New ordinal relative fuzzy entropy

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Abstract

In real life, occurrences of a series of things are supposed to come in an order. Therefore, it is necessary to regard sequence as a crucial factor in managing different kinds of things in fuzzy environment. However, few related researches have been made to provided a reasonable solution to this demand. Therefore, how to measure degree of uncertainty of ordinal fuzzy sets is still an open issue. To address this issue, a novel ordinal relative fuzzy entropy is proposed in this paper taking orders of propositions into consideration in measuring level of uncertainty in fuzzy environment. Compared with previously proposed entropies, effects on degrees of fuzzy uncertainty brought by sequences of sequential propositions are embodied in values of measurement using proposed method in this article. Moreover, some numerical examples are offered to verify the correctness and validity of the proposed entropy.

Keywords: Order Fuzzy environment Degree of uncertainty Entropy

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1. Introduction

How to measure degree of uncertainty of uncertain information in a fuzzy environment is a hot topic which attracts lots of attention from researchers globally. Many meaningful and useful theories have been developed to obtain truly useful information from uncertainty, such as Dempster - Shafer evidence theory [1, 2], complex mass function [3–5], D-numbers [6–8], Z-numbers [9–12], soft theory [13–16], fuzzy theory [17–20] and so on [21–24]. Due to the effectiveness of those theories in handling uncertainty, they have been applied into different kinds of fields and applications, like pattern recognition [25–27], decision making [28–32] and so on [33–36]. Proposals of these actual applications contribute to extract important information from a uncertain situation.

However, preparations before processing uncertain information must include a measure on the level of uncertainty of a whole system. At present, the most efficient tool in giving an indicator of the degree of uncertainty of a existing system is to utilize entropies. Lots of relevant works have been made, such as Deng entropy [37], motion entropy [38], interval entropy [39] and non-additive entropy [40]. Nevertheless, all of the proposed entropies do not take orders of propositions contained in a fuzzy system as a factor in affecting the level of uncertainty of given system. There is a lack of relevant researches on measuring degrees of uncertainty of an ordinal fuzzy system. Therefore, in this paper, a new ordinal relative fuzzy entropy is proposed to appropriately give an accurate description on situations of an ordinal fuzzy system, which could be utilized in intuitionistic, pythagorean, fermatean and orthopair environment. Compared with other existing proposed entropies, it could reflect the influences brought by sequences of propositions which fits rules and criterions of actual situations.

This paper is organized as follows. Some basic concepts are introduced in the section of preliminaries. In the next section, the proposed method is elaborately illustrated and details is clearly explained. And in the section of numerical examples, 5 examples in different fuzzy environments are provided to verify the rationality and validity of the proposed entropy. In the last, conclusions are made to summarize the contributions of the proposed method.

2. Preliminaries

In this section, some relevant concepts are briefly introduced and explained. Lots of related works have been made to solve a considerable amount of problems involved with different fields of applications [41–45].

2.1. Fuzzy sets [46]

Definition 1. Let P be a FS in a finite universe of discourse which is called H. And the mathematics form of a FS, P, in H can be defined as:

$$P = \{x, \mu(x)\} \tag{1}$$

$\mu(x)$ represents the membership degree of $x \in H$.

2.2. Intuitionistic fuzzy sets [47]

Definition 2. Let A be an IFS in a finite universe of discourse which is called X. And the mathematics form of an IFS, A, in X can be defined as:

$$A = \{\langle x, \mu_A(x), v_A(x)|x \in X\} \tag{2}$$

Besides, the properties the IFS satisfies can be defined as:

$$\mu_A(x) : X \to [0,1] \tag{3}$$

$\mu_A(x)$ represents the membership degree of $x \in X$.

$$v_A(x) : X \to [0,1] \tag{4}$$

v_A represents the non - membership degree of $x \in X$. Besides, a constraint which the two parameters meet is defined as:

$$0 \leq \mu_A(x) + v_A(x) \leq 1 \tag{5}$$

For an IFS defined in X, a degree of hesitance $\pi(x)$ is defined as:

$$\pi(x) = 1 - \mu_A(x) - v_A(x) \tag{6}$$

The value of $\pi(x)$ reflects the degree of hesitance of $x \in X$.

2.3. Pythagorean fuzzy sets [48, 49]

Definition 3. Let B be a PFS in a finite universe of discourse which is called Z. And the mathematics form of a PFS, B, in Z can be defined as:

$$B = \{\langle x, B_Y(x), B_N(x) | x \in Z\} \tag{7}$$

Besides, the properties the PFS satisfies can be defined as:

$$B_Y(x) : Z \to [0,1] \tag{8}$$

$B_Y(x)$ represents the membership degree of $x \in Z$.

$$B_N(x) : Z \to [0,1] \tag{9}$$

$B_N(x)$ represents the non - membership degree of $x \in Z$. Besides, a constraint which the two parameters meet is defined as:

$$0 \leq B_Y^2(x) + B_N^2(x) \leq 1 \tag{10}$$

For an PFS defined in Z, a parameter $K^2(x) = B_Y^2(x) + B_N^2(x)$ is given, then the degree of hesitance, $B_H(x)$, is defined as:

$$B_H(x) = \sqrt{1 - K^2(x)} \tag{11}$$

The value of $B_H(x)$ reflects the degree of hesitance of $x \in Z$.

2.4. Fermatean fuzzy sets [50]

Definition 4. Let C be a FFS in a finite universe of discourse which is called Q. And the mathematics form of a FFS, C, in Q can be defined as:

$$C = \{\langle x, \alpha_F(x), \beta_F(x)\rangle | x \in Q\} \tag{12}$$

Besides, the properties the FFS satisfies can be defined as:

$$\alpha_F(x) : Q \to [0,1] \tag{13}$$

$\alpha_F(x)$ represents the membership degree of $x \in Q$.

$$\beta_F(x) : Q \to [0,1] \tag{14}$$

$\beta_F(x)$ represents the non - membership degree of $x \in Q$. Besides, a constraint which the two parameters meet is defined as:

$$0 \leq \alpha_F(x)^3 + \beta_F(x)^3 \leq 1 \tag{15}$$

For an *FFS* defined in Q, a degree of hesitance, $\pi_F(x)$, is defined as:

$$\pi_F(x) = \sqrt[3]{1 - \alpha_F(x)^3 - \beta_F(x)^3} \tag{16}$$

The value of $\pi_F(x)$ reflects the degree of hesitance of $x \in Q$.

2.5. Orthopair Fuzzy Sets [51]

Definition 5. Let D be an *OFS* in a finite universe of discourse which is called R. And the mathematics form of an *OFS*, D, in R can be defined as:

$$D = \{\langle x, D^+(x), D^-(x) | x \in R\} \tag{17}$$

Besides, the properties the *OFS* satisfies can be defined as:

$$D^+(x) : R \to [0,1] \tag{18}$$

$D^+(x)$ represents the membership degree of $x \in R$.

$$D^-(x) : R \to [0,1] \tag{19}$$

$D^-(x)$ represents the non - membership degree of $x \in R$. Besides, a constraint which the two parameters meet is defined as:

$$0 \leq (D^+(x))^n + (D^-(x))^n \leq 1 \tag{20}$$

For an *OFS* defined in R, a degree of hesitance, $\pi_O(x)$, is defined as:

$$\pi_O(x) = \sqrt[n]{1 - (D^+(x))^n - (D^-(x))^n} \tag{21}$$

The value of $\pi_O(x)$ reflects the degree of hesitance of $x \in R$.

2.6. *Shannon entropy [52]*

The Shannon entropy is denoted as Sh and it is defined as:

$$Sh = -\sum_{i=1}^{n} p_i log_b p_i \tag{22}$$

n is the number of targets contained in a system and the icon p_i represents the probability of a certain things to happen. When the base of the logarithm, b, is equal to 2, then the unit of Shannon entropy is bit.

2.7. *De Luca and Termini's fuzzy set entropy [53]*

Definition 6. A fuzzy set $E = \{x, \mu(x)\}$ is given, where $\mu(x)$ represents the membership of certain propositions, then the corresponding entropy is defined as:

$$DT(E) = \frac{1}{n}\sum_{i=1}^{n}[\mu_E(x_i)log^{\mu_E(x_i)} + (1 - \mu_E(x_i))log^{(1-\mu_E(x_i))}] \tag{23}$$

2.8. *Pal Nikhil and Pal Sankar's fuzzy exponential entropy [54]*

Definition 7. A fuzzy set $F = \{x, \mu(x)\}$ is given, where $\mu(x)$ represents the membership of certain propositions, then the corresponding entropy is defined as:

$$PP(F) = \frac{1}{n(\sqrt{e} - 1)}\sum_{i=1}^{n}[\mu_F(x_i)e^{(1-\mu_F(x_i))} + (1 - \mu_F(x_i))e^{\mu_F(x_i)} - 1] \tag{24}$$

2.9. Zhang and Jiang's intuitionistic fuzzy entropy [55]

Definition 8. An intuitionistic fuzzy set $G = \{x, \mu_G(x), v_G(x)\}$ is given, where $\mu_G(x)$ represents the membership of certain propositions and $v_G(x)$ represents the non - membership of certain propositions, then the corresponding entropy is defined as:

$$ZJ(G) = -\frac{1}{n}\sum_{i=1}^{n}[\frac{\mu_G(x)+1-v_G(x)}{2}log_2^{\frac{\mu_G(x)+1-v_G(x)}{2}} + \frac{v_G(x)+1-\mu_G(x)}{2}log_2^{\frac{v_G(x)+1-\mu_G(x)}{2}}] \tag{25}$$

2.10. Hung and Yang's intuitionistic fuzzy entropy [56]

Definition 9. An intuitionistic fuzzy set $H = \{x, \mu_H(x), v_H(x)\}$ is given, where $\mu_H(x)$ represents the membership of certain propositions and $v_H(x)$ represents the non - membership of certain propositions, then the corresponding entropy is defined as:

$$HY(x) = -(\mu_H(x)log^{\mu_H(x)} + v_H(x)log^{v_H(x)} + \pi_H(x)log^{\pi_H(x)}) \tag{26}$$

2.11. Xu's pythagorean fuzzy entropy [57]

Definition 10. A pythagorean fuzzy set $I = \{x, I_Y(x), I_N(x)\}$ is given, where $I_Y(x)$ represents the membership of certain propositions and $I_N(x)$ represents the non - membership of certain propositions, then the corresponding entropy is defined as:

$$X(I) = \frac{1}{n}\sum_{i=1}^{n}[1-(1-I_H(x_i))|I_Y(x_i)-I_N(x_i)|] \tag{27}$$

2.12. Yang's pythagorean fuzzy entropy [58]

Definition 11. A pythagorean fuzzy set $J = \{x, J_Y(x), J_N(x)\}$ is given, where $J_Y(x)$ represents the membership of certain propositions and $J_N(x)$ represents the non - membership of certain propositions, then the corresponding entropy is defined as:

$$Y(J) = \frac{1}{n}\sum_{i=1}^{n} -(J_Y(x)^2 log^{J_Y(x)^2} + J_N(x)^2 log^{J_N(x)^2} + J_H(x)^2 log^{J_H(x)^2}) \quad (28)$$

3. Generalized entropies for Fermatean fuzzy sets and Orthopair fuzzy sets

Due to the lack of proper entropies to measure degree of uncertainty of Fermatean fuzzy sets and Orthopair Fuzzy Sets, some existing entropies are generalized to provide a solution to give a measurement of the two fuzzy sets mentioned before. In this section, Zhang and Jiang's intuitionistic fuzzy entropy [55] and Yang's pythagorean fuzzy entropy [58] are selected to customised as specific entropies for Fermatean fuzzy sets. Besides, Hung and Yang's intuitionistic fuzzy entropy [56] and Xu's pythagorean fuzzy entropy [57] are chosen to offer a solution in measuring level of uncertainty of Orthopair Fuzzy Sets.

3.1. Fermatean fuzzy entropies

Definition 11. A fermatean fuzzy set $L = \{\langle x, \alpha_L(x), \beta_L(x)\rangle\}$ is given, where $\alpha_L(x)$ represents the membership of certain propositions and $\beta_L(x)$ represents the non - membership of certain propositions, then the corresponding entropies are defined as:

$$ZJ_{Fermatean}(L) = -\frac{1}{n}\sum_{i=1}^{n}[\frac{\alpha_L(x)+1-\beta_L(x)}{2}log^{\frac{\alpha_L(x)+1-\beta_L(x)}{2}} + \frac{\beta_L(x)+1-\alpha_L(x)}{2}log^{\frac{\beta_L(x)+1-\alpha_L(x)}{2}}] \tag{29}$$

$$Y(L) = \frac{1}{n}\sum_{i=1}^{n} -(\alpha_L(x)^2 log^{\alpha_L(x)^2} + \beta_L(x)^2 log^{\beta_L(x)^2} + \pi_L(x)^2 log^{\pi_L(x)^2}) \tag{30}$$

3.2. Orthopair fuzzy entropies

Definition 12. An orthopair fuzzy set $M = \{\langle x, M^+(x), M^-(x)\rangle\}$ is given, where $M^+(x)$ represents the membership of certain propositions and $M^-(x)$ represents the non - membership of certain propositions, then the corresponding entropies are defined as:

$$HY_{Orthopair}(M) = -(M^+(x)log^{M^+(x)} + M^-(x)log^{M^-(x)} + \pi_O(x)log^{\pi_O(x)}) \tag{31}$$

$$X(M) = \frac{1}{n}\sum_{i=1}^{n}[1-(1-\pi_O(x))|M^+(x)-M^-(x)|] \tag{32}$$

4. Proposed new ordinal relative fuzzy entropy

Although entropies have been widely used to measure the degree of uncertainty of a system which contains a sequence of fuzzy sets [59–62], none of the previously proposed entropies take order of propositions into consideration which is an obvious defect in disposing problems related to actual situations. Because every thing comes in an order, one thing happened then the other thing is able to take place. A new entropy is proposed to measure the level of uncertainty of different kinds of fuzzy sets considering effects brought by sequences of propositions.

4.1. Sequential fuzzy sets

All of the elements contained in an ordinal system of fuzzy sets come in an order. And the relations among them are determined by their sequences to some extent. For an ordinal system of fuzzy sets, $\Theta_{Ordinal} = \{P_1, P_2, P_3, P_4\}$, the proposition P_1 must appear before the occurrences of propositions P_2, P_3, P_4. On the other side, propositions P_2, P_3, P_4 can not take the place of proposition P_1 also. Then, all of the properties the ordinal system of fuzzy sets are supposed to satisfy are defined as:

- For a certain ordinal system of fuzzy sets, the sequence of every proposition is already confirmed and can not be changed. Once the order is altered, then the system is replaced by a new one.
- The only relation among propositions contained in an ordinal system of fuzzy sets is their order. And no other relationship exists among appointed propositions.
- With the increase of the number of propositions confirmed, the degree of uncertainty of the whole ordinal system of fuzzy sets is further determined.

And a simple case is provided to illustrate all of the properties the ordinal system of fuzzy sets satisfy.

Case 1: Assume there is an ordinal system of fuzzy sets, $\Theta_{Ordinal=\{P_1,P_2,P_3\}}$, three distinctive propositions are contained in this system. Then, the proposition P_1 is supposed to happen at first, then the second one to take place is the proposition P_2. After confirmation of propositions P_1, P_2, only the proposition P_3 can be added into the whole fuzzy system. If the sequence of all the propositions contained in the system is disturbed, then the conditions of them are supposed to be changed.

4.2. Distributed weights for fuzzy sets in proposed entropy

Because every proposition contained in fuzzy system comes in an order, the level of uncertainty of the system is further confirmed. And it is necessary to consider the role of every step of confirming a proposition is different, a confirmation of a proposition is supposed to have a direct and indirect effect on other values of propositions. Besides, assume the number of propositions contained in the system is a and the sequence of a certain proposition is b. Therefore, the whole process of determining the values of weights of different propositions is defined as:

(1) The weights of specific propositions is $a - b + 1$, and the process of calculation is defined as:

$$Weight_{P_b} = a - b + 1 \tag{33}$$

(2) The original values of propositions contained in a fuzzy system is denoted as $\langle x, Pre, Aft \rangle$. In fuzzy sets, the element Pre represents $\mu(x)$ and Aft is considered equal to 0. Besides, the element Pre represents $\mu_A(x)$ and Aft represents $v_A(x)$ in intuitionistic environment. More than that, when taking pythagorean fuzzy sets into consideration, the element Pre represents $\alpha_B(x)$ and Aft represents $\beta_B(x)$. In the last, in orthopair environment, the element Pre represents $D^+(x)$ and Aft represents $D^-(x)$. Then the process of the calculation of obtaining the

intermediate values is defined as:

$$Mass_{P_b}^{Pre} = Weight_{P_b} * Pre_{P_b} \tag{34}$$

$$Mass_{P_b}^{Aft} = Weight_{P_b} * Aft_{P_b} \tag{35}$$

(3) A step of normalization of intermediate values is designed, and the calculation of process for elements Pre and Aft are defined as:

$$Value_{P_b}^{Pre_{Final}} = \frac{Mass^{Pre_{P_b}}}{2 * \sum_{t=1}^{n} Mass^{Pre_{P_t}}} \tag{36}$$

$$Value_{P_b}^{Aft_{Final}} = \frac{Mass^{Aft_{P_b}}}{2 * \sum_{t=1}^{n} Mass^{Aft_{P_t}}} \tag{37}$$

Note : The cause of carrying the step of normalization is that the degree of importance of a certain proposition decreases with the increase of the number of its sequence, so the index of hesitance of it is exaggerated to reduce its impact in determining the degree of uncertainty of the whole system. And in order to ensure the values after modification still meets properties of kinds of fuzzy sets, both of the modified values are divided by 2.

4.3. *Relative fuzzy entropy*

Obviously, previously proposed entropy can not reflect influences brought by orders of propositions. Because all the entropies regard every fuzzy set as an individual instead of seeking for their underlying relationships and influences to the whole system with every step of confirming a part of it. To remain consistent with the operation of assigning different weights to propositions, it is expected to manifest effects those modified values may

Table 1: Details of the three fuzzy propositions in Case 2

P_1^{Final}	P_2^{Final}	P_3^{Final}
$\{\langle x_1, 0.52, 0.43\rangle\}$	$\{\langle x_2, 0.34, 0.41\rangle\}$	$\{\langle x_3, 0.58, 0.22\rangle\}$

lead. In order to solve all of the problems mentioned above, a variable k is restricted range between 1 and $a-1$ and then a relative fuzzy entropy is defined as:

$$RFE(P_b, P_c) = Pre_{Final_{P_b}} \times log^{(\frac{Pre_{Final_{P_b}}}{Pre_{Final_{P_c}}}+e)} + Aft_{Final_{P_b}} \times log^{(\frac{Aft_{Final_{P_b}}}{Aft_{Final_{P_c}}}+e)}, \quad b > c \tag{38}$$

If a denominator of an index of a log function and an addition, e, is cancelled, then the proposed entropy can degenerate into the form of shannon entropy [52], which means this proposed method remains consistent with traditional information entropy.

Case 2 : Assume three propositions, P_1, P_2, P_3, are contained in an ordinal fuzzy system. And all of them satisfy all of the properties sequential fuzzy sets have. Besides the details of the fuzzy sets corresponding to propositions mentioned before are given in Table 1.

Utilizing the values provided in Table 1, for proposition P_1, the mass of its relative fuzzy entropy can be calculated and the process of calculation can be given as:

$RFE(P_1, P_2) = Pre_{Final_{P_1}} \times log^{(\frac{Pre_{Final_{P_1}}}{Pre_{Final_{P_2}}}+e)} + Aft_{Final_{P_1}} \times log^{(\frac{Aft_{Final_{P_1}}}{Aft_{Final_{P_2}}}+e)} = 1.9078$

$RFE(P_1, P_3) = Pre_{Final_{P_1}} \times log^{(\frac{Pre_{Final_{P_1}}}{Pre_{Final_{P_3}}}+e)} + Aft_{Final_{P_1}} \times log^{(\frac{Aft_{Final_{P_1}}}{Aft_{Final_{P_3}}}+e)} = 1.9204$

4.4. Individual ordinal relative fuzzy entropy

For previously proposed fuzzy entropy including relative fuzzy entropy, their calculation on values of degree of uncertainty of a fuzzy system is non-directional, which dose not conform to actual situations and manifest influences brought by sequences of propositions contained in a fuzzy system. In order to embody the features of an ordinal system, an individual ordinal relative fuzzy entropy is proposed to adapt to this specific situation. For instance, assume there are three propositions, P_1, P_2, P_3, with respect to P_1, its value of individual ordinal relative fuzzy entropy can be only calculated through obtaining total mass of $RFE(P_1, P_2)$ and $RFE(P_1, P_3)$. Besides, for proposition P_2, only the process $P_2 \rightarrow P_3$ can be taken into calculation. As for P_3, its value of individual ordinal relative fuzzy entropy is regarded as 0. Because before the appearance of the last proposition P_3 and fuzzy system is ordinal, the whole system has already been confirmed. Then the calculation of individual ordinal relative fuzzy entropy is defined as:

$$IORFE(P_b, P_c) = \sum_{c=b+1}^{n} RFE(P_b, P_c) \tag{39}$$

Case 3: Assume three propositions, P_1, P_2, P_3, are contained in an ordinal fuzzy system. And all of them satisfy all of the properties sequential fuzzy sets have. Besides the details of the fuzzy sets corresponding to propositions mentioned before are given in Table 2.

Utilizing the values provided in Table 2, the mass of its individual relative fuzzy entropy can be calculated and the process of calculation can be given as:

$IORFE(P_1, P_c) = \sum_{c=1+1}^{n} RFE(P_1, P_c) = RFE(P_1, P_2) + RFE(P_1, P_3) = 1.5681 + 1.5257 = 3.0938$

Table 2: Details of the three fuzzy propositions in Case 3

P_1^{Final}	P_2^{Final}	P_3^{Final}
$\{\langle x_1, 0.35, 0.44\rangle\}$	$\{\langle x_2, 0.26, 0.38\rangle\}$	$\{\langle x_3, 0.56, 0.29\rangle\}$

$IORFE(P_2, P_c) = \sum_{c=2+1}^{n} RFE(P_2, P_c) = RFE(P_2, P_3) = 1.1981$

$IORFE(P_3, P_c) = 0$

Because the fuzzy system given is ordinal, the process of calculation is also expected to be directional. With a value of individual ordinal relative fuzzy entropy calculated, the mass manifests the level of uncertainty at this stage of the system. The number of propositions confirmed is more, the degree of uncertainty of the system is further determined which is represented by the obtained values of individual ordinal relative fuzzy entropy. The proposed fuzzy entropy appropriately measures situations of every component in the whole fuzzy system.

4.5. Complete ordinal relative fuzzy entropy

The complete ordinal relative fuzzy entropy can be regarded as a synthesis of individual ordinal relative fuzzy entropy which measures conditions of every stage of the fuzzy system. Therefore, the complete ordinal relative fuzzy entropy takes conditions of every phase of the whole system into account. Then the process of calculation of complete ordinal relative fuzzy entropy is defined as:

$$CORFE(P_b, P_c) = \sum_{b=1}^{n} IORFE(P_b, P_c) \tag{40}$$

Case 4: Assume three propositions, P_1, P_2, P_3, are contained in an ordinal fuzzy system. And all of them satisfy all of the properties sequen-

Table 3: Details of the three fuzzy propositions in Case 4

P_1^{Final}	P_2^{Final}	P_3^{Final}
$\{\langle x_1, 0.44, 0.32\rangle\}$	$\{\langle x_2, 0.56, 0.33\rangle\}$	$\{\langle x_3, 0.43, 0.27\rangle\}$

tial fuzzy sets have. Besides the details of the fuzzy sets corresponding to propositions mentioned before are given in Table 3.

Utilizing the values provided in Table 3, the mass of its individual relative fuzzy entropy can be calculated and the process of calculation can be given as:

$CORFE(P_b, P_c) = \sum_{b=1}^{n} IORFE(P_b, P_c) = RFE(P_1, P_2) + RFE(P_1, P_3) + RFE(P_2, P_3) + RFE(P_3, P_c) = 1.3984 + 1.4663 + 1.7770 + 0 = 4.6417$

So, this is a final measurement of the given ordinal fuzzy system.

4.6. Measurement about tradition fuzzy system using ordinal relative fuzzy entropy

In the sections discussed above, they provide a solution on how to measure the condition of the ordinal fuzzy system which takes orders of propositions as an important factor in measurement on degree of uncertainty of given system. However, the proposed ordinal entropy can be also utilized to measure the degree of uncertainty of traditional fuzzy systems when considering every kind of sequence of propositions. In other words, if all of the different combinations of sequences are taken into consideration, the level of uncertainty of a unordered fuzzy system can be also calculated in the form of a synthesis of all kinds of situations. Then, the detailed process of calculation is defined as:

- List all kinds of sequences of propositions contained in an ordinal fuzzy system.

Table 4: Details of the three fuzzy propositions in Case 5

P_1	P_2	P_3
$\{\langle x_1, 0.43, 0.45\rangle\}$	$\{\langle x_2, 0.52, 0.41\rangle\}$	$\{\langle x_3, 0.24, 0.37\rangle\}$

- Calculate a sum of all values of different ordinal fuzzy system using complete ordinal relative fuzzy entropy.
- Get an average according to the sum obtained above and the number of propositions existing in a fuzzy system, and the final mass is an evaluation of the classic unordered fuzzy system.

Case 5: Assume three propositions, P_1, P_2, P_3, are contained in an ordinal fuzzy system in an intuitionistic environment. And all of them satisfy all of the properties sequential fuzzy sets have. Besides the details of the fuzzy sets corresponding to propositions mentioned before are given in Table 4. The details of calculation according to defined process are given below.

All possible combinations can be listed as:

$\{P_1, P_2, P_3\}, \{P_1, P_3, P_2\}, \{P_2, P_1, P_3\}, \{P_2, P_3, P_1\}, \{P_3, P_1, P_2\}, \{P_3, P_2, P_1\}$

A different sequence of propositions in an ordinal fuzzy system means completely status of values of propositions. And on the base of definition of complete ordinal relative fuzzy entropy, their values can be obtained and listed as:

$CORFE_{System1}(P_b, P_c) = 5.1671$, $CORFE_{System2}(P_b, P_c) = 3.6493$,

$CORFE_{System3}(P_b, P_c) = 4.9177$, $CORFE_{System4}(P_b, P_c) = 3.8328$,

$CORFE_{System5}(P_b, P_c) = 3.7105$, $CORFE_{System6}(P_b, P_c) = 4.0399$.

And the number of different systems is 6 and an average of the level of ordinal fuzzy system can be obtained by:

Table 5: Results produced by the three entropies

Zhang and Jiang's entropy [55]	Hung and Yang's intuitionistic entropy [56]	Proposed entropy
0.1070	2.1382	4.2195

$CORFE_{Unordered} = \frac{\sum_{i=1}^{6} CORFE_{Systemi}(P_b,P_c)}{6} = \frac{5.1671+3.6493+4.9177+3.8328+3.7105+4.0399}{6} = 4.2195$

Then, this is the final evaluation of the unordered fuzzy system.

Because the system is in an intuitionistic environment, then the degree of uncertainty of the given fuzzy system can be measured by Zhang and Jiang's intuitionistic fuzzy entropy [55] and Hung and Yang's intuitionistic entropy [56]. Then, their values of measurement made the three fuzzy entropies including proposed ordinal entropy are given in Table 5.

It can be easily obtained that the proposed ordinal relative fuzzy entropy can be also used to measure level of uncertainty of a traditional fuzzy system when considering all kinds of combinations of sequences. The effectiveness of proposed entropy in handling different kinds of situations is validated in this section.

5. Numerical examples

In this section, 5 examples of fuzzy sets in different kinds of environment are provided to verify the better validity and correctness of proposed ordinal relative fuzzy entropy in measuring degree of uncertainty of ordinal fuzzy system compared with previously proposed entropies.

Example 1: Assume there are three propositions, P_1, P_2, P_3, are contained in an ordinal fuzzy system. Their original values are listed in Table 6 and the detailed process of calculation is given below.

First, obtain the weights corresponding to specific propositions.

Table 6: Original values of given propositions in Example 1

P_1	P_2	P_3
$\{\langle x_1, 0.48\rangle\}$	$\{\langle x_2, 0.56\rangle\}$	$\{\langle x_3, 0.66\rangle\}$

Table 7: Values of measurement given by three different values in Example 1

Sequence	Proposed entropy	Pal and Pal's entropy [54]	De and Termini's entropy [53]
$\{P_1, P_2, P_3\}$	1.3328	0.3786	-2.0193
$\{P_1, P_3, P_2\}$	1.4055	0.3786	-2.0193
$\{P_2, P_1, P_3\}$	1.4594	0.3786	-2.0193
$\{P_2, P_3, P_1\}$	1.5832	0.3786	-2.0193
$\{P_3, P_1, P_2\}$	1.6811	0.3786	-2.0193
$\{P_3, P_2, P_1\}$	1.7316	0.3786	-2.0193

$Weight_{P_1} = a - b_{P_1} + 1 = 3$, $Weight_{P_2} = a - b_{P_2} + 1 = 2$, $Weight_{P_3} = a - b_{P_3} + 1 = 1$

Second, get the intermediate values of propositions.

$Mass_{P_1}^{Pre} = Weight_{P_1} * Pre_{P_1} = 1.44$, $Mass_{P_2}^{Pre} = Weight_{P_2} * Pre_{P_2} = 1.12$, $Mass_{P_3}^{Pre} = Weight_{P_3} * Pre_{P_3} = 0.66$

Third, get a step of normalization according to the definition of normalization.

$Value_{P_1}^{Pre_{Final}} = \frac{Mass^{Pre_{P_1}}}{2*\sum_{t=1}^{n} Mass^{Pre_{P_t}}} = 0.2236$, $Value_{P_2}^{Pre_{Final}} = \frac{Mass^{Pre_{P_2}}}{2*\sum_{t=1}^{n} Mass^{Pre_{P_t}}} = 0.1739$

$Value_{P_3}^{Pre_{Final}} = \frac{Mass^{Pre_{P_3}}}{2*\sum_{t=1}^{n} Mass^{Pre_{P_t}}} = 0.1024$

Forth, find out all kinds of combinations of the three propositions and they are listed below.

$\{P_1, P_2, P_3\}$, $\{P_1, P_3, P_2\}$, $\{P_2, P_1, P_3\}$, $\{P_2, P_3, P_1\}$, $\{P_3, P_1, P_2\}$, $\{P_3, P_2, P_1\}$

Fifth, calculate the values of degree of uncertainty of the ordinal fuzzy system according to definition of ordinal relative fuzzy entropy. And the values of measurement given by three different values are given in Table 7.

It can be easily concluded that these previously proposed entropy can not reflect the influences brought by order of propositions contained in an ordinal fuzzy system. However, the values obtained by proposed entropy fluctuate according to the sequence of propositions, which conforms to actual situations.

Example 2: Assume there are three propositions, P_1, P_2, P_3, are contained in an ordinal fuzzy system in intuitionistic environment. Their original values are listed in Table 8 and the detailed results obtained are shown in Table 9.

Table 8: Original values of given propositions in Example 2

P_1	P_2	P_3
$\{\langle x_1, 0.47, 0.43\rangle\}$	$\{\langle x_2, 0.52, 0.34\rangle\}$	$\{\langle x_3, 0.25, 0.65\rangle\}$

Table 9: Values of measurement given by three different values in Example 2

Sequence	Proposed entropy	Zhang and Jiang's entropy [55]	Hung and Yang's entropy [56]
$\{P_1, P_2, P_3\}$	4.6529	0.9522	2.7869
$\{P_1, P_3, P_2\}$	4.2509	0.9522	2.7869
$\{P_2, P_1, P_3\}$	4.4752	0.9522	2.7869
$\{P_2, P_3, P_1\}$	4.1179	0.9522	2.7869
$\{P_3, P_1, P_2\}$	4.1399	0.9522	2.7869
$\{P_3, P_2, P_1\}$	4.1823	0.9522	2.7869

It can be easily obtained that the proposed entropy is able to manifest the effects brought by sequences of propositions and the values obtained by the proposed method fluctuate with the change of orders of propositions. On the contrary, the two previously proposed entropy can not reflect influences orders of propositions may cause.

Example 3: Assume there are three propositions, P_1, P_2, P_3, are contained in an ordinal fuzzy system in pythagorean environment. Their original values are listed in Table 10 and the detailed results obtained are shown in Table 11.

Table 10: Original values of given propositions in Example 3

P_1	P_2	P_3
$\{\langle x_1, 0.23, 0.56\rangle\}$	$\{\langle x_2, 0.45, 0.34\rangle\}$	$\{\langle x_3, 0.53, 0.32\rangle\}$

Table 11: Values of measurement given by three different values in Example 3

Sequence	Proposed entropy	Xu's fuzzy entropy [57]	Yang's fuzzy entropy [58]
$\{P_1, P_2, P_3\}$	3.9817	0.8622	0.9240
$\{P_1, P_3, P_2\}$	4.3255	0.8622	0.9240
$\{P_2, P_1, P_3\}$	4.0899	0.8622	0.9240
$\{P_2, P_3, P_1\}$	4.8144	0.8622	0.9240
$\{P_3, P_1, P_2\}$	4.2680	0.8622	0.9240
$\{P_3, P_2, P_1\}$	4.5497	0.8622	0.9240

In this example, changes of sequences' influences on level of uncertainty of a given fuzzy system is presented by proposed entropy. The effectiveness of proposed method in handling fuzziness in pythagorean environment has been proven. Besides, the other two previously proposed entropy can not reflect this kind of effects cause by orders of propositions.

Example 4: Assume there are three propositions, P_1, P_2, P_3, are contained in an ordinal fuzzy system in fermatean environment. Their original values are listed in Table 12 and the detailed results obtained are shown in Table 13.

It can be easily concluded that the proposed method appropriately reflect influences brought by sequences of propositions in a given fuzzy system. On the opposite, obviously, the generalization of the two previously proposed entropy can not properly reflect effects caused by orders of propo-

Table 12: Original values of given propositions in Example 4

P_1	P_2	P_3
$\{\langle x_1, 0.56, 0.32\rangle\}$	$\{\langle x_2, 0.43, 0.33\rangle\}$	$\{\langle x_3, 0.40, 0.37\rangle\}$

Table 13: Values of measurement given by three different values in Example 4

Sequence	Proposed entropy	Generalized Zhang and Jiang's entropy [55]	Generalized Yang's entropy [58]
$\{P_1, P_2, P_3\}$	4.1064	0.9833	0.9260
$\{P_1, P_3, P_2\}$	4.1372	0.9833	0.9260
$\{P_2, P_1, P_3\}$	4.2716	0.9833	0.9260
$\{P_2, P_3, P_1\}$	3.8051	0.9833	0.9260
$\{P_3, P_1, P_2\}$	4.2934	0.9833	0.9260
$\{P_3, P_2, P_1\}$	3.8033	0.9833	0.9260

sitions.

Example 5: Assume there are three propositions, P_1, P_2, P_3, are contained in an ordinal fuzzy system in orthopair environment. Their original values are listed in Table 14 and the detailed results obtained are shown in Table 15.

Table 14: Original values of given propositions in Example 5

P_1	P_2	P_3
$\{\langle x_1, 0.36, 0.54\rangle\}$	$\{\langle x_2, 0.44, 0.37\rangle\}$	$\{\langle x_3, 0.65, 0.22\rangle\}$

In this example, the effectiveness of proposed method in considering influences brought by sequences of propositions contained in an ordinal fuzzy system has been proven. However, the two generalization of previously proposed entropy can not reflect this specific phenomenon.

Table 15: Values of measurement given by three different values in Example 5

Sequence	Proposed entropy	Hung and Yang's entropy [56]	Xu's fuzzy entropy [57]
$\{P_1, P_2, P_3\}$	4.1591	2.8537	0.8983
$\{P_1, P_3, P_2\}$	4.5900	2.8537	0.8983
$\{P_2, P_1, P_3\}$	4.4464	2.8537	0.8983
$\{P_2, P_3, P_1\}$	4.4266	2.8537	0.8983
$\{P_3, P_1, P_2\}$	4.3313	2.8537	0.8983
$\{P_3, P_2, P_1\}$	3.9835	2.8537	0.8983

6. Conclusion

In real life, everything is supposed to take places in an underlying sequence and each of them has certain relations with others and influences on situations of other incidents. The proposed entropy regards orders of propositions contained in an ordinal fuzzy system as a crucial factor in measuring degree of uncertainty of the given fuzzy system, which is the main contribution of this paper. On the contrary, those entropies which are already existed can not properly reflect real world and operation laws of different things. Without doubt, the effectiveness in measuring level of uncertainty of the given ordinal fuzzy system which conforms to actual situations. Numerical examples also provide strong evidences and support in verifying validity and correctness of the proposed entropy.

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