $\Psi_1 \equiv |0\rangle$ representing the zero bit state $|0\rangle$ (since $2^0 = 1$). In essence, one has a realization of the integer/state correspondence typical of QM such that the Collatz paths from n to 1 are encoded in terms of quantum transitions among the states Ψ_n, and leading effectively to an overall downward cascade to Ψ_1. The QM oscillator approach explains naturally why the Collatz conjecture fails for negative integers because there are no states below the ground state.

1 Introduction

In the Wikipedia page it states that the Collatz conjecture is named after Lothar Collatz, who introduced the idea in 1937, two years after receiving his doctorate. [1]. It is also known as the $3n+1$ problem [2], the $3n+1$ conjecture, the Ulam conjecture (after Stanislaw Ulam), Kakutani's problem (after Shizuo Kakutani), the Thwaites conjecture (after Sir Bryan Thwaites), Hasse's algorithm (after Helmut Hasse), or the Syracuse problem. The sequence of numbers involved is sometimes referred to as the hailstone sequence or hailstone numbers (because the values are usually subject to multiple descents and ascents like hailstones in a cloud.

As of 2020, the conjecture has been checked by computer for all starting values up to $2^{68} \simeq 2.95 \times 10^{20}$ [3]. One may add that the computational results in [4] can also be considered to be another large sample test (around 160000 elements) of the convergence of the Collatz algorithm and obtained within a large interval of natural numbers given by $[3^2 \pm 1,\ 4^{8192} \pm 1]$. A vast number of articles have been devoted to proving and disproving the conjecture ever since. For more recent ones see [8].

The conjecture is based on a very simple iteration process. For any positive even integer greater than 1, the rule is $n \rightarrow \frac{n}{2}$. And for n odd, one takes $n \rightarrow 3n+1$, leading to an even number. The Collatz conjecture states that starting from any integer $n > 1$, the Collatz iteration process **always** ends up at 1, after a finite number of steps. This is not the case for negative integers since there are nontrivial cycles. For instance $\{-5, -14, -7, -20, -10, -5, \ldots\}$ and one does not reach -1 . The Collatz process has the trivial cycle $\{1, 4, 2, 1, 4, 2, \ldots\}$ if one continues iterating after reaching 1. A computer program would never end in this case, so once we reach the final destination at 1, the process should stop.

Inspired by the work of one of us (RCD) [4], [5] on Boolean Hypercubes, Natural Vector Spaces and the Collatz conjecture, we shall show next how one can establish a dictionary between operators associated with the QM harmonic oscillator and the Collatz process which reveals very important clues as to why the Collatz conjecture most likely is true.

More precisely, we show that it is the very special decomposition of $3n+1 = n+2n+1$ which leads to the operator representation for the process $\Psi_n \rightarrow \Psi_{3n+1}$ to have the key **diagonal** form $(\mathbf{I} + \mathcal{L}^+ + \mathcal{P})_{n=odd}$, where $\mathbf{I}$ is the identity operator, $\mathcal{L}^+$ is a ladder operator that increases the bit number by one, and $\mathcal{P}_{n=odd}$ is a projection operator that maps any state Ψ_n into the ground state $\Psi_1 \equiv |0\rangle$ representing the zero bit state $|0\rangle$ (since $2^0 = 1$).

A rigorous analysis reveals that the Collatz conjecture is most likely true, if the composition of a chain of $\mathbf{L}_{\frac{n}{2}}; \mathbf{L}_{3n+1}$ operators (written as L_* in condensed notation) leads to the null-eigenfunction conditions $(\mathbf{L}_*\mathbf{L}_* \ldots \mathbf{L}_* - \mathcal{P})\Psi_n =$

0, and excludes the existence of nontrivial cycles described by the condition $(\mathbf{L}_* \mathbf{L}_* \ldots \mathbf{L}_* - \mathbf{I})\Psi_n = 0$.

A polynomial approach to the Collatz conjecture has been studied by [7]. By using polynomials based on a binary numeral system the authors [7] show that the degree of the polynomials, on average, decreases after a finite number of steps of the Collatz operation, which provides a weak proof of the conjecture by using induction with respect to the degree of the polynomials. It is warranted to explore this polynomial approach with our present work based on the QM oscillator and the Hermite polynomials.

2 The QM oscillator and the Collatz Conjecture

2.1 Dictionary between the QM oscillator and the Collatz process

The following dictionary between the QM oscillator and the Collatz process reveals very important clues as to why the Collatz conjecture might be true. Given an arbitrary number n it admits the binary expansion $\sum_{l=0}^{N-1} a_{nl} 2^l$, and whose binary coefficients are $a_{nl} = \{0, 1\}$. The one-to-one correspondence between the number n and the state Ψ_n is

$$n \ = \ \sum_{l=0}^{N-1} \ a_{nl} \ 2^l \ \leftrightarrow \ |\Psi_n > \ \equiv \ \sum_{l=0}^{N-1} \ a_{nl} \ |l> \tag{1}$$

where the superposition of bit states $|l>$, which are associated with the numbers 2^l, ranges from $l = 0$ to $N - 1$. The state $|\Psi_n>$ is a *superposition* of bit states $|l>$ and is a close relative of the coherent state $|z> \equiv e^{-\frac{|z|^2}{2}} \sum e^{z a^\dagger} |0>$ with z a complex number. The state $|\Psi_n>$ can be represented by a *column* vector whose entries are $\{a_{n0}|0>, a_{n1}|1>, a_{n2}|2>, \ldots, a_{nN-1}|N-1>\}$.

When n is even, the first Collatz iteration yields $\frac{n}{2}$, therefore one has a transition from the state $|\Psi_n>$ to the state $|\Psi_{\frac{n}{2}}>$ given by

$$|\Psi_{\frac{n}{2}} > \ \equiv \ \sum_{l=0}^{N-1} \ a_{nl} \ |l-1> \ = \ \sum_{l=1}^{N-1} \ a_{nl} \ |l-1> \tag{2}$$

since dividing by two amounts to removing one bit $\frac{1}{2} 2^l = 2^{l-1}$, and the first binary coefficient $a_{n0} = 0$ is always *zero* when n is even.

When n is odd, the first Collatz iteration yields $3n+1$. The binary expansion of $3n + 1$ (an even number) is

$$3n + 1 \ = \ 1 \ + \ \sum_{l=0}^{N-1} \ a_{nl} \ 2^l \ + \ \sum_{l=0}^{N-1} \ a_{nl} \ 2^{l+1} \tag{3}$$

therefore one has a transition from the state $|\Psi_n>$ to the state $|\Psi_{3n+1}>$ given by

$$|\Psi_{3n+1}> \ \equiv \ |0> \ + \ \sum_{l=0}^{N-1} a_{nl} \ |l> \ + \ \sum_{l=0}^{N-1} a_{nl} \ |l+1> \tag{4}$$

since multiplying by two amounts to adding one bit $2 \ 2^l = 2^{l+1}$.

Given the binary expansion of any even number n in eq-(1), the first binary coefficient of an even number n is always $a_{n0} = 0$ zero, and in general one will have a set of non-zero binary coefficients at the specific locations $l_1, l_2, l_3, \ldots$, Namely $a_{nl_1} = a_{nl_2} = a_{nl_3} = \ldots = 1$ and the rest of the binary coefficients are zero. One may have all the binary coefficients to be non-vanishing except the first one $a_{n0} = 0$. Or one may have all the binary coefficients to be vanishing except the last one $a_{nN-1} = 1$. And so forth. Using the well known relations involving the action of the creation and annihilation operators on the energy eigenstates $|l>$ $(l = 0, 1, 2, ...)$ of a harmonic oscillator $\mathbf{a}^\dagger |l> = \sqrt{l+1} \ |l+1>$ and $\mathbf{a}|l> = \sqrt{l} \ |l-1>$, one learns that the operator $\mathbf{L}_{\frac{n}{2}}$ which maps the state $|\Psi_n>$ to $|\Psi_{\frac{n}{2}}>$ (when n is even) is given by

$$\mathbf{L}_{\frac{n}{2}} \ = \ Diag \left(\ \mathbf{1}, \ \mathbf{1}, \ldots, \frac{\mathbf{a}}{\sqrt{l_1}}, \ \mathbf{1}, \ \mathbf{1}, \ldots, \ \frac{\mathbf{a}}{\sqrt{l_2}}, \mathbf{1}, \ \mathbf{1}, \ldots, \ \frac{\mathbf{a}}{\sqrt{l_3}}, \ \ldots \ \right) \tag{5}$$

which is a diagonal $N \times N$ matrix and whose entries are comprised of the annihilation operator $\mathbf{a}$ (up to judicious numerical coefficients) and the identity operator $\mathbf{1}$. The location of the unit operators $\mathbf{1}$ in eq-(5) correspond to the location of the *vanishing* binary coefficients. And the $\mathbf{a}$ operators will *reduce* the bits by one unit at each specific location where the binary coefficients are non-vanishing. For this reason we may rewrite $\mathbf{L}_{\frac{n}{2}} = \mathcal{L}^-$, where $\mathcal{L}^-$ plays the role of a ladder operator that reduces the bits by one unit; i.e. it reduces the size of the column vector by one.

The operator $\mathbf{L}_{3n+1}$ that maps $|\Psi_n>$ to $|\Psi_{3n+1}>$ (when n is odd) is given by

$$\mathbf{L}_{3n+1} \ = \ Diag \, (\ \mathbf{1}, \ \mathbf{1}, \ \mathbf{1}, \ldots, \mathbf{1}) \ + \\ Diag \left(\mathbf{a}^\dagger, \ \mathbf{1}, \ \mathbf{1}, \ \ldots, \ \frac{\mathbf{a}^\dagger}{\sqrt{l_1+1}}, \ \mathbf{1}, \ \mathbf{1}, \ \ldots, \frac{\mathbf{a}^\dagger}{\sqrt{l_2+1}}, \ \mathbf{1}, \ \mathbf{1}, \ \ldots, \ \frac{\mathbf{a}^\dagger}{\sqrt{l_3+1}}, \ \ldots \ \right) + \\ Diag \, (\ \mathbf{1}, \ \mathbf{1}, \ \mathbf{1}, \ \ldots \mathbf{a}^{l_1+1}, \ \mathbf{1}, \ \mathbf{1}, \ \ldots \ \mathbf{a}^{l_2+1}, \mathbf{1}, \ \mathbf{1}, \ \ldots \ \mathbf{a}^{l_3+1}, \ \ldots \ ) \tag{6}$$

which is a diagonal $N \times N$ matrix whose entries are comprised of creation $\mathbf{a}^\dagger$ and annihilation operators $\mathbf{a}$, in addition to the identity operators $\mathbf{1}$. We may rewrite $\mathbf{L}_{3n+1}$ in condensed notation as $\mathbf{I} + \mathcal{L}^+ + \mathcal{P}_{odd}$, respectively. $\mathbf{I}$ is the identity operator comprised of ones along the diagonal. $\mathcal{L}^+$ is the ladder operator that increases the bits by one unit (increases the size of the column vector by one). And $\mathcal{P}_{odd}$ is the **projection operator** that maps the state Ψ_n (for n odd) to the ground state $\mathcal{P}_{odd}\Psi_n = \Psi_1 = |0>$.

Let us explain in detail the origins of the diagonal entries of $\mathcal{P}_{odd}$. For n odd, the binary coefficient $a_{n0} = 1$ is not zero, so the first unit $\mathbf{1}$ operator appearing in $\mathcal{P}_{odd}$ will act on the ground state $|0>$ giving $|0>$, as desired. The following $\mathbf{1}$'s in $\mathcal{P}_{odd}$ appear at the locations of the vanishing binary coefficients, until one hits the first non-vanishing binary coefficient at the l_1-th entry. The next non-vanishing locations are situated at the l_2-th, l_3-th, ... entries, respectively. Whereas, the location of the remaining $\mathbf{1}$'s in between correspond to the respective vanishing binary coefficients. Finally, the action of the operators $\mathbf{a}^{l_1+1}; \mathbf{a}^{l_2+1}; \mathbf{a}^{l_3+1}, \ldots$ on the bit states $|l_1\rangle; |l_2\rangle; |l_3\rangle; \ldots$, respectively, is going to be zero. Therefore, the overall effect of the projection operator $\mathcal{P}_{odd}$ on Ψ_{odd} gives $|\Psi_1\rangle = |0\rangle$, as expected, due to the fact the $a_{n0} = 1 \neq 0$.

Therefore, one has in condensed notation

$$\mathbf{L}_{\frac{n}{2}} \equiv \mathcal{L}^-, \quad \mathbf{L}_{3n+1} \equiv \mathbf{I} + \mathcal{L}^+ + \mathcal{P}_{odd} \tag{7}$$

For n even, the **projection operator** that maps the state Ψ_n to the ground state $\mathcal{P}_{even}\Psi_n = \Psi_1 = |0>$ is

$$\mathcal{P}_{even} \equiv Diag\left(\mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \ldots \frac{\mathbf{a}^{l_1}}{\sqrt{l_1!}},\ \mathbf{1},\ \mathbf{1},\ \ldots\ \mathbf{a}^{l_2+1}, \mathbf{1},\ \mathbf{1},\ \ldots\ \mathbf{a}^{l_3+1},\ \ldots \right) \tag{8a}$$

The action of $\frac{\mathbf{a}^{l_1}}{\sqrt{l_1!}}\ |l_1\rangle = |0\rangle$ is what projects Ψ_n (n even) into the ground state; while the action of $\mathbf{a}^{l_2+1}|l_2\rangle = 0$; $\mathbf{a}^{l_3+1}|l_3\rangle = 0, \ldots$ is what projects out the other bit states. The action of the unit operators $\mathbf{1}$ in eq-(8a) is **null** because the location of the unit operators corresponds to the location of the vanishing binary coefficients.

Some important remarks are in order. Strictly speaking, the operators $\mathcal{P}_n$ that map Ψ_n to Ψ_1 are not projectors in the usual sense, since projection operators Π_i satisfy $\Pi_i\Pi_j = 0$ when $i \neq j$; $\mathbf{I} = \sum \Pi_i$, and $(\Pi_i)^2 = \Pi_i$ for all i. Since the operators $\mathcal{P}_n$ "collapse" any state down to Ψ_1, they should be coined "collapsors" (or "Collatzors" if one wishes to play with words).

Secondly, the expression for the projection operator $\mathcal{P}_{even}$ in eq-(8a) is **not** unique. One could have chosen instead of eq-(8a) the following projection operator

$$\mathcal{P}'_{even} \equiv Diag\left(\mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \ldots \mathbf{a}^{l_1+1},\ \mathbf{1},\ \mathbf{1},\ \ldots\ \frac{\mathbf{a}^{l_2}}{\sqrt{l_2!}}, \mathbf{1},\ \mathbf{1},\ \ldots\ \mathbf{a}^{l_3+1},\ \ldots \right) \tag{8b}$$

that maps the state $|\Psi_n\rangle$ to the ground state $\mathcal{P}'_{even}|\Psi_n\rangle = |\Psi_1\rangle = |0\rangle$. And one could continue to find yet another projection operator

$$\mathcal{P}''_{even} \equiv Diag\left(\mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \ldots \mathbf{a}^{l_1+1},\ \mathbf{1},\ \mathbf{1},\ \ldots\ \mathbf{a}^{l_2+1}, \mathbf{1},\ \mathbf{1},\ \ldots\ \frac{\mathbf{a}^{l_3}}{\sqrt{l_3!}},\ \ldots \right) \tag{8c}$$

and so forth. For this reason one must choose a selection criteria which avoids any ambiguities in the expression for the projections $\mathcal{P}_{even}$ in the even n case.

We shall choose the expression provided by eq-(8a) which is based on the location of the **first** non-vanishing binary coefficient. In this way, the expression for the $\mathcal{P}_{odd}$ projection operator in the odd n case is the one displayed by the last term in eq-(6). The reason being that for n odd, the first non-vanishing binary coefficient is $a_{n0} = 1$.

To sum up, the following operators

$$\mathbf{I}, \ \mathcal{P}_{even}, \ \mathcal{P}_{odd}, \ \mathcal{L}^+, \ \mathcal{L}^-, \ \mathbf{L}_{3n+1} \ = \ \mathbf{I} + \mathcal{L}^+ + \mathcal{P}_{odd}, \ \mathbf{L}_{\frac{n}{2}} \ = \ \mathcal{L}^- \tag{9}$$

are the ones involved in the whole Collatz process. Except for the identity operator, all the other operators are state-dependent. Note that despite that these operators are state dependent they are still **linear** involving powers of $\mathbf{a}, \mathbf{a}^\dagger$, because the latter powers are realized as higher order linear differential operators acting on the states Ψ_n, and which in turn, are given by a superposition of energy eigenstates of the harmonic oscillator.

The full Collatz symbolic process $C_{s(n)}[n] = 1$ of an integer n, where $s(n)$ is the total number of steps required for n to reach 1, is represented (reformulated) as a concatenation of $\mathbf{L}_{(1)}, \mathbf{L}_{(2)}, \ldots \mathbf{L}_{(s(n))}$ operators acting on Ψ_n, and whose net final effect is to end at the ground state. For n even, one has

$$\mathbf{L}_{(s(n))} \ \mathbf{L}_{(s(n)-1)} \ \ldots \ \mathbf{L}_{(2)} \ \mathbf{L}_{(1)} \ |\Psi_n\rangle \ = \ \mathcal{P}_{even} \ |\Psi_n\rangle \ = \ \Psi_1 \ \equiv |0\rangle \tag{10a}$$

And for n odd

$$\mathbf{L}_{(s(n))} \ \mathbf{L}_{(s(n)-1)} \ \ldots \ \mathbf{L}_{(2)} \ \mathbf{L}_{(1)} \ |\Psi_n\rangle \ = \ \mathcal{P}_{odd} \ |\Psi_n\rangle \ = \ \Psi_1 \ \equiv |0\rangle \tag{10b}$$

The column vector representing Ψ_1 is comprised of a single entry $|0\rangle$.

Each step in the concatenation process will select also **state dependent** operators written as

$$\mathbf{L}_{(1)}, \ \mathbf{L}_{(2)}, \ \ldots, \mathbf{L}_{(s(n))} \tag{11}$$

for simplicity due to the complexity of the Collatz iteration process. For example, given a positive integer n, the iteration process generates the sequence of numbers (a Collatz path)

$$n \ \rightarrow \ \{n \equiv n_0, \ n_1, \ n_2, \ n_3, \ \ldots, 8, \ 4, \ 2, \ 1\} \tag{12a}$$

and the corresponding eigenstates are

$$\{\Psi_n, \ \Psi_{n_1}, \ \Psi_{n_2}, \ \Psi_{n_3}, \ \ldots, \Psi_8, \ \Psi_4, \ \Psi_2, \ \Psi_1 = |0\rangle\} \tag{12b}$$

Therefore when one writes $L_{(i)}$ it represents the operator acting on the state $\Psi_{n_i} \neq \Psi_i$, because $n_i \neq i$. Extreme caution must be taken with the notation and the indices.

The key point now is to show how one performs the sequence of operations involving the string of operators in eqs-(10a, 10b). One must, firstly, start with the operation

$$\mathbf{L}_{(1)}\ |\Psi_n\rangle \ = \ \mathbf{L}[\Psi_n]\ |\Psi_n\rangle \ = \ |\Psi_{n_1}\rangle \tag{12c}$$

Then continue

$$\mathbf{L}_{(2)}\ |\Psi_{n_1}\rangle \ = \ \mathbf{L}[\Psi_{n_1}]\ |\Psi_{n_1}\rangle \ = \ |\Psi_{n_2}\rangle \tag{12d}$$

and proceed

$$\mathbf{L}_{(3)}\ |\Psi_{n_2}\rangle \ = \ \mathbf{L}[\Psi_{n_2}]\ |\Psi_{n_2}\rangle \ = \ |\Psi_{n_3}\rangle \tag{12d}$$

and so forth until reaching $|\Psi_1\rangle = |0\rangle$. As stated above, the string of linear operators are state-dependent and this explains the above notation $\mathbf{L}[\Psi_n]; \mathbf{L}[\Psi_{n_1}]; \mathbf{L}[\Psi_{n_2}]; \ldots$. These operators are represented by operator-valued entries of diagonal matrices of **different** sizes because the binary representation of the sequence of integers (12a) stops at **different** powers of two : $2^{N-1}, 2^{N_1-1}, 2^{N_2-1}, 2^{N_3-1}, \ldots$ and corresponding to the binary expansion of $n, n_1, n_2, n_3, \ldots$. $N, N_1, N_2, N_3, \ldots$ are the dimensions of the corresponding Boolean hypercubes.

For this reason we have to explain how to **compose** a string of operators associated with diagonal matrices of **different** sizes. In the Appendix we provide some examples of how to perform such composition of operators by **adding** a judicious string of unit operators $\mathbf{1}$'s to the left, and/or to the right, of the diagonal entries such that **all** the operators are now represented by diagonal matrices of the same size. The common size of all the diagonal matrices is provided by the value of the maximum dimension N_{max} of the Boolean hypercube associated with the binary expansion of the maximum number $n_{max} = \sum_{l=0}^{N_{max}-1}\ a_{n_{max} l}\ 2^l$ attained in the Collatz chain (path, sequence) (12a). By the same token, the projection operators $\mathcal{P}_n$ in the right-hand side of eqs-(10a, 10b) must also be represented by diagonal matrices of the same size N_{max} to match the size of the matrices in the left hand side. It is also necessary to **embed** all the column vectors into a column vector of maximal size N_{max} as well by adding extra zeros.

After doing so, one may write eqs-(10a,10b) in the following **symbolic** form which reflect the state-dependence of the operators,

$$\mathbf{L}[\Psi_{n_{s(n)}}]\ \mathbf{L}[\Psi_{n_{s(n)-1}}]\ \ldots\ \mathbf{L}[\Psi_{n_3}]\ \mathbf{L}[\Psi_{n_2}]\ \mathbf{L}[\Psi_{n_1}]\ |\Psi_n\rangle \ = \ \mathcal{P}_n\ |\Psi_n\rangle \ = \ \Psi_1\ \equiv |0\rangle \tag{13}$$

and where $\mathcal{P}$ is either the projection operator for the $n =$ even, or the $n =$ odd number case. From eq-(13) one infers the relations for all values of n

$$\Big(\ \mathbf{L}[\Psi_{n_{s(n)}}]\ \mathbf{L}[\Psi_{n_{s(n)-1}}]\ \ldots\ \mathbf{L}[\Psi_{n_3}]\ \mathbf{L}[\Psi_{n_2}]\ \mathbf{L}[\Psi_{n_1}]\ -\ \mathcal{P}_n\ \Big)\ \Psi_n\ =\ 0 \tag{14}$$

The main result of this work is that if, and only if, the Collatz conjecture is true the concatenation process of the operators must obey the relations (14) for all values of n even, odd, respectively. To verify these relations for *all* values of n is **extremely** difficult. By **construction**, the projection operator acting on Ψ_n **always** obeys $\mathcal{P}[\Psi_n]\Psi_n = \Psi_1 = |0\rangle$, for all values of n. The reason we rewrite (13) in the form (14) will be explained below.

2.2 Operator Identities

On the other hand, if the Collatz conjecture is true, this is equivalent to stating $(\mathbf{L}_*\mathbf{L}_* \ldots \mathbf{L}_* - \mathcal{P})\Psi_n = 0$ as mentioned earlier in eq-(14). In the first case, this leads to the following **operator** relations (identities) $\mathbf{L}_*\mathbf{L}_* \ldots \mathbf{L}_* - \mathcal{P} = 0$ which are very restrictive. With the definitions of the projection operators appearing in eqs-(6,8a) one finds (see Appendix) that only in very special cases these **operator** relations (identities) are satisfied. However, one can find many other **different** expressions for the projection operators and having the most general form

$$\mathcal{P}_{n=odd} \ = \ Diag\left(\ P_{n0}(\mathbf{a}, \mathbf{a}^\dagger), \ P_{n1}(\mathbf{a}, \mathbf{a}^\dagger), \ P_{n2}(\mathbf{a}, \mathbf{a}^\dagger), \ \ldots, \ P_{nN-1}(\mathbf{a}, \mathbf{a}^\dagger) \ \right) \tag{15}$$

$$\mathcal{P}_{n=even} \ = \ Diag\left(\ \tilde{P}_{n0}(\mathbf{a}, \mathbf{a}^\dagger), \ \tilde{P}_{n1}(\mathbf{a}, \mathbf{a}^\dagger), \ \tilde{P}_{n2}(\mathbf{a}, \mathbf{a}^\dagger), \ \ldots, \ \tilde{P}_{nN-1}(\mathbf{a}, \mathbf{a}^\dagger) \ \right) \tag{16}$$

The diagonal entries of the most general projection operators are comprised of **polynomial** functions of the $\mathbf{a}, \mathbf{a}^\dagger$ operators. These projection operators must satisfy the key conditions

$$\mathcal{P}_{n=odd}|\Psi_{n=odd}\rangle \ = \ |\Psi_1\rangle \ = \ |0\rangle, \ \ \mathcal{P}_{n=even}|\Psi_{n=even}\rangle \ = \ |\Psi_1\rangle \ = \ |0\rangle \tag{17}$$

such that every positive finite integer n has a one-to-one correspondence with the state Ψ_n obeying the conditions (17) associated with the full Collatz cascading process downwards to the ground state.

Despite that $\mathbf{a}, \mathbf{a}^\dagger$ do not commute one can still reorder the monomials $(\mathbf{a}^{p_1})(\mathbf{a}^\dagger)^{p_2})$ such that the annihilation operators appear to the right of the creation operators as it is customary in QFT. This reordering can be done in a pairwise step fashion by using the commutation relations $[\mathbf{a}, \mathbf{a}^\dagger] = \mathbf{1} \Rightarrow \mathbf{a}\mathbf{a}^\dagger = \mathbf{1} + \mathbf{a}^\dagger\mathbf{a} = \mathbf{1} + \mathbf{N}$, where the number operator is defined as $\mathbf{N} \equiv \mathbf{a}^\dagger\mathbf{a}$, such that $\mathbf{N}|l\rangle = l|l\rangle$. For example any power of $(\mathbf{a}\mathbf{a}^\dagger)^p$ can be rewritten as $(\mathbf{1} + \mathbf{N})^p$ leading to $(\mathbf{1} + \mathbf{N}) \ldots (\mathbf{1} + \mathbf{N})|l\rangle = (1 + l)^p|l\rangle$, due to number operator properties $\mathbf{N}|l\rangle = l|l\rangle$ for all values of l.

Given the key properties of the number operator, like $[\mathbf{N}, \mathbf{a}] = -\mathbf{a}, \ [\mathbf{N}, \mathbf{a}^\dagger] = \mathbf{a}^\dagger$, we shall propose the simplest ansatz by setting **all** of the polynomial entries $P_{nl}(\mathbf{a}, \mathbf{a}^\dagger); \tilde{P}_{nl}(\mathbf{a}, \mathbf{a}^\dagger)$ in eqs-(15,16) to be of the form $\{\frac{\mathbf{a}^l}{\sqrt{l!}}; A_{nl}(\mathbf{N})\mathbf{a}^{l+1}; B_{nl}(\mathbf{N})\}$ where A_{nl}, B_{nl} are polynomials in the number operator $\mathbf{N} = \mathbf{a}^\dagger\mathbf{a}$. In this fashion the direct connection to number theory is more transparent. One will have the following cases to consider in eq-(17) : $\frac{\mathbf{a}^l}{\sqrt{l!}}|l\rangle = |0\rangle$, which is the desired goal. For instance, when n is odd, the first non-vanishing binary coefficient corresponds to the value of $l = 0$, and one will have $\frac{\mathbf{a}^l}{\sqrt{l!}}|l\rangle = |0\rangle$ because $0! = 1$ and $\mathbf{a}^0 = \mathbf{1}$. The action $A_{nl}(\mathbf{N})\mathbf{a}^{l+1}|l\rangle = 0$ annihilates those states; while the action of the operators $B_{nl}(\mathbf{N})\}$ will be zero since they correspond to the locations of the vanishing binary coefficients in the bit-states expansion of $|\Psi_n\rangle$. In this way one can always ensure that the conditions of eq-(17) are satisfied.

Before proceeding, let us formally define the inverse of the projection operators $\mathcal{P}_n(\mathbf{a}, \mathbf{a}^\dagger)$ from the condition $\mathcal{P}_n^{-1}\mathcal{P}_n = \mathcal{P}_n\mathcal{P}_n^{-1} = \mathbf{I}$, for all n, where one performs the inverse operation in each one of the diagonal entries P_{nl} which comprise $\mathcal{P}_n$ by the usual Taylor series expansion, taking into account that a proper radius of convergence domain must be specified. For instance, the power series

$$N^{-1} \ = \ \frac{1}{N} \ = \ \frac{1}{1-(1-N)} \ = \ 1 \ + \ (1-N) \ + \ (1-N)^2 \ + \ (1-N)^3 \ \ldots \tag{18}$$

convergences when $|1 - N| < 1 \Rightarrow 0 < N < 2$. One can enlarge the radius of convergence by writing $N = M - (M - N)$ (with $M > 1$) so that the power series converges when $|1 - \frac{N}{M}| < 1 \Rightarrow 0 < \frac{N}{M} < 2 \Rightarrow 0 < N < 2M$. When N is replaced by a polynomial $P(\mathbf{a}, \mathbf{a}^\dagger)$ the inverse operator can be expanded in a similar Taylor series.

Let us now provide an specific example to see how one can implement the operator identities $\mathbf{L}_*\mathbf{L}_* \ldots \mathbf{L}_* = \mathcal{P}$ if, and only if, one recurs to the most **general** form of projection operators given by eqs-(15,16) and obeying (17). Upon writing the operators in symbolic notation $\mathbf{L}[\Psi_{odd}] = \mathbf{I} + \mathcal{L}^+ + \mathcal{P}_{odd}$, and $\mathbf{L}[\Psi_{even}] = \mathcal{L}^-$, the Collatz sequence $\{3, 10, 5, 16, 8, 4, 2, 1\}$ yields the following relation

$$\mathcal{L}_2^- \ \mathcal{L}_4^- \ \mathcal{L}_8^- \ \mathcal{L}_{16}^- \ (\ \mathbf{I} \ + \ \mathcal{L}_5^+ \ + \ \mathcal{P}_5 \ ) \ \mathcal{L}_{10}^- \ (\ \mathbf{I} \ + \ \mathcal{L}_3^+ \ + \ \mathcal{P}_3 \ ) \ = \ \mathcal{P}_3 \tag{19}$$

The subscripts in the operators in (19) denote on what specific states Ψ_{n_i} the operators are acting. From the Collatz chain process in eq-(19) one can then infer the following nested sequence of operations

$$\mathcal{L}_2^- \ = \ \mathcal{P}_2, \ \ \mathcal{L}_2^- \ \mathcal{L}_4^- \ = \ \mathcal{P}_4 \ \Rightarrow \ \mathcal{L}_4^- \ = \ \mathcal{P}_2^{-1} \ \mathcal{P}_4 \tag{20a}$$

$$\mathcal{L}_2^- \ \mathcal{L}_4^- \ \mathcal{L}_8^- \ = \ \mathcal{P}_8 \ \Rightarrow \ \mathcal{L}_8^- \ = \ \mathcal{P}_4^{-1} \ \mathcal{P}_8 \tag{20b}$$

$$\mathcal{L}_2^- \ \mathcal{L}_4^- \ \mathcal{L}_8^- \ \mathcal{L}_{16}^- \ = \ \mathcal{P}_{16} \ \Rightarrow \ \mathcal{L}_{16}^- \ = \ \mathcal{P}_8^{-1} \ \mathcal{P}_{16} \tag{20c}$$

$$\mathcal{P}_{16} \ (\mathbf{I} \ + \ \mathcal{L}_5^+ \ + \ \mathcal{P}_5) \ = \ \mathcal{P}_5 \ \Rightarrow \ (\mathbf{I} \ + \ \mathcal{L}_5^+ \ + \ \mathcal{P}_5) \ = \ \mathcal{P}_{16}^{-1} \ \mathcal{P}_5 \tag{20d}$$

$$\mathcal{P}_5 \ \mathcal{L}_{10}^- \ = \ \mathcal{P}_{10} \ \Rightarrow \ \mathcal{L}_{10}^- \ = \ \mathcal{P}_5^{-1} \ \mathcal{P}_{10} \tag{20e}$$

$$\mathcal{P}_{10} \ (\mathbf{I} \ + \ \mathcal{L}_3^+ \ + \ \mathcal{P}_3) \ = \ \mathcal{P}_3 \ \Rightarrow \ (\mathbf{I} \ + \ \mathcal{L}_3^+ \ + \ \mathcal{P}_3) \ = \ \mathcal{P}_{10}^{-1} \ \mathcal{P}_3 \tag{20f}$$

And finally, after assuming that the products of operators are **associative**, and by performing the "telescoping" product of all of the relevant terms in eqs-(20)

one arrives straightforwardly to the operator relation (identity) of eq-(19) due to pairwise cancellations of the form $\mathcal{P}\mathcal{P}^{-1} = \mathbf{I}$.

Note that despite that the ladder operators are **linear** in $\mathbf{a}, \mathbf{a}^\dagger$, eqs-(20) are not as simple as they seem because each diagonal projection operator $\mathcal{P}_n$, and the diagonal ladder operators $\mathcal{L}_n^\pm$, must be expanded into their "binary" components : $\mathcal{P}_n = \{P_{n0}, P_{n1}, \ldots, P_{nN-1}\}$, $\mathcal{L}_n^\pm = \{L_{n0}^\pm, L_{n1}^\pm, \ldots, L_{nN-1}^\pm\}$. Hence, eqs-(20) involve an intricate set of nested (loop) relations involving all of the **polynomials** in the $\mathbf{a}, \mathbf{a}^\dagger$ operators displayed explicitly by eqs-(15,16). This is the reason why the ansatz in setting **most** of the polynomials as functions of the number operator $\mathbf{N} = \mathbf{a}^\dagger\mathbf{a}$ will simplify matters considerably.

To conclude, we propose that a plausible proof of the Collatz conjecure could be attained based on this construction which relies on the binary decomposition of numbers, the QM oscillator algebra and an infinite associative Loop algebra comprised of the infinite number of $\{\mathbf{I}, \mathcal{L}^\pm, \mathcal{P}\}$ operators, corresponding to each value of n (even, odd). In a similar vein that an n-th order polynomial equation of the form $P_n(x) = 0$ always admits n roots $x_1, x_2, \ldots, x_n$ (real, complex), for all values of n, we are postulating here that if one can prove that operator-valued polynomial $P_{nl}(\mathbf{a}, \mathbf{a}^\dagger), \tilde{P}_{nl}(\mathbf{a}, \mathbf{a}^\dagger)$ solutions **always** exist to the operator identities, and satisfying the conditions in eq-(17), for all values of n, then the Collatz conjecture is true. Below we shall propose a "simpler" approach by just asking what are the conditions required to prove that there are no nontrivial cycles in the Collatz process, and whether or not solutions exist for these operator-valued polynomials.

2.3 Null Eigenfunctions

Another approach that can be explored if one finds the previous operator avenue too cumbersome, is to find a different set of projection operators that are more closely related to those displayed in eqs-(6,8a). And instead of imposing the previous operator identities one has the following Null Eigenfunctions conditions $(\mathbf{L}_*\mathbf{L}_* \ldots \mathbf{L}_* - \mathcal{P})\Psi_n = 0$, with zero eigenvalue, on each state Ψ_n. This is possible because the operators themselves are state dependent. The composition of the $\mathbf{L}$ operators leads to higher order linear differential equations of the functions Ψ_n by recalling the definitions of the operators

$$\mathbf{a} \ = \ \sqrt{\frac{m\omega}{2\hbar}} \ (\hat{x} \ + \frac{i}{m\omega} \ \hat{p}), \quad \mathbf{a}^\dagger \ = \ \sqrt{\frac{m\omega}{2\hbar}} \ (\hat{x} \ - \frac{i}{m\omega} \ \hat{p}) \tag{21}$$

in the QM harmonic oscillator, obeying $[\mathbf{a}, \mathbf{a}^\dagger] = \mathbf{1}$, where m is the particle's mass, ω is the angular frequency, and $\hbar$ is the reduced Planck's constant $(\frac{h}{2\pi})$.

The Heisenberg uncertainty relations reveal that the momentum operator $\hat{p}$ in (21) can be realized as a differential operator $\hat{p} \leftrightarrow -i\hbar\frac{d}{dx}$. And vice versa, the $\hat{x}$ operator can be realized in terms of momentum derivatives $\frac{d}{dp}$. Because the $\mathbf{L}, \mathcal{P}$ operators are explicitly constructed in terms of the $\mathbf{1}, \mathbf{a}, \mathbf{a}^\dagger$ operators, and which in turn, can be realized in terms of x and $\frac{d}{dx}$, the full operator $(\mathbf{L}_*\mathbf{L}_*\ldots\mathbf{L}_* - \mathcal{P})$ turns into a (higher order) linear differential operator acting on Ψ_n.

Consequently, in this case it is appropriate to say that Ψ_n is a null eigenfunction of the full operator encompassed inside the parenthesis, with zero eigenvalue. This implies that the Ψ_n's (for all n) are the null eigenfunctions of the infinite set of (higher order) linear differential equations $(\mathbf{L}_*\mathbf{L}_*\ldots\mathbf{L}_* - \mathcal{P})\Psi_n = 0$, with zero eigenvalue. Since the Ψ_n's themselves are a superposition of the QM harmonic oscillator eigenstates (bits), the latter linear differential equations can be decomposed themselves into an even larger set of linear differential equations involving the harmonic oscillator energy eigenstates $\phi_l(x), (l = 0, 1, 2, \ldots)$, which are given by the product of Gaussian exponentials e^{-x^2} times the Hermite polynomials $H_l(x)$. Hence, one ends with a very large number of differential constraints on the harmonic oscillator energy eigenstates that can only be satisfied, if, and only if, the diagonal entries of $\mathbf{L}_*\mathbf{L}_*\ldots\mathbf{L}_*$ have the following form for n odd

$$\left((\mathbf{a}\mathbf{a}^\dagger)^\alpha;\ Q_{n1}(\mathbf{a},\mathbf{a}^\dagger);\ \ldots,\ Q_{nl_1}(\mathbf{a},\mathbf{a}^\dagger)\mathbf{a}^{l_1+1};\ Q_{nl_1+1}(\mathbf{a},\mathbf{a}^\dagger);\ldots,Q_{nl_2}(\mathbf{a},\mathbf{a}^\dagger)\mathbf{a}^{l_2+1},\ \ldots\right) \tag{22}$$

where Q_{nl} are polynomials in $\mathbf{a},\mathbf{a}^\dagger$ and α is an integer.

Because the first binary coefficient is nonvanishing for n odd, the action of the first entry of (22) on any Ψ_n is $(\mathbf{a}\mathbf{a}^\dagger)^\alpha|0\rangle = |0\rangle$. The action of the remaining entries of (22) on the bits components $|l\rangle$ of Ψ_n are trivially zero at the location of the vanishing binary coefficients (in the binary expansion of Ψ_n). Whereas the action at the location of the non-vanishing binary coefficients is also zero because

$$Q_{nl_1}(\mathbf{a},\mathbf{a}^\dagger)\mathbf{a}^{l_1+1}\ |l_1\rangle\ =\ 0,\ \ Q_{nl_2}(\mathbf{a},\mathbf{a}^\dagger)\mathbf{a}^{l_2+1}\ |l_2\rangle\ =\ 0,\ \ldots \tag{23}$$

Consequently, the only non-vanishing entry-result of the composition of the operators $\mathbf{L}_*\mathbf{L}_*\ldots\mathbf{L}_*|\Psi_n\rangle$ turns out to be $(\mathbf{a}\mathbf{a}^\dagger)^\alpha|0\rangle = |0\rangle$, and which agrees precisely with $\mathcal{P}_{n=odd}|\Psi_n\rangle = |\Psi_1\rangle = |0\rangle$.

When n is even, the diagonal entries of $\mathbf{L}_*\mathbf{L}_*\ldots\mathbf{L}_*$ must have the following form

$$\left(\tilde{Q}_{n0}(\mathbf{a},\mathbf{a}^\dagger),\ \tilde{Q}_{n1}(\mathbf{a},\mathbf{a}^\dagger),\ \ldots,\ (\mathbf{a}\mathbf{a}^\dagger)^\beta\frac{\mathbf{a}^{l_1}}{\sqrt{l_1!}},\ \tilde{Q}_{nl_1+1}(\mathbf{a},\mathbf{a}^\dagger),\ \ldots,\ Q_{nl_2}(\mathbf{a},\mathbf{a}^\dagger)\mathbf{a}^{l_2+1},\ \ldots\right) \tag{24}$$

with β an integer. Following the same arguments as above, one learns that the only non-vanishing result from the action of $\mathbf{L}_*\mathbf{L}_*\ldots\mathbf{L}_*|\Psi_n\rangle$ is in the entry

$$(\mathbf{a}\mathbf{a}^\dagger)^\beta\frac{\mathbf{a}^{l_1}}{\sqrt{l_1!}}\ |l_1\rangle\ =\ (\mathbf{a}\mathbf{a}^\dagger)^\beta\ |0\rangle\ =\ |0\rangle \tag{25}$$

and which agrees with $\mathcal{P}_{n=even}|\Psi_n\rangle = |\Psi_1\rangle = |0\rangle$. (A word of caution, the values of $l_1, l_2, \ldots$ in eq-(22) do not necessarily match those in eqs-(24). Instead of writing $l'_1, l'_2, \ldots$ in (24) we just dropped the primes).

In order to prove eqs-(22,24) one has to introduce suitable Polynomials (to be determined afterwards) $\{B_{nl}(\mathbf{a}, \mathbf{a}^\dagger), C_{nl}(\mathbf{a}, \mathbf{a}^\dagger)\}$ in the diagonal entries of the early expressions for the projection operators in eqs-(6,8a), and corresponding to the location of the non-vanishing binary coefficients, as follows

$$\mathcal{P}_{odd} = Diag\left(\mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \ldots B_{nl_1}\ \mathbf{a}^{l_1+1},\ \mathbf{1},\ \mathbf{1},\ \ldots\ B_{nl_2}\ \mathbf{a}^{l_2+1}, \mathbf{1},\ \mathbf{1},\ \ldots\ B_{nl_3}\ \mathbf{a}^{l_3+1},\ \ldots \right) \tag{26}$$

$$\mathcal{P}_{even} = Diag\left(\mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \ldots \frac{\mathbf{a}^{l_1}}{\sqrt{l_1!}},\ \mathbf{1},\ \mathbf{1},\ \ldots\ C_{nl_2}\ \mathbf{a}^{l_2+1}, \mathbf{1},\ \mathbf{1},\ \ldots\ C_{nl_3}\ \mathbf{a}^{l_3+1},\ \ldots \right) \tag{27}$$

In order to simplify the task enormously one may choose again the ansatz by setting the polynomials to be functions $B_{nl}(\mathbf{N}), C_{nl}(\mathbf{N})$ of the number operator $\mathbf{N} = \mathbf{a}^\dagger\mathbf{a}$ only. The expressions of these Polynomials $\{B_{nl}(\mathbf{N}), C_{nl}(\mathbf{N})\}$ are determined in three steps : firstly, one inserts the modified expression for the projection operators provided by eqs-(26,27); secondly, one performs the composition of the operators $\mathbf{L}_*\mathbf{L}_*\ldots\mathbf{L}_*$ and equates it with the diagonal entries of eqs-(22,24) (for n odd, even). This procedure leads to a set of equations from which one can deduce the functional form of the polynomials $\{B_{nl}(\mathbf{N}), C_{nl}(\mathbf{N})\}$. Thirdly, the expressions of the **unspecified** polynomials $Q_{nl}(\mathbf{N}); \tilde{Q}_{nl}(\mathbf{N})$ in eqs-(22,24) are finally determined in terms of $\{B_{nl}(\mathbf{N}), C_{nl}(\mathbf{N})\}$. The reason eqs-(22,24,26,27) have such diagonal form in terms of the polynomials $Q_{nl}(\mathbf{a}, \mathbf{a}^\dagger), \tilde{Q}_{nl}(\mathbf{a}, \mathbf{a}^\dagger)$ is because the $\mathbf{a}^\dagger$ terms are always present whenever the operators $\mathbf{L}_{3n+1} = \mathbf{I} + \mathcal{L}^+ + \mathcal{P}$ appear in the composition $\mathbf{L}_*\mathbf{L}_*\ldots\mathbf{L}_*$.

2.4 Cycle Identities

The Collatz conjecture would be false if there are nontrivial cycles besides the trivial one $\{1, 4, 2, 1, 4, 2, 1, \ldots\}$. If there exists at least one nontrivial cycle, this implies that there is at least one value of n such that Ψ_n satisfies $(\mathbf{L}_*\mathbf{L}_*\ldots\mathbf{L}_* - \mathbf{I})\Psi_n = 0$, with zero eigenvalue, where the identity operator $\mathbf{I}$ is the one responsible for the cycle. This very special null eigenfunction condition would lead, again, to many differential constraints imposed on the harmonic oscillator energy eigenstates appearing in the binary decomposition of Ψ_n. Such differential constraints could be satisfied if, and only if, the diagonal entries of $\mathbf{L}_*\mathbf{L}_*\ldots\mathbf{L}_*$ have the following form for n odd

$$\left((\mathbf{a}\mathbf{a}^\dagger)^{\alpha_o};\ Q'_{n1}(\mathbf{a}, \mathbf{a}^\dagger);\ \ldots,\ \frac{(\mathbf{a}\mathbf{a}^\dagger)^{\alpha_{l_1}}}{(1+l_1)^{\alpha_{l_1}}};\ Q'_{nl_1+1}(\mathbf{a}, \mathbf{a}^\dagger);\ldots, \frac{(\mathbf{a}\mathbf{a}^\dagger)^{\alpha_{l_2}}}{(1+l_1)^{\alpha_{l_2}}},\ \ldots \right) \tag{28}$$

where Q'_{nl} are polynomials in $\mathbf{a}, \mathbf{a}^\dagger$ and the exponents α's (corresponding to the locations of the non-vanishing binary coefficients) are integers. When n is even, the diagonal entries of $\mathbf{L}_* \mathbf{L}_* \ldots \mathbf{L}_*$ must have the following form

$$\left(\tilde{Q}'_{n0}(\mathbf{a}, \mathbf{a}^\dagger),\ \tilde{Q}'_{n1}(\mathbf{a}, \mathbf{a}^\dagger),\ \ldots,\ \frac{(\mathbf{a}\mathbf{a}^\dagger)^{\beta_{l_1}}}{(1+l_1)^{\beta_{l_1}}},\ \tilde{Q}'_{nl_1+1}(\mathbf{a}, \mathbf{a}^\dagger),\ \ldots,\ \frac{(\mathbf{a}\mathbf{a}^\dagger)^{\beta_{l_2}}}{(1+l_2)^{\beta_{l_2}}},\ \ldots \right) \tag{29}$$

where the exponents β's are positive integers and $\tilde{Q}'_{nl}$ are polynomials in $\mathbf{a}, \mathbf{a}^\dagger$. Once again, to simplify the task enormously one may choose the ansatz by setting the polynomials to be functions $Q'_{nl}(\mathbf{N}), \tilde{Q}'_{nl}(\mathbf{N})$ of the number operator $\mathbf{N} = \mathbf{a}^\dagger \mathbf{a}$ only.

One may note that the action of $\frac{(\mathbf{a}\mathbf{a}^\dagger)^p}{(1+l)^p}$ on the bit states $|l\rangle$ yelds $\frac{(1+l)^p}{(1+l)^p}|l\rangle = |l\rangle$ and effectively behaves like a unit operator. Whereas the action of the entries involving the Polynomials $Q'_{nl}, \tilde{Q}'_{nl}$ yields trivially zero because they correspond to the locations of the vanishing binary coefficients (in the expansion of Ψ_n).

The key question is : can one find a judicious choice of yet another set of different diagonal projection operators $\mathcal{P}_n$ (for n odd, even) and such that the diagonal entries of $\mathbf{L}_* \mathbf{L}_* \ldots \mathbf{L}_*$ match those in eqs-(28,29) ? If so, this will lead to the existence of a nontrivial cycle. However, if the Collatz conjecture is true this implies that one **cannot** find such a judicious set of projection operators $\mathcal{P}_n$ for any values of n. Therefore, the question of whether or not there are nontrivial cycles is a "simpler" way in proving/disproving the conjecture.

Concluding, we found that the Collatz conjecture can be encoded in the infinite number of equations ($\mathbf{L}_* \mathbf{L}_* \ldots \mathbf{L}_* - \mathcal{P}$)$|\Psi_n\rangle = 0$, associated to every positive integer, even or odd. We studied the particular case about the existence of (non-trivial) null eigenfunctions, and the more restrictive case of the operator identities $\mathbf{L}_* \mathbf{L}_* \ldots \mathbf{L}_* - \mathcal{P} = 0$. In the latter case, because one has an infinite number of operators at our disposal, in principle, one could have an infinite number of operator identities $\mathbf{L}_* \mathbf{L}_* \ldots \mathbf{L}_* - \mathcal{P} = 0$, if, and only if, all of these relations are mutually **consistent** (as shown in eqs-(20)). The null egenfunctions and operator identities cases, both require finding the suitable expressions for the projection operators $\mathcal{P}_n$ that map every state Ψ_n to the ground state Ψ_1, and which in turn, are provided in terms of a family of Polynomials of the $\mathbf{a}, \mathbf{a}^\dagger$ operators. To simplify the calculations one may postulate the ansatz that **most** of the polynomials are functions of the number operator $\mathbf{N} = \mathbf{a}^\dagger \mathbf{a}$. If these two (extremely laborious) procedures turn out to be satisfactory, then the Collatz conjecture is true.

If one replaced the $3n+1$ iteration (n odd) for $3n+b$, with b odd and greater than 1, the operator representation for the process $\Psi_n \to \Psi_{3n+b}$ will **no** longer have the **diagonal** form $(\mathbf{I} + \mathcal{L}^+ + \mathcal{P})_{n=odd}$, and which is intrinsically tied up to the very special decomposition of $3n+1 = n+2n+1$, but it is going to have another expression given by $\mathbf{L}_{3n+b} = (\mathbf{I} + \mathcal{L}^+ + \mathcal{P})_{n=odd} + \Delta_{n,b-1}$, where $\Delta_{n,b-1}$ is now an off-diagonal operator-valued matrix whose entries are comprised of $\mathbf{1}, \mathbf{a}, \mathbf{a}^\dagger$, and that maps Ψ_n to Ψ_{b-1}.

Consequently, due to the presence of this extra off-diagonal operator-valued matrix $\Delta_{n,b-1}$, it is now possible to find null eigenfunctions $(\mathbf{L}_* \mathbf{L}_* \ldots \mathbf{L}_* - \mathbf{I})\Psi_n = 0$ and obtain nontrivial cycles for very specific values of b and n. In this case, the generalization of the Collatz conjecture would be false. For example, when $b = n$ one has the trivial cycle $\{b, 4b, 2b, b, 4b, 2b, b, \ldots\}$. Nowak [6] has found that when $a = 2000003$, any iteration process of $n = n_o$ that is **not** a multiple of 2000003 will end up in the cycle involving the Mersenne prime $127 = 2^7 - 1$. This cycle based on 127 has a length of 15126 steps and a maximum value (number) of 48382644622. Nowak uses the iteration $\frac{3n+b}{2}$ for n odd, and $\frac{n}{2}$ for n even as usual.

To finalize we should add that an heuristic explanation as to why there are no divergences in the Collatz process can be found if one proceeds with the Syracuse iteration (proposed by Hasse) $n \to \frac{3n+1}{2}$, when n is odd, and $n \to \frac{n}{2}$ when n is even. The former leads to a dilation with a factor of $\frac{3}{2}$. The latter leads to a contraction by a factor of $\frac{1}{2}$. Because there are an equal number of even and odd numbers, on average, the overall scaling factor is $\frac{3}{2} \times \frac{1}{2} = \frac{3}{4} < 1$, leading to a contraction, and the Collatz process does not diverge. If one uses, for example, the iteration $\frac{5n+1}{2}$ for n odd, the overall scaling factor would be $\frac{5}{2} \times \frac{1}{2} = \frac{5}{4} > 1$, and the process should diverge.

APPENDIX

In the Appendix we provide two examples of how to perform the composition of operators by **adding** a judicious string of unit operators **1**'s to the left, and/or to the right, of the diagonal entries such that **all** the operators are now represented by diagonal matrices of the same size. It is also necessary to **embed** all the column vectors into a column vector of maximal size N_{max} as well by adding extra zeros.

Let us consider the Collatz process $\{8, 4, 2, 1\}$ associated with the states $\{\Psi_8, \Psi_4, \Psi_2, \Psi_1 = |0\rangle\}$. The projection operator $\mathcal{P}[\Psi_8] = Diag(\mathbf{1}, \mathbf{1}, \mathbf{1}, \frac{\mathbf{a}^3}{\sqrt{3!}})$ maps Ψ_8 into Ψ_1, where Ψ_8 is represented by the column vector of maximal size whose entries are $\{0, 0, 0, |3\rangle\}$.

The lowering operators $\mathcal{L}^-$ that reduce the bits by one unit are

$$\mathbf{L}[\Psi_8] = Diag(\mathbf{1}, \mathbf{1}, \mathbf{1}, \frac{\mathbf{a}}{\sqrt{3}}); \quad \mathbf{L}[\Psi_4] = Diag(\mathbf{1}, \mathbf{1}, \frac{\mathbf{a}}{\sqrt{2}}); \quad \mathbf{L}[\Psi_2] = Diag(\mathbf{1}, \mathbf{a}) \tag{A.1}$$

Since the last two diagonal operators do not have the same size as the first diagonal operator of maximal size, we **add** the sufficient number of **1**'s entries to their **left** and arrive at

$$\mathbf{L}[\Psi_4] \to Diag(\mathbf{1}, \mathbf{1}, \mathbf{1}, \frac{\mathbf{a}}{\sqrt{2}}), \quad \mathbf{L}[\Psi_2] \to Diag(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{a}) \tag{A.2}$$

Now one can verify that the products obey the relation

$$Diag(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{a})\ Diag(\mathbf{1},\mathbf{1},\mathbf{1},\frac{\mathbf{a}}{\sqrt{2}})\ Diag(\mathbf{1},\mathbf{1},\mathbf{1},\frac{\mathbf{a}}{\sqrt{3}})\ =\ Diag(\mathbf{1},\mathbf{1},\mathbf{1},\frac{\mathbf{a}^3}{\sqrt{3!}})\ =\ \mathcal{P}[\Psi_8] \qquad (A.3)$$

and, therefore one has the operator relation $\mathbf{L}_*\mathbf{L}_*\ldots\mathbf{L}_* - \mathcal{P} = 0$ in this very special case. It is also necessary to **embed** the column vector Ψ_4 with entries $\{0,0,|2\rangle\}$ into a column vector with entries $\{0,0,0,|2\rangle\}$; and the column vector Ψ_2 with entries $\{0,|1\rangle)$ into the one with entries $\{0,0,0,|1\rangle\}$. In this way the latter column vectors will have the same number of entries as Ψ_8 whose entries are $\{0,0,0,|3\rangle\}$.

In the second example we shall see the case where we **add** the sufficient number of **1**'s entries to their **right**, instead. Let us consider the Collatz process $\{3,10,5,16,8,4,2,1\}$ associated with the states $\{\Psi_3,\Psi_{10},\Psi_5,\Psi_{16},\Psi_8,\Psi_4,\Psi_2,\Psi_1 = |0\rangle\}$. One can read-off the maximum value given by 16 in the Collatz chain. The binary representation of 16 is 2^4, thus the diagonal matrix of maximal size will have $4+1=5$ entries since Ψ_{16} is represented by a column vector whose entries are $\{0,0,0,0,|4\rangle\}$.

Because Ψ_3 is represented by a column vector whose entries are $\{|0\rangle,|1\rangle\}$ one needs to add 3 extra zeros $\{|0\rangle,|1\rangle\},0,0,0\}$ in order to match the same number of entries as Ψ_{16} which will appear in the Collatz chain process.

The expression for the operator $\mathbf{L}[\Psi_3]$ that maps Ψ_3 into Ψ_{10} has the $\mathbf{I} + \mathcal{L}^+ + \mathcal{P}$ form given by

$$\mathbf{L}[\Psi_3]\ =\ Diag(\mathbf{1},\mathbf{1})\ +\ Diag(\mathbf{a}^\dagger,\ \frac{\mathbf{a}^\dagger}{\sqrt{2}})\ +\ Diag(\mathbf{1},\mathbf{a}^2) \qquad (A.4)$$

A careful inspection of the Collatz chain process reveals that now we must **add** three **1**'s entries to their **right** , leading to diagonal operators of the maximal size

$$\mathbf{L}[\Psi_3]\ \to\ Diag(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\ +\ Diag(\mathbf{a}^\dagger,\ \frac{\mathbf{a}^\dagger}{\sqrt{2}},\mathbf{1},\mathbf{1},\mathbf{1})\ +\ Diag(\mathbf{1},\mathbf{a}^2,\mathbf{1},\mathbf{1},\mathbf{1}) \qquad (A.5)$$

and that matches the number of entries of the column vector of maximal size Ψ_{16}, whose entries are $\{0,0,0,0,|4\rangle\}$. Finally one can show that $\mathbf{L}[\Psi_3]\Psi_3 = \Psi_{10}$ after using the binary composition rules

$$|1\rangle + |1\rangle\ =\ |2\rangle;\ \ |2\rangle + |2\rangle\ =\ |3\rangle,\ \ |l\rangle + |l\rangle\ =\ |l{+}1\rangle,\ \ |l{+}1\rangle + |l{+}1\rangle\ =\ |l{+}2\rangle,\ \ldots$$

with

$$\Psi_3 \equiv \{|0\rangle,|1\rangle,0,0,0\},\quad \Psi_{10} \equiv \{0,|1\rangle,0,|3\rangle,0\}, \qquad (A.6)$$

In other words, one has a map from $\Psi_3 = \{|0\rangle,|1\rangle,0,0,0\}$ into $\Psi_{10} = \{0,|1\rangle,0,|3\rangle,0\}$ corresponding to the first Collatz iteration $\mathcal{C}(3) = 10$. Despite this very tedious process, one can show that the composition of all the **7** string of operators acting on Ψ_3 will lead finally, and effectively, to the projection operator acting on $\Psi_3 : \mathcal{P}[\Psi_3]\Psi_3 = \Psi_1 = |0\rangle$.

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