Notes on Quaternions

Joseph Palazzo

Abstract

This paper is in answer to Roger Penrose's highly acclaimed book, The Road to Reality [1]*, in particular section 11.3 on the geometry of quaternions. Instead of a muddled and perplexing exposition by the notable physicist, we offer a very simple explanation.*

We leave to the reader to find her way into Penrose's book only to single out his remarks at the beginning of chapter 11, that on 16 October 1843, the great physicist William Rowan Hamilton was "… one who puzzled long and deeply over this matter… the answer came to him, and he was so excited by his discovery that he immediately carved his fundamental equations

$$i^2 = j^2 = k^2 = ijk = -1$$

on a stone of Dublin's Brougham Bridge."

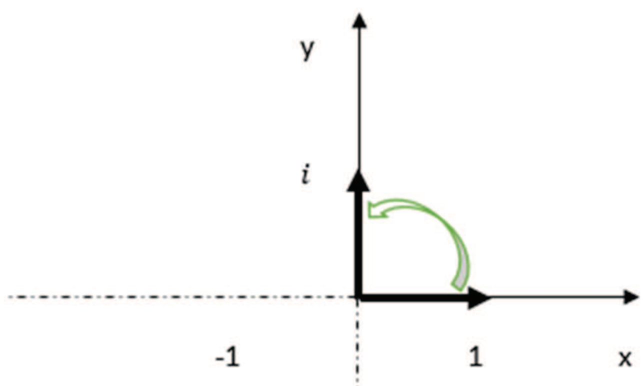
We begin with the complex number designated as,

$$z = x + y\boldsymbol{i}$$

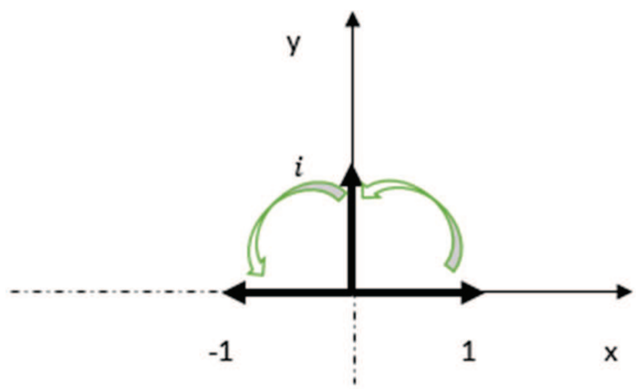
Where $\boldsymbol{i}^2 = -1$

First let us look at complex numbers and how the imaginary number can be conceived on a 2D Cartesian plane, often referred as an Argand diagram. To illustrate its geometric interpretation, we will break up its diagram into two parts (A) and (B).

(A) We start with a unit vector lying on the x-axis. We then rotate this unit vector onto the y-axis (a counter-clockwise rotation = π/2), where it picks up a factor of i.



(B) Then we rotate the unit vector again to the x-axis by another counter-clockwise π/2, (total rotation = π), where it picks up another i, so $\boldsymbol{i}^2 = -1$.



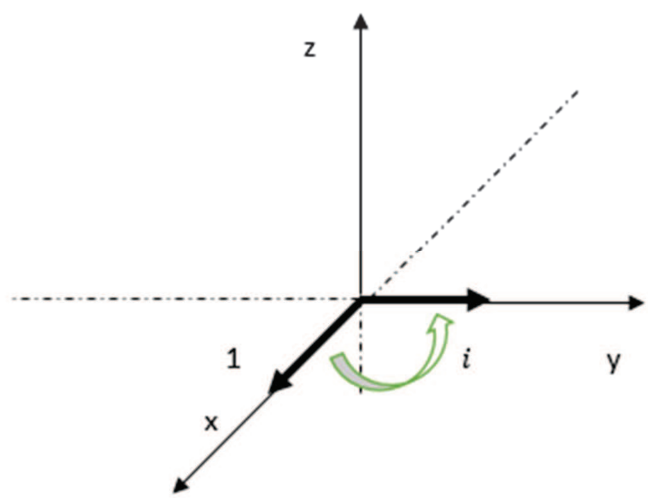
Ignoring the rotations we have described, the same diagram can also be interpreted as a reflection of the unit vector on the x-axis about the y-axis, where upon that action, the unit vector becomes -1.

Now a quaternion is defined as,

$$q = t + u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}$$

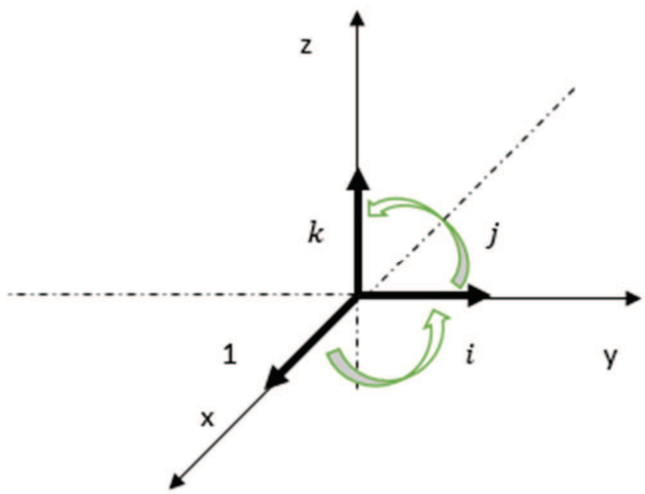
Where $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$

(C) We extend the Argand diagram to 3D, again with a unit vector lying on the x-axis.



We proceed in the same fashion: first a counter-clockwise rotation π/2 of the unit vector onto the y-axis, where it picks up an $\boldsymbol{i}$.

(D) We then make a π/2 counter-clockwise rotation of the unit vector onto the z-axis, where it now picks up a $\boldsymbol{j}$.



Note: $\boldsymbol{ij} = \boldsymbol{k}$

Some identities about quaternions, which are useful (keep track of the order of multiplication):

(1) Repeating the above: $\boldsymbol{ij} = \boldsymbol{k}$

(2) Multiply (1) throughout by $\boldsymbol{k}$: $\boldsymbol{ijk} = \boldsymbol{k}^2 = -\mathbf{1}$

(3) Multiply (1) throughout by $\boldsymbol{j}$: $\boldsymbol{ijj} = \boldsymbol{kj} \rightarrow \boldsymbol{i} = -\boldsymbol{kj}$

(4) Multiply (1) throughout by $\boldsymbol{i}$: $\boldsymbol{iij} = \boldsymbol{ik} \rightarrow \boldsymbol{j} = -\boldsymbol{ik}$

(5) Consider $\boldsymbol{ji} = (-\boldsymbol{ik})(-\boldsymbol{kj}) = -\boldsymbol{ij}$

We see that we have anti-commutation relationship between the quaternions. The others are: $\boldsymbol{ik} = -\boldsymbol{ki}$ and $\boldsymbol{jk} = -\boldsymbol{kj}$ **.**

Another way to remember these relations is to consider cyclic permutations of $\boldsymbol{ijk}$**.**

(i) $\boldsymbol{ijk}$ can be read as $\boldsymbol{i} = \boldsymbol{jk}$ or $\boldsymbol{ij} = \boldsymbol{k}$

(ii) Similarly $\boldsymbol{kij}$ as $\boldsymbol{k} = \boldsymbol{ij}$ or $\boldsymbol{ki} = \boldsymbol{j}$

(iii) And $\boldsymbol{jki}$ as $\boldsymbol{j} = \boldsymbol{ki}$ or $\boldsymbol{jk} = \boldsymbol{i}$

Reference:

[1] *Penrose R., (2004) The Road to Reality, Alfred A. Knopf, New York.*