CHORDAL BIPARTITE GRAPHS ARE RANK DETERMINED

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Abstract. A *partial matrix* A is a rectangular array with entries in $\mathbb{F} \cup \{*\}$, where $\mathbb{F}$ is the ground field, and $*$ is a placeholder symbol for entries which are not specified. The *minimum rank* $\operatorname{mr}(A)$ is the smallest value of the ranks of all matrices obtained from A by replacing the $*$ symbols with arbitrary elements in $\mathbb{F}$. For any bipartite graph G with vertices (U, V), one defines the set $M(G)$ of partial matrices in which the row indexes are in U, the column indexes are in V, and the (u, v) entry is specified if and only if u, v are adjacent in G. We prove that, if G is chordal bipartite, then the minimum rank of any matrix in $M(G)$ is determined by the ranks of its fully specified submatrices. This result was conjectured by Cohen, Johnson, Rodman, Woerdeman in 1989.

1. Introduction

In the *low rank matrix completion* problem, one is given a matrix in which several entries are unknown, and the task is to fill in these unknown entries so that the rank of the resulting matrix is minimal possible. This problem and its approximate version are being intensively discussed in modern literature [5, 6, 8, 9], and their potential applications include collaborative filtering [1], computer vision [34], machine learning [20], phase retrieval [7], and recommendation systems [27].

In the most relevant case of the matrices over the real numbers, the low rank matrix completion problem is NP-hard [26] and even $\exists\mathbb{R}$-complete [29]. Therefore, no simple combinatorial algorithm is expected to solve this problem, and, in particular, one cannot find the minimum rank of a general partial matrix by solely looking at the ranks of its maximal specified minors. In fact, the minimum rank of

$$\begin{pmatrix} * & 1 & 1 \\ 1 & * & 1 \\ \varepsilon & 1 & * \end{pmatrix} \tag{1.1}$$

is 2 for $\varepsilon \neq 1$, and it equals 1 for $\varepsilon = 1$. However, if $\varepsilon \neq 0$, then every fully specified submatrix of (1.1) has rank one, so the knowledge of the ranks of these submatrices is indeed insufficient for the determination of the minimum rank of (1.1).

Remark 1.1. In what follows, we work over an arbitrary ground field $\mathbb{F}$.

Cohen, Johnson, Rodman, Woerdeman [10] take a deeper study of the phenomenon discussed above. For any bipartite graph G with a fixed splitting of the vertices into the parts U and V, one defines the set of partial matrices $M(G)$ in which the row indexes correspond to the vertices in U, the column indexes correspond to V, and the (u, v) entry is specified if and only if u, v are adjacent in G.

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Definition 1.2. The matrices in $M(G)$ are said to be *subordinate* to G.

Definition 1.3. A bipartite graph G is *rank determined* if, for any partial matrices A, B in $M(G)$, the equality of the ranks of every pair of the corresponding fully specified submatrices of A, B implies the equality of the minimum ranks of A, B.

The matrices of the form (1.1) are subordinate to C_6 (the cycle on six vertices). Therefore, if we take the matrix (1.1) with $\varepsilon = 1$ as A and a similar matrix with some $\varepsilon \neq 0, 1$ in the role of B, we get a witness that C_6 does not satisfy Definition 1.3. Therefore, the graph C_6 is not rank determined unless $|\mathbb{F}| = 2$.

Remark 1.4. This argument fails over $\mathbb{F}_2$ because we cannot find $\varepsilon \neq 0, 1$. In fact, every bipartite graph is rank determined over $\mathbb{F}_2$ because the equality of the ranks of just the corresponding 1×1 submatrices as in Definition 1.3 implies $A = B$.

The authors of [10] construct an example similar to (1.1) and show that the cycles C_{2k} are not rank determined, provided that $k \geqslant 3$ is an integer and $|\mathbb{F}| \neq 2$. They note that the property of a graph being rank determined is inherited by the induced subgraphs, which is true simply because the additions of the zero rows and columns to a given matrix do not affect its rank.

Definition 1.5. A bipartite graph G is called *chordal bipartite* if no subgraph induced by six or more vertices of G is a cycle.

Therefore, a bipartite graph G can be rank determined with respect to a field $\mathbb{F}$ with $|\mathbb{F}| \neq 2$ only if G is chordal bipartite. Cohen, Johnson, Rodman, Woerdeman presumed that the converse is also true, see Conjecture 3.3 in [10].

Conjecture 1.6. *Every chordal bipartite graph is rank determined.*

This paper is devoted to the proof of Conjecture 1.6. In what follows, we write $\operatorname{mr}(A)$ to denote the minimum rank of a partial matrix A. If i is a row index of A, the notation $A(i|\cdot)$ stands for the i-th row of A. Similarly, if j is a column index of A, then $A(\cdot|j)$ is the j-th column of A, and $A(i|j)$ is the entry at the intersection of the i-th row and j-th column of A. The *support* of the j-th column of A is the set of all row indexes $\hat{\imath}$ such that $A(\hat{\imath}|j)$ is a specified entry.

2. Related work

An earlier paper of Woerdeman [31] confirms Conjecture 1.6 for *non-separable bipartite graphs*, that is, for those bipartite graphs that have no induced matchings of more than one edge. This result follows by an explicit formula for the minimum rank of the so called *triangular* partial matrices, which can be equivalently defined as the family subordinate to the non-separable bipartite graphs.

Definition 2.1. A partial matrix T is *triangular* if, for any pair of columns of T, the support of one of these columns is a subset of the support of the other column.

One easy special case of the result in [31] is as follows.

Proposition 2.2. *We have*

$$\operatorname{mr}\begin{pmatrix} A & B \\ C & \star \end{pmatrix} = \operatorname{rk}\begin{pmatrix} A & B \end{pmatrix} + \operatorname{rk}\begin{pmatrix} A \\ C \end{pmatrix} - \operatorname{rk} B,$$

where A, B, C are matrices over $\mathbb{F}$, and $\star$ is a block of non-specified entries.

In fact, an inductive application of Proposition 2.2 results in an explicit formula for the minimum rank of a triangular partial matrix of arbitrary size. Also, this gives a fast algorithm for the minimum rank computation in the triangular case and motivates the notion of the *triangular minimum rank*.

Definition 2.3. A *relaxation* of a partial matrix A is a matrix obtained from A by replacing some family of specified entries of A by the $*$ placeholders.

Definition 2.4. Let A be a partial matrix. The *triangular minimum rank* $\operatorname{tmr}(A)$ is the largest value of $\operatorname{mr}(T)$ over all triangular relaxations T of A.

These considerations motivate the following strengthening of Conjecture 1.6.

Conjecture 2.5 ([10, 11, 12, 13, 15, 16, 17, 21, 22, 25, 32, 33])**.** *If A is a partial matrix subordinate to a chordal bipartite graph, then* $\operatorname{tmr}(A) = \operatorname{mr}(A)$.

If A' is a relaxation of A, then $\operatorname{mr}(A') \leqslant \operatorname{mr}(A)$ immediately by Definition 2.3. Therefore, Definition 2.4 implies $\operatorname{tmr}(A) \leqslant \operatorname{mr}(A)$, and Conjecture 2.5 posits the opposite inequality for partial matrices subordinate to chordal bipartite graphs.

We proceed with a short survey of the work on Conjecture 2.5. As said above, the initial paper [31] confirms it for non-separable bipartite graphs. Cohen, Johnson, Rodman, Woerdeman [10] prove Conjecture 2.5 under an additional assumption $\operatorname{tmr}(A) = 1$. Woerdeman [32] confirms this conjecture for a banded partial matrix A, which appears if the corresponding bipartite graph (U, V, E) admits bijective enumerations $\varphi : \{1, \ldots, n\} \to U$ and $\psi : \{1, \ldots, m\} \to V$ such that the properties

$$i \leqslant k,\ \ j \geqslant l,\ \ \{\varphi_i, \psi_j\} \in E,\ \ \{\varphi_k, \psi_l\} \in E$$

imply $\{\varphi_{\hat\imath}, \psi_{\hat\jmath}\} \in E$ for all $\hat\imath \in \{i, \ldots, k\}$, $\hat\jmath \in \{l, \ldots, j\}$. As explained in [32], this is a strengthening of an earlier result of Bartelt, Johnson, Rodman, Woerdeman [3] that confirmed Conjecture 2.5 for tridiagonal partial matrices, and this includes one more specific case of the validity of Conjecture 2.5 analyzed by Gohberg, Kaashoek, Woerdeman [13]. Rodman [28] computes the rank of a block diagonal partial matrix and shows that the families of all graphs satisfying Conjectures 1.6 and 2.5 are closed under the unions. Bernstein, Blekherman, Sinn [4] prove Conjecture 2.5 for those partial matrices A whose specified entries form a *generic* family in $\mathbb{C}$, and hence the complex number version of Conjecture 2.5 holds for almost all choices of A.

Cohen, Pereira [11] work on the analogue of Conjecture 2.5 for symmetric partial matrices, and they prove the equivalence of this analogue to the original version. Another paper of Cohen, Pereira [12] revisits the proof of the statement converse to Conjecture 1.6 and gives further information on the minimum ranks of partial matrices with the block cyclic structure. Harrison [17] and Johnson, Whitney [22] propose several new concepts and open problems surrounding Conjecture 2.5. Grossmann, Woerdeman [16] introduce the function of the fractional minimal rank of a partial matrix, which is denoted fmr and satisfies $\operatorname{tmr} \leqslant \operatorname{fmr} \leqslant \operatorname{mr}$, and they discuss a version of Conjecture 2.5 relaxed to the condition $\operatorname{fmr}(A) = \operatorname{mr}(A)$ instead of $\operatorname{tmr}(A) = \operatorname{mr}(A)$. Woerdeman [33] formulates another relaxation of Conjecture 2.5 and proves it in a special case. McKee [25] introduces the concept of a biclique comparability graph, which allows him to reformulate Conjecture 2.5 and obtain several further results on the topic. Motivated by Conjectures 1.6 and 2.5, Bakonyi, Bono [2] show that it is possible to add an edge to any vertex of a chordal bipartite graph so that the resulting graph is also chordal bipartite.

3. An outline of the proof

In what follows, we prove Conjecture 2.5 and hence Conjecture 1.6 as well. The forthcoming Section 4 collects several notational conventions, standard results on chordal bipartite graphs and other preliminaries required for our proof, which is presented in the remaining Sections 5 and 6.

Our argument can be seen as a combination of two approaches, which leads to an elaborate inductive argument presented in an infinite descent form in Section 6. One approach, suggested in the paper [4], is based on the analysis of the matrix obtained after a relaxation of a bisimplicial entry or after the removal of the corresponding row or column of a given partial matrix. The other approach was mentioned in [2], and, conversely, it requires the analysis of further specifications of a given matrix.

In Section 5, we collect several observations related to the first approach mentioned above. One of these observations is a straightforward generalization of the Gaussian elimination process to the case of partial matrices, and the others are devoted to situations when the removal of a row or column of a given matrix does not change its tmr and mr ranks or at least does not affect the property of a given matrix to be or not to be a counterexample to Conjecture 2.5.

Section 6 recovers the full generality of our argument and completes the proof. The main technical tool is Claim 6.14, which, essentially, allows one to either

(i) replace a given *strongly bisimplicial* entry (i, j) by a $*$ or

(ii) specify every entry of the $\hat{\jmath}$-th column, whenever $\hat{\jmath} \in \operatorname{Supp} A(i|\cdot)$ and $\hat{\jmath} \neq j$,

without breaking the potential property of a given partial matrix A to be a counterexample to Conjecture 2.5. We use this result to reduce the potential family of counterexamples to matrices having several simple forms presented in Remark 6.9 and discussed in Claims 6.11–6.13, which allows one to conclude the argument.

4. Basic results and notation

This section collects several definitions and preliminary results that we use later in our proof. We recall that we work over an arbitrary ground field $\mathbb{F}$, and an (i, j) entry of a partial matrix A is called *specified* whenever $A(i|j)$ is an element of $\mathbb{F}$. Otherwise, we write $A(i|j) = *$ to state that the (i, j) entry of A is *not specified.*

Definition 4.1. A partial matrix is called *fully specified* if it has no $*$ entry.

Definition 4.2. We write $\operatorname{Supp} v$ to denote the *support* of a partial vector v, which is the set of all indexes pointing to the specified entries of v.

We proceed with some terminology taken from graph theory. We recall that an edge e of a bipartite graph G is *bisimplicial* if the union of the neighborhoods of the endpoints of e induces a complete bipartite subgraph of G, see [4, 14]. A natural correspondence of partial matrices and graphs suggests the following definition.

Definition 4.3. An (i, j) entry of a partial matrix A is *bisimplicial* if the $(\hat{\imath}, \hat{\jmath})$ entry of A is specified for all $\hat{\imath} \in \operatorname{Supp} A(\cdot|j)$ and $\hat{\jmath} \in \operatorname{Supp} A(i|\cdot)$.

Definition 4.4. An (i, j) entry of a partial matrix A is *strongly bisimplicial* if

(1) the (i, j) entry of A is specified,

(2) for any $\hat{\jmath} \in \operatorname{Supp} A(i|\cdot)$, we have $\operatorname{Supp} A(\cdot|j) \subseteq \operatorname{Supp} A(\cdot|\hat{\jmath})$,

(3) the columns of A with indexes in $\operatorname{Supp} A(i|\cdot)$ form a triangular partial matrix.

Remark 4.5. It is clear from Definitions 4.3 and 4.4 that every strongly bisimplicial entry of a given partial matrix is bisimplicial as well.

The following terminology is commonly used in theory of $(0, 1)$ matrices [18, 24].

Definition 4.6. A partial matrix A is *totally balanced* if $A \in M(G)$ for some chordal bipartite graph G, that is, if A is subordinate to a chordal bipartite graph.

Definition 4.7. If a partial matrix B is a relaxation of A in the notation of Definition 2.3, then A is called a *specification* of B.

Remark 4.8. In terms of Definition 4.7, one has $\operatorname{tmr}(B) \leqslant \operatorname{tmr}(A)$, $\operatorname{mr}(B) \leqslant \operatorname{mr}(A)$.

We need several standard results from theory of chordal bipartite graphs.

Proposition 4.9 ([2, 14])**.** *Let A be a totally balanced partial matrix, and let j be one of the column indexes of A. Then the specification of A obtained by replacing every $*$ entry in the j-th column with an element of $\mathbb{F}$ is totally balanced as well.*

Proof. Any induced subgraph of a chordal bipartite graph (U, V, E) is chordal bipartite, and the addition to U of a new vertex whose neighborhood is V also preserves the property of being chordal bipartite [2, 14]. So the removal of the j-th column of A leaves the resulting matrix totally balanced, and the subsequent addition of a fully specified column at the j-th place does not affect this property either. $\square$

Proposition 4.10 ([2, 14])**.** *The relaxation of a totally balanced partial matrix obtained by replacing one of its bisimplicial entries with a $*$ is totally balanced.*

Proof. The removal of a bisimplicial edge from a chordal bipartite graph produces a chordal bipartite graph, see Propositon 2.1 in [2] for a more recent account or Theorem 1, Corollary 5 in [14] for an earlier appearance. $\square$

The following result is much harder, and it is usually proved by a reduction to the so-called Γ-free orderings of totally balanced matrices [19, 24, 30].

Theorem 4.11 ([19, 24, 30])**.** *If a totally balanced partial matrix A contains at least one specified entry, then A contains at least one strongly bisimplicial entry.*

Proof. Every partite set of a chordal bipartite graph is known to contain a vertex v such that the neighborhoods of the neighbors of v form a chain under inclusion, see Theorem 3.2 in [19] or Theorem 1 in [30] for recent references and Theorem 3 in [24] for detailed proofs. In other words, the matrix A in the formulation of the current theorem has a row (its index is denoted with i) such that the supports of the columns of A with the indexes in $\operatorname{Supp} A(i|\cdot)$ form a chain under inclusion. Since A has specified entries, we can assume that this chain is non-empty, and then we take an index $j \in \operatorname{Supp} A(i|\cdot)$ for which $\operatorname{Supp} A(\cdot|j)$ is a subset of any $\operatorname{Supp} A(\cdot|\hat{\jmath})$ with $\hat{\jmath} \in \operatorname{Supp} A(i|\cdot)$, and then the entry (i, j) is strongly bisimplicial. $\square$

5. Further auxiliary results

This section collects several further auxiliary statements which do not seem to appear in existing literature, so they require being given detailed proofs. The following is a natural restriction of Gaussian elimination to partial matrices.

Lemma 5.1. *Let A be a partial matrix satisfying* $\operatorname{Supp} A(\cdot|j) \subseteq \operatorname{Supp} A(\cdot|\hat{\jmath})$ *for some pair $j \neq \hat{\jmath}$ of column indexes. For $\lambda \in \mathbb{F}$, we consider the matrix A' obtained from A by placing the value $A(i|j)+\lambda A(i|\hat{\jmath})$ at every (i, j) entry with $i \in \operatorname{Supp} A(\cdot|j)$. Then* $\operatorname{mr}(A') = \operatorname{mr}(A)$ *and* $\operatorname{tmr}(A') = \operatorname{tmr}(A)$.

Proof. If C is a completion of A, then we define the matrix C' by adding, to the j-th column of C, its $\hat{\jmath}$-th column multiplied by λ. This is an elementary transformation of columns, so we have $\operatorname{rk}(C') = \operatorname{rk}(C)$. The assumptions of the lemma show that C' is a completion of A', and hence $\operatorname{mr}(A') \leqslant \operatorname{mr}(A)$. By switching the roles of A and A', we get $\operatorname{mr}(A) \leqslant \operatorname{mr}(A')$ and conclude that $\operatorname{mr}(A) = \operatorname{mr}(A')$.

In order to check $\operatorname{tmr}(A') = \operatorname{tmr}(A)$, we assume without loss of generality that

$$\min\{\operatorname{tmr}(A'),\, \operatorname{tmr}(A)\} = \operatorname{tmr}(A) \tag{5.1}$$

and use the letter r to denote the quantity in (5.1). Then the matrix A has a triangular relaxation T with $\operatorname{mr}(T) = r$, and we consider two separate cases:

(C1) if $\operatorname{Supp} T(\cdot|j)$ is a subset of $\operatorname{Supp} T(\cdot|\hat{\jmath})$, then we consider the triangular relaxation T' of A' whose specified entries are located at the same positions as those in T. The matrices T and T' are connected by the transformation as in the formulation of the lemma, so we get $\operatorname{mr}(T) = \operatorname{mr}(T') = r$ by the first paragraph of this proof. This implies $\operatorname{tmr}(A') \leqslant r$ and hence $\operatorname{tmr}(A') = \operatorname{tmr}(A) = r$ in view of (5.1);

(C2) if $\operatorname{Supp} T(\cdot|j)$ includes $\operatorname{Supp} T(\cdot|\hat{\jmath})$ as a subset, then we have

$$\operatorname{Supp} T(\cdot|\hat{\jmath}) \subseteq \operatorname{Supp} T(\cdot|j) \subseteq \operatorname{Supp} A(\cdot|\hat{\jmath}),$$

where the second inclusion follows by the assumption of the lemma. Therefore, we can define the relaxation $\overline{T}$ of A as the one having the specified entries precisely at the same locations as T except that the $\hat{\jmath}$-th column of $\overline{T}$ has the support $\operatorname{Supp} T(\cdot|j)$ instead of $\operatorname{Supp} T(\cdot|\hat{\jmath})$. In other words, we changed one of the columns of a triangular partial matrix T to the one whose support equals the support of some other column of T, so the partial matrix $\overline{T}$ is still triangular. Therefore, $\overline{T}$ is a triangular relaxation of A, which implies $\operatorname{mr}(\overline{T}) \leqslant \operatorname{tmr}(A) = r$. Since T is itself a relaxation of $\overline{T}$, we get $\operatorname{mr}(\overline{T}) \geqslant \operatorname{mr}(T) = r$ and hence $\operatorname{mr}(\overline{T}) = r$, which reduces the situation to the already confirmed case (C1).

Since T is triangular, either (C1) or (C2) is true, hence $\operatorname{tmr}(A') = \operatorname{tmr}(A)$. □

One further notational convention is required to proceed.

Remark 5.2. In Lemma 5.3 below and in what follows, the blocks of block structured matrices are not assumed to be actually present unless explicitly stated otherwise. In other words, we allow the blocks to be of the sizes $0 \times n$, $m \times 0$ or 0×0, and if it is the case, these blocks are said to be *void*, and their ranks are assumed to equal zero. A *row vector* and *column vector* are, respectively, the blocks of the sizes $1 \times d$ or $d \times 1$ with $d \geqslant 0$, and whenever $d = 0$ these blocks are thought of as the (zero) elements of the corresponding 0-dimensional vector spaces over $\mathbb{F}$.

We proceed with a situation when the removal of several rows of a given partial matrix does not affect its minimum rank.

Lemma 5.3. *Let A be a partial matrix of the form*

$$\begin{pmatrix} F_1 & \star \\ F_2 & A_1 \end{pmatrix} \tag{5.2}$$

with $\star$ being a block of $*$'s. If F_1, F_2 are fully specified and A_1 is an arbitrary partial matrix, and the columns of F_2 are linearly independent, then $\mathrm{mr}(A) = \mathrm{mr}(F_2|A_1)$.

Proof. Since the columns of F_2 are linearly independent, every row of F_1 is a linear combination of the rows of F_2. According to Lemma 5.1, the replacement of F_1 with the zero matrix does not change the mr rank of A. Therefore, the removal of the rows containing F_1 does not affect these ranks either. □

The following statement gives a situation where the removal of a row of a partial matrix does not affect the difference between its mr and tmr ranks.

Lemma 5.4. *Let A be a partial matrix of the form*

$$\begin{pmatrix} f_1 & \star \\ F_2 & A_1 \end{pmatrix}$$

with f_1 being a fully specified row vector, F_2 being a fully specified matrix, A_1 being an arbitrary partial matrix, and $\star$ being a row vector without any specified entries. Then $\mathrm{mr}(F_2|A_1) - \mathrm{tmr}(F_2|A_1) = \mathrm{mr}(A) - \mathrm{tmr}(A)$.

Proof. If f_1 belongs to the linear span of the rows of F_2, then, as in the previous lemma, the result follows from Lemma 5.1 because the replacement of f_1 with the zero vector does not affect the tmr and mr ranks of A.

If f_1 is not spanned by the rows of F_2, then the elementary transformations of the columns corresponding to the left blocks reduce the situation to the case when A has a column with a 1 in the first row and zeros everywhere else. By Lemma 5.1, these transformations affect the tmr and mr ranks of neither A nor $(F_2|A_1)$, so both tmr and mr reduce exactly by one when passing from A to $(F_2|A_1)$. □

In the following situation, the difference between the mr and tmr ranks of a partial matrix does not reduce with the removal of a row.

Lemma 5.5. *Let A be a partial matrix of the form*

$$\begin{pmatrix} f_1 & s & \star \\ F_2 & f_4 & A_1 \\ F_3 & * & A_2 \end{pmatrix} \tag{5.3}$$

with f_1 and f_4 being fully specified row and column vectors, respectively. Also, we assume that $s \in \mathbb{F}$ is a single specified entry, the F_2, F_3 blocks are fully specified, A_1, A_2 are arbitrary partial matrices, and the $\star$'s stand for the blocks without any specified entry. If f_4 is not spanned by the columns of F_2, then the removal of the first row of A leads to the matrix A' satisfying $\mathrm{mr}(A') - \mathrm{tmr}(A') \geqslant \mathrm{mr}(A) - \mathrm{tmr}(A)$.

Proof. In view of Lemma 5.1, we can assume without loss of generality that the columns corresponding to the leftmost blocks of A are linearly independent. If, nevertheless, the corresponding columns of A' are linearly dependent, then the elementary transformations reduce the situation to the case when A has a column with a 1 in the first row and zeros everywhere else. Similarly to the previous lemma, we conclude that the removal of the first row of A reduces both the mr and tmr ranks by one, so the assertion of the lemma holds true.

Therefore, it remains to consider the case when the columns corresponding to the leftmost blocks of A' are linearly independent. Moreover, for any choice of a fully specified vector v_2 of the appropriate size, we get that

$$\text{the columns of } \begin{pmatrix} F_2 & f_4 \\ F_3 & v_2 \end{pmatrix} \text{ are linearly independent} \tag{5.4}$$

because f_4 is not spanned by the columns of F_2. Now we consider a completion

$$B' = \begin{pmatrix} F_2 & f_4 & U_1 \\ F_3 & u_2 & U_2 \end{pmatrix} \tag{5.5}$$

of the matrix A' realizing its minimum rank, that is, satisfying

$$\operatorname{rk}(B') = \operatorname{mr}(A'). \tag{5.6}$$

Also, we set up the new matrix

$$A'' = \begin{pmatrix} f_1 & s & \star \\ F_2 & f_4 & U_1 \\ F_3 & u_2 & U_2 \end{pmatrix}$$

and get

$$\operatorname{mr}(A) \leqslant \operatorname{mr}(A'') = \operatorname{rk}(B') = \operatorname{mr}(A') \leqslant \operatorname{mr}(A), \tag{5.7}$$

where the first inequality follows because A'' is a specification of A. Further, the first equality in (5.7) is true by Lemma 5.3, which is applicable because of (5.4). The second equality is true by the condition (5.6), and, finally, the last inequality in (5.7) is valid because A' is a submatrix of A. Therefore, all the quantities in (5.7) are equal, which implies $\operatorname{mr}(A') = \operatorname{mr}(A)$ and shows that the $\operatorname{mr} - \operatorname{tmr}$ difference could not have decreased when passing from A to A'. □

We proceed with another situation similar to Lemmas 5.4 and 5.5.

Lemma 5.6. *Let A be a partial matrix of the form*

$$\begin{pmatrix} F_1 & v_1 & \star & \star \\ F_2 & o & o & A_1 \\ F_3 & \star & v_2 & A_2 \end{pmatrix}$$

with F_1, F_2, F_3 being fully specified matrices. Also, we assume that v_1, v_2 are arbitrary partial column vectors, and A_1, A_2 are arbitrary partial matrices. The o's stand for the fully specified vectors of all zeros, and the $\star$'s are matrices with all $*$'s. Then there exists a partial matrix A' with less columns than A such that

$$\operatorname{mr}(A') - \operatorname{tmr}(A') \geqslant \operatorname{mr}(A) - \operatorname{tmr}(A).$$

Proof. Using Lemma 5.4, we remove all those rows of A which contain the $*$ entries of v_1. Therefore, we can assume without loss of generality that

$$v_1 \text{ is a fully specified vector.} \tag{5.8}$$

Similarly, Lemma 5.1 allows us to assume that

$$\text{the columns formed by the leftmost blocks of } A \text{ are linearly independent.} \tag{5.9}$$

Special case. If, nevertheless, the columns in (5.9) become linearly dependent after the removal of the F_1 block, then the elementary transformations reduce the situation to the case when A has a column with several non-zeros in the F_1 block and with all zeros at the F_2 and F_3 blocks. Then we use Lemma 5.1 to perform the Gaussian elimination on the rows of the $(F_1|v_1)$ block, which is fully specified by (5.8), and hence we end up with a matrix containing a column one of whose entries is nonzero and all the others are zeros. The removal of this column and the row containing its nonzero entry reduces both the tmr and mr ranks of A exactly by one, which implies the desired assertion.

Now we can focus on the situation when the assumption of our special case is false, and, in fact, we can further assume that

$$\text{the columns of } \begin{pmatrix} F_2 & o \\ F_3 & v_2 \end{pmatrix} \text{ are linearly independent} \tag{5.10}$$

because otherwise, in view of Lemma 5.1, the column of A containing the v_2 block can be removed without changing the ranks. We are going to finalize the proof similarly to the previous lemma, so we set up the new matrices

$$A' = \begin{pmatrix} F_2 & o & A_1 \\ F_3 & v_2 & A_2 \end{pmatrix} \text{ and } A'' = \begin{pmatrix} F_1 & v_1 & \star & \star \\ F_2 & o & o & A_1 \\ F_3 & v_2 & v_2 & A_2 \end{pmatrix}$$

and get

$$\operatorname{mr}(A) \leqslant \operatorname{mr}(A'') = \operatorname{mr}(A') \leqslant \operatorname{mr}(A), \tag{5.11}$$

where the first inequality follows because A'' is a specification of A. The equality in (5.11) is true by Lemma 5.3, which is applicable because of (5.10). Finally, the last inequality in (5.11) is valid because A' is a submatrix of A. Therefore, all the quantities in (5.11) are equal, which implies $\operatorname{mr}(A') = \operatorname{mr}(A)$ and shows that the $\operatorname{mr} - \operatorname{tmr}$ difference could not have decreased when passing from A to A'. □

We finalize the section with one observation on triangular partial matrices.

Definition 5.7. An $m \times r$ partial matrix A has the *full rank property* if $\operatorname{mr}(A) = r$.

Lemma 5.8. *Let $A = (T|A')$ be a block partial matrix in which the T block is triangular. If T does not possess the full rank property, then one of the columns of T can be removed from A without reducing the* tmr *and* mr *ranks of A.*

Proof. If T has one column, then the lack of the full rank property implies that all specified entries of T are zeros, which immediately shows that this column does not affect the rank of A. Now we assume that T has $r > 1$ columns and proceed by the induction on r. In particular, we can assume without loss of generality that

(5.12) the matrix formed by any $r - 1$ columns of T has the full rank property.

Since T is triangular, we can find an index j so that the support s of the j-th column of T is a subset of the support of any other column of T. For the submatrix T' formed by the rows of T with the indexes in s, we have the two options:

- if the j-th column of T' is a linear combination of the other columns of T', then Lemma 5.1 shows that the mr and tmr ranks of A do not change if all entries of the j-th column get replaced by zeros. Therefore, the removal of the j-th column does not affect the tmr and mr ranks of A as desired;
- if the j-th column of T' does not belong to the linear span of the other columns of T', then the j-th column of any completion C of T cannot be spanned by the other columns of C. Therefore, the rank of C reduces by one with the removal of the j-th column, and hence the condition (5.12)

implies $\operatorname{rk} C = (r-1)+1 = r$, which corresponds to the full rank property of T and contradicts the assumptions of the lemma.

These options cover all possibilities, so the proof is complete. $\square$

6. The proof

We proceed with the proof of Conjecture 2.5. Our argument goes by infinite descent using the following weight function on the set of all partial matrices.

Definition 6.1. If A is a partial matrix, then $\omega(A)$ is the triple (w_1, w_2, w_3), where

(W1) w_1 is the total number of columns of A,
(W2) w_2 is the number of those columns of A which have at least one $*$ entry,
(W3) w_3 is the total number of specified entries in A.

Definition 6.2. We work with the lexicographic ordering of the set of weights of partial matrices. Namely, we write $(u_1, u_2, u_3) \leqslant (w_1, w_2, w_3)$ if and only if

$$(u_1 < w_1) \text{ OR } (u_1 = w_1) \wedge (u_2 < w_2) \text{ OR } (u_1 = w_1) \wedge (u_2 = w_2) \wedge (u_3 \leqslant w_3).$$

Our argument proceeds by contradiction.

Remark 6.3. In what follows, we assume that Conjecture 2.5 is false.

Since the relation in Definition 6.2 is a well ordering [23, page 20], the set of counterexamples to Conjecture 2.5 admits an element with minimal possible weight.

Definition 6.4. In what follows, $\mathcal{A}$ is a totally balanced partial matrix such that

(M1) $\operatorname{tmr}(\mathcal{A}) < \operatorname{mr}(\mathcal{A})$,
(M2) for any totally balanced A' with $\operatorname{tmr}(A') < \operatorname{mr}(A')$, we have $\omega(\mathcal{A}) \leqslant \omega(A')$.

We need several further auxiliary definitions in order to proceed.

Definition 6.5. A specified entry (i, j) of a partial matrix A is *distinguished* if

(D1) the j-th column of A is not fully specified, and
(D2) all other specified entries of the i-th row of A lie in fully specified columns.

Definition 6.6. A specified entry (i, j) of a partial matrix A is *regular* if

(R1) it is not distinguished, and
(R2) the j-th column of A is not fully specified.

Therefore, all specified entries of a partial matrix A are split into three disjoint types, (1) the distinguished ones, (2) the regular ones, (3) those in fully specified columns. We need several further related definitions.

Definition 6.7. The j-th column of a partial matrix A is *distinguished* if it contains at least one distinguished entry.

Definition 6.8. The j-th column of a partial matrix A is *ordinary* if it is neither fully specified nor distinguished.

Similarly, all columns of a partial matrix A are split into three disjoint types, (1) the distinguished ones, (2) the ordinary ones, (3) the fully specified ones.

Remark 6.9. In other words, we represent $\mathcal{A}$ as

$$\mathcal{A} = \begin{pmatrix} F_1 & D & \star \\ F_2 & R_1 & R_2 \end{pmatrix}$$

in which the columns corresponding to F_1 are fully specified, those corresponding to D are distinguished, the remaining columns are ordinary, and a specified entry e of $\mathcal{A}$ is distinguished if and only if e is a specified entry of the D block.

One more related notion is in order.

Definition 6.10. If the j-th column of A is not fully specified, then the *regular support* of this column is the set of all i such that the (i, j) entry of A is regular.

Now we are ready to proceed with the core of the proof, which takes the rest of this section and is separated into several claims.

Claim 6.11. *Let (i, j) be a distinguished entry of $\mathcal{A}$. Then we can obtain another matrix satisfying the assumptions of Definition 6.4 by subsequently*

(1) replacing every regular entry of the j-th column of $\mathcal{A}$ with a zero,

(2) assigning an appropriate value in $\mathbb{F}$ to every distinguished entry of the j-th column of $\mathcal{A}$ (that is, this value may or may not equal the initial one).

Proof. Let ρ be the regular support of the j-th column of $\mathcal{A}$. If the restriction of this j-th column to ρ belongs to the linear span of the corresponding restrictions of the fully specified columns of $\mathcal{A}$, then the result follows from Lemma 5.1. Otherwise, we get a contradiction to the minimality of $\mathcal{A}$ due to Lemma 5.5. $\square$

Claim 6.12. *At least one column of $\mathcal{A}$ is ordinary.*

Proof. Otherwise, the representation in Remark 6.9 transforms into

$$\mathcal{A} = \begin{pmatrix} F_1 & D \\ F_2 & Z \end{pmatrix}$$

with F_1 and F_2 being fully specified, where D is a matrix whose specified entries are all distinguished, and Z is a matrix whose specified entries are all regular. Claim 6.11 allows us to assume that every specified entry of Z is zero, so we get

$$\operatorname{rk}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \leqslant \operatorname{mr}(\mathcal{A}) \leqslant \operatorname{rk}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} + 1 \tag{6.1}$$

because $\operatorname{mr}(D) \leqslant 1$ and $\operatorname{mr}(Z) = 0$. If the correct value of $\operatorname{mr}(\mathcal{A})$ corresponds to the lower bound in (6.1), then the fully specified columns of $\mathcal{A}$ constitute a triangular relaxation realizing the minimum rank of $\mathcal{A}$, which implies $\operatorname{tmr}(A) = \operatorname{mr}(A)$ and contradicts to the condition (M1) in Definition 6.4. Otherwise, we use Lemma 5.1 and conclude that $\mathcal{A}$ should have a distinguished column c whose restriction to $\operatorname{Supp} c$ does not belong to the linear span of the corresponding restrictions of the fully specified columns of $\mathcal{A}$. In this case, the matrix formed by the fully specified columns of $\mathcal{A}$ together with c gives a triangular relaxation realizing the minimum rank of $\mathcal{A}$, which gives a similar contradiction to Definition 6.4. $\square$

Claim 6.13. *At least three columns of $\mathcal{A}$ are not fully specified.*

Proof. If $\mathcal{A}$ has at most two columns that are not fully specified, then $\mathcal{A}$ is either triangular or has no ordinary columns, so the result follows from Claim 6.12. $\square$

Now we are ready to overcome the main technical difficulty of our proof. We recall that we use the convention of Remark 5.2, and any line of the blocks in the partition (6.2) in Claim 6.14 below can be void unless it follows from the assumptions that the corresponding line is present. More precisely, we assume that the leftmost line of blocks is exactly one column because it is stated that f_0 and f_2 are column vectors, and other conditions forcing several lines of blocks to be present are marked (i) and (ii) in the formulation below.

Claim 6.14. *Suppose that $\mathcal{A}$ has the form*

$$\begin{pmatrix} f_0 & F_1 & \star & \star \\ f_2 & F_3 & F_4 & A_1 \\ \star & A_2 & A_3 & A_4 \end{pmatrix} \tag{6.2}$$

with column vectors f_0, f_2 and matrices F_1, F_3, F_4 all being fully specified. The A_1, A_2, A_3, A_4 blocks are arbitrary partial matrices, and the $\star$'s are the blocks of all $*$'s of appropriate sizes. Suppose that

(i) the upper line of blocks in (6.2) *is not void, which means, in other words, that the f_0 and F_1 blocks have at least one row each,*

(ii) the rightmost line of blocks in (6.2) *is not void, which means, in other words, that the A_1 and A_4 blocks have at least one column each.*

Additionally, we assume that the removal from (6.2) *of the rightmost line of blocks as in (ii) leaves a matrix that possesses the full rank property. Then*

(1) the F_4 and A_3 blocks have no columns,

(2) the A_2 block is fully specified.

Proof. In the proof below, we assume $\operatorname{tmr}(\mathcal{A}) = r$.

It is clear from (6.2) that every entry of the f_0 block is bisimplicial, so the matrix obtained after the replacement of one of these entries by a $*$ remains totally balanced by Proposition 4.10. In fact, the subsequent replacement of every other entry of f_0 with a $*$ still leaves the resulting matrix A' totally balanced[1]. Also, if the bottom line of blocks in (6.2) was void, then Lemma 5.4 would allow us to remove the upper line of blocks as well, which would give an immediate contradiction to the minimality of $\mathcal{A}$ because of the assumption (i) of the current claim. Therefore, the passing from $\mathcal{A}$ to A' changes neither the total number of columns nor the number of fully specified columns of the matrix, and at the same time the total number of specified entries reduces because of the assumption (i). This implies $\omega(A') < \omega(\mathcal{A})$ by Definition 6.2, and hence we get $\operatorname{mr}(A') = \operatorname{tmr}(A')$ by the minimality of $\mathcal{A}$. Since A' is a relaxation of $\mathcal{A}$, we have $\operatorname{tmr}(A') \leqslant \operatorname{tmr}(\mathcal{A}) = r$ and consequently $\operatorname{mr}(A') \leqslant r$, which means that there exists a fully specified matrix

$$B' = \begin{pmatrix} \varphi_0 & F_1 & \beta_1 & \beta_2 \\ f_2 & F_3 & F_4 & \Phi_1 \\ \beta_3 & \Phi_2 & \Phi_3 & \Phi_4 \end{pmatrix}$$

with $\operatorname{rk}(B') = r$ which has the same block structure as (6.2) and agrees with every specified entry of $\mathcal{A}$ except possibly several entries in the upper left block. We set up the new matrix

$$A'' = \begin{pmatrix} f_0 & F_1 & \beta_1 & \star \\ f_2 & F_3 & F_4 & A_1 \\ \star & \Phi_2 & \Phi_3 & A_4 \end{pmatrix}$$

[1]This can be observed either by the induction on the length of f_0 or because the duplication of some vertex of a given chordal bipartite graph leaves it chordal bipartite.

which is a specification of $\mathcal{A}$. We remark that the equality $A'' = \mathcal{A}$ is possible only if the assertion of the claim is true; our strategy to complete the proof is now to reach a contradiction from $A'' \neq \mathcal{A}$. Indeed, the condition $A'' \neq \mathcal{A}$ implies

$$\omega(A'') < \omega(\mathcal{A}) \tag{6.3}$$

because A'' has more fully specified columns than $\mathcal{A}$. Also, the matrix A'' is totally balanced by Proposition 4.9, which, together with (6.3) and Definition 6.4, implies that $\operatorname{tmr}(A'') = \operatorname{mr}(A'')$. Since A'' is a specification of $\mathcal{A}$, we get $\operatorname{mr}(A'') \geqslant \operatorname{mr}(\mathcal{A}) > \operatorname{tmr}(\mathcal{A}) = r$. Therefore, the matrix A'' admits a triangular relaxation[2]

$$T'' = \begin{pmatrix} f_0' & F_1 & \beta_1 & \star \\ f_2' & F_3 & F_4 & T_1 \\ \star & \Phi_2 & \Phi_3 & T_4 \end{pmatrix} \tag{6.4}$$

of the minimal rank at least $r+1$. If f_0' has all entries $*$'s, then T'' is a relaxation of B' and hence $\operatorname{mr}(T'') \leqslant \operatorname{rk}(B') = r$, which contradicts to the previous sentence. Therefore, the upper left block of (6.4) contains at least one specified entry, which forces T_4 to be the matrix of all $*$'s because T'' is triangular. In fact, this allows us to further assume $f_0' = f_0$ and $f_2' = f_2$ because the corresponding specification does not break the triangular form of T''. So we have

$$T'' = \begin{pmatrix} f_0 & F_1 & \beta_1 & \star \\ f_2 & F_3 & F_4 & T_1 \\ \star & \Phi_2 & \Phi_3 & \star \end{pmatrix} \tag{6.5}$$

without loss of generality.

Now we go back to the assumptions of the current claim and recall that the matrix $\overline{A}$ obtained from $\mathcal{A}$ by the removal of the rightmost line of blocks has the full rank property. By the minimality of $\mathcal{A}$, the matrix $\overline{A}$ has a triangular relaxation

$$\overline{B} = \begin{pmatrix} \psi_0 & \Psi_1 & \star \\ f_2 & F_3 & F_4 \\ \star & \Psi_2 & \Psi_3 \end{pmatrix}$$

that still has the full rank property. Now we see that the matrix

$$\mathcal{B} = \begin{pmatrix} \psi_0 & \Psi_1 & \star & \star \\ f_2 & F_3 & F_4 & T_1 \\ \star & \Psi_2 & \Psi_3 & \star \end{pmatrix}$$

is a triangular relaxation of $\mathcal{A}$. Since $(f_2|F_3|F_4)$ is fully specified, the minimum rank of $\mathcal{B}$ is computed with the use of Proposition 2.2. We obtain

$$\operatorname{mr}(\mathcal{B}) = \operatorname{mr}(\overline{B}) + \operatorname{mr}(f_2|F_3|F_4|T_1) - \operatorname{rk}(f_2|F_3|F_4). \tag{6.6}$$

The full rank property of $\overline{B}$ guarantees that its minimum rank cannot change with taking any specification, which implies

$$\operatorname{mr}(\overline{B}) = \operatorname{mr} \begin{pmatrix} f_0 & F_1 & \beta_1 \\ f_2 & F_3 & F_4 \\ \star & \Phi_2 & \Phi_3 \end{pmatrix} \tag{6.7}$$

[2]The two middle vertical lines of blocks in (6.4) are taken fully specified because their full specification does neither reduce the rank nor break the triangular structure.

and hence, in view of Proposition 2.2, a comparison of (6.5), (6.6), (6.7) shows that $\operatorname{mr}(\mathcal{B}) = \operatorname{mr}(T'')$. Since we had $\operatorname{mr}(T'') > r$, this implies $\operatorname{mr}(\mathcal{B}) > r = \operatorname{tmr}(\mathcal{A})$, which is the desired contradiction because $\mathcal{B}$ is a triangular relaxation of $\mathcal{A}$. □

We proceed with two important corollaries of the results above.

Claim 6.15. *A strongly bisimplicial entry of $\mathcal{A}$ cannot be regular.*

Proof. According to Definition 4.4, we can represent $\mathcal{A}$ in the form

$$\begin{pmatrix} s & f_1 & \star \\ f_2 & F_3 & A_1 \\ \star & A_2 & A_4 \end{pmatrix} \tag{6.8}$$

with s being the current strongly bisimplicial entry, and where the vectors f_1, f_2 are fully specified, the matrix F_3 is fully specified, and A_2 is triangular. The matrices A_1 and A_4 are arbitrary, and we can assume that the columns of the A_4 block are not void because otherwise $\mathcal{A}$ would be triangular in contrary to Definition 6.4. Also, the matrix obtained from (6.8) by removing the columns involving A_4 has the full rank property by Lemma 5.8. Therefore, we can apply Claim 6.14 (assuming that the columns corresponding to F_4 and A_3 in (6.2) are void), and hence we see that A_2 is fully specified, so the upper left entry in (6.8) is distinguished. □

Claim 6.16. *Let j be the index of some distinguished column of $\mathcal{A}$, and let s be the regular support of this j-th column of $\mathcal{A}$. Then, for any column index $\hat{\jmath} \neq j$, the condition $s \subseteq \operatorname{Supp} A(\cdot|\hat{\jmath})$ implies that the $\hat{\jmath}$-th column of $\mathcal{A}$ is fully specified.*

Proof. If this is not the case, the matrix $\mathcal{A}$ represents as

$$\begin{pmatrix} F_1 & v_1 & \star & \star \\ F_2 & f_4 & f_5 & A_1 \\ F_3 & \star & v_2 & A_2 \end{pmatrix} \tag{6.9}$$

with the fully specified columns separated in the leftmost blocks, and where the second and third columns of blocks are the j-th and $\hat{\jmath}$-th columns, respectively. The upper line of rows represents the set of all rows which contain the distinguished entries of the j-th column, and the middle line of rows is the regular support s as in the formulation of the claim. We also see from these assumptions that f_4 and f_5 are fully specified. According to Claim 6.11, we can assume that f_4 is a zero vector, and we consider the two possible options for f_5 separately:

- if f_5 is a linear combination of the columns of F_2, then Lemma 5.1 allows us to assume that f_5 is a zero vector. So we get under the assumptions of Lemma 5.6 and obtain a contradiction to the minimality of $\mathcal{A}$;
- if f_5 is not a linear combination of the columns of F_2, then the matrix obtained from (6.9) by removing its rightmost blocks has the full rank property. In order to proceed with the use of Claim 6.14, we swap the two leftmost blocks of the columns in (6.9), which makes $\mathcal{A}$ take the form

$$\begin{pmatrix} v_1 & F_1 & \star & \star \\ f_4 & F_2 & f_5 & A_1 \\ \star & F_3 & v_2 & A_2 \end{pmatrix} \tag{6.10}$$

which corresponds to the block partition in (6.2). The application of Claim 6.14 is possible because the rightmost blocks of (6.10) are not void by Claim 6.13, and, as said above, their removal leaves the matrix that possesses the full rank property. It remains to note that, in fact, the use of Claim 6.14 leads to an immediate contradiction because the $\hat{\jmath}$-th row of $\mathcal{A}$ shows that the conclusion (1) of Claim 6.14 cannot be satisfied.

Both cases lead to contradictions, so the proof is complete. □

We proceed the argument and take the representation

$$\mathcal{A} = \begin{pmatrix} F_1 & D & \star \\ F_2 & R_1 & R_2 \end{pmatrix}$$

as in Remark 6.9. By Claim 6.12, the matrix $(R_1|R_2)$ has at least one specified entry, so we apply Theorem 4.11 to find a strongly bisimplicial entry (i, j) in $(R_1|R_2)$.

Also, we write J to denote the support of the i-th row of $(R_1|R_2)$, and the notation $s_{\hat{\jmath}}$ is to stand for the regular support of the $\hat{\jmath}$-th column of $\mathcal{A}$. Let $\delta \in J$ be a distinguished column index of $\mathcal{A}$ (we remark that such a δ may or may not actually exist). According to Claim 6.16, the support of no other column of $(R_1|R_2)$ can contain s_δ, which means, since (i, j) is strongly bisimplicial, that

$$s_{\hat{\jmath}} \subseteq s_\delta$$

for all $\hat{\jmath} \in J$. In particular, this means that the above mentioned choice of δ, if possible, is unique, and the columns with indexes in J form a triangular submatrix of $\mathcal{A}$. Therefore, the entry (i, j) is strongly bisimplicial relative to the whole matrix $\mathcal{A}$ as well, so we see it from Claim 6.15 that (i, j) cannot be regular, but in fact it should be regular because (i, j) is taken in $(R_1|R_2)$. This contradiction shows that the assumption in Remark 6.3 is false and concludes the argument.

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