Solution of the Diophantine Brocard equation

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Abstract

It is shown that Brocard's Diophantine equation can have a solution only if the factorial is represented as a product of two natural numbers that differ by 2, or as a product of four consecutive natural numbers. Then a theorem was proven stating that the product of m consecutive natural numbers cannot be represented as a product of two natural numbers differing by 2 if m≠4. After this, it was proven that it is impossible to represent a factorial greater than 7! in the form of the product of four consecutive natural numbers and two natural numbers differing by 2, it follows that Brocard's equation has no solutions, with the exception of the well-known three factorials.

1 Introduction

The Diophantine Brocard equation is a mathematical problem in which it is required to find integer values of n and m for which $n! + 1 = m^2$. This mathematical problem was formulated by Henri Brocard in two papers in 1876 and 1885 [1, 2]. Later in 1913 this problem was reintroduced by Srinivasa Ramanuja [3, 4] because he did not know about the papers of Henri Brocard. To date, only three solutions of the Brocard equation are known:

$4! + 1 = 5^2,\ 5! + 1 = 11^2$ and $7! + 1 = 71^2$.

2 Representations of factorial

2.1 Representation of factorial as a product of two natural numbers differing by 2

Since Diophantine Brocard–Ramanujan equation is related to the square of a natural number, we first show the representation of the square of a natural number.

Lemma 2.1. The square of any natural number greater than 1 is expressed as a product of two natural numbers differing by 2 according to the formula

(2.1) $$a^2 = (a-1)(a+1) + 1.$$

Lemma 1 implies that the solution of the Brocard–Ramanujan problem exists only in the case when

(2.2) $$n! = (a-1)(a+1).$$

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Further, if we accept the designation $a - 1 = b$, then equation (2.2) will look like

(2.3) $$n! = b(b + 2).$$

Note: if $n! = b(b + 2)$, then $(b + 1)^2 - 1 = n!$

On the basis of equation (2.3), we can assert that to obtain the Brocard–Ramanujan equality $n! + 1 = m^2$, the factorial must be equal to the product of two natural numbers differing by 2. Based on Lemma 1 and equality (2.3), we formulate the following Lemma.

Lemma 2.2. If there is a natural number b such that $n! = b(b + 2)$, then the equality $n! + 1 = (b + 1)^2$ will certainly be obtained.

Note that only the above three factorials, which are solutions of the Diophantine Brocard–Ramanujan equation, have a representation in the form of equality (2.3):

I) $4! = 4 \cdot (4 + 2)$; II) $5! = 10 \cdot (10 + 2)$; III) $7! = 70 \cdot (70 + 2)$.

2.2 Representation of factorial as a product of four consecutive natural numbers

All three factorials 4!, 5!, 7!, which are solutions to the Brocard problem, can be represented as four consecutive natural factors

$4! = (4 - 3) \cdot (4 - 2) \cdot (4 - 1) \cdot 4 = 1 \cdot 2 \cdot 3 \cdot 4 = 24;$

$5! = (5 - 3) \cdot (5 - 2) \cdot (5 - 1) \cdot 5 = 2 \cdot 3 \cdot 4 \cdot 5 = 120.$

$7! = 7 \cdot (7 + 1) \cdot (7 + 2) \cdot (7 + 3) = 7 \cdot 8 \cdot 9 \cdot 10 = 5040.$

For 4!, 5!, the representation in the form of four consecutive natural factors is formed in a natural way, and for 7! this representation is created by rearranging the prime factors of 5040.

3. Main theorems

Considering that the factorial is the product of successive natural numbers, we conducted research into the laws of the product of successive natural numbers. Based on the results of a study of the laws of the product of successive natural numbers, the author formulated the following theorem.

Theorem 3.1. The product of *m* consecutive natural numbers cannot be represented as the product of two natural numbers differing by 2 if $m \neq 4$, except in rare cases when the product of m consecutive natural numbers has a representation as the product of four consecutive natural numbers.

Before proving Theorem 3.1, we first prove the following theorem.

Theorem 3.2. The product of any four consecutive natural numbers is represented as the product of two natural numbers that differ by 2.

To prove Theorem 3.2, consider a natural number represented as a product of four consecutive natural numbers, then group the first factor of the natural number with its fourth factor, and group the middle factors together, then open the brackets, then we get

(3.1) $a(a+1)(a+2)(a+3) = (a^2+3a)(a^2+2a+a+2) = (a^2+3a)(a^2+3a+2).$

Further, taking the notation $a^2+3a = b,$ we present equation (3.1) in the form

(3.2) $a(a+1)(a+2)(a+3) = b(b+2).$

As can be seen from equations (3.1) and (3.2), we have obtained a natural number represented as the product of two natural numbers that differ by 2, which means Theorem 3.2 is correct and it has been proven.

Further, taking into account that the product of consecutive natural numbers, the number of which is greater than 4 (m>4), can be represented as a product of four consecutive natural numbers and a natural number, representable as a product of one or more natural numbers, which is a continuation of four consecutive natural numbers, let's write the following identity

(3.3) $a(a+1)(a+2)(a+3)[(a+4)(a+5)\cdot\ldots\cdot(a+e)] = b(b+1)(b+2)(b+3),$

where $a, b, e \in \mathbb{N}$.

From here we get

(3.4) $\frac{b(b+1)(b+2)(b+3)}{a(a+1)(a+2)(a+3)} = (a+4)(a+5)\cdot\ldots\cdot(a+e).$

To prove Theorem 3.1, we represent the right side of equation (3.3) in the form

(3.5) $b(b+1)(b+2)(b+3) = (a+d)(a+1+d)(a+2+d)(a+3+d),$ где $d = 0, 1, 2, .$

Then equation (3.4) taking into account (3.5) will have the following form

(3.6) $\frac{(a+d)(a+1+d)(a+2+d)(a+3+d)}{a(a+1)(a+2)(a+3)} = (a+4)(a+5)\cdot \ldots \cdot (a+e)$.

Equation (3.6) means that if a natural number, representable as the product of consecutive natural numbers, the number of which is greater than 4, also has a representation as the product of four consecutive natural numbers, then when dividing it by a natural number, representable as the product of four consecutive natural numbers, the result of division must be representable as a product of successive natural numbers, which are a continuation of successive factors of the divisor; in other cases, for *m>4* and natural numbers *a* and *d*, we obtain the following inequality,

(3.7) $\frac{(a+d)(a+1+d)(a+2+d)(a+3+d)}{a(a+1)(a+2)(a+3)} \neq (a+4)(a+5)\cdot \ldots \cdot (a+e)$.

To prove the above statement, namely the validity of inequality (3.7), consider two cases $1 \le d \le 3$ and $3 < d$.

Case $1 \le d \le 3$.

For $d = 1$ we have

$$\frac{(a+1)(a+1+1)(a+2+1)(a+3+1)}{a(a+1)(a+2)(a+3)} = \frac{(a+1)(a+2)(a+3)(a+4)}{a(a+1)(a+2)(a+3)} = \frac{(a+4)}{a} < (a+4).$$

For $d = 2$ we have

$$\frac{(a+2)(a+1+2)(a+2+2)(a+3+2)}{a(a+1)(a+2)(a+3)} = \frac{(a+2)(a+3)(a+4)(a+5)}{a(a+1)(a+2)(a+3)} = \frac{(a+4)(a+5)}{a(a+1)} \neq (a+4).$$

It is clear that the identity $\frac{(a+4)(a+5)}{a(a+1)} = (a+4)$ has a natural solution if $a(a+1) = a+5$, but $a(a+1) \neq a+5$, since $a^2 + a \neq a+5;\ a^2 \neq 5$.

For $d = 3$ we have

$$\frac{(a+3)(a+1+3)(a+2+3)(a+3+3)}{a(a+1)(a+2)(a+3)} = \frac{(a+3)(a+4)(a+5)(a+6)}{a(a+1)(a+2)(a+3)} = \frac{(a+4)(a+5)(a+6)}{a(a+1)(a+2)}.$$

The result of division must be the product of consecutive natural numbers, so we will consider the following two cases:

1) If $\frac{(a+4)(a+5)(a+6)}{a(a+1)(a+2)} = a+4$ then $a(a+1)(a+2) = (a+5)(a+6)$, taking this into account, transforming both sides of the identity,

$a(a+1)(a+2) = (a^2+a)(a+2) = a^3+2a^2+a^2+2a = a^3+3a^2+2a;$

$(a+5)(a+6) = a^2+6a+5a+30 = a^2+11a+30,$

we get

$a^3+3a^2+2a = a^2+11a+30$ or $a^3+2a^2-9a \neq 30.$

2) If $\frac{(a+4)(a+5)(a+6)}{a(a+1)(a+2)} = (a+4)(a+5)$ then $a(a+1)(a+2) = (a+6).$

However, $a(a+1)(a+2) \neq a+6$, since

$(a^2+a)(a+2) = a^3+2a^2+a^2+2a = a^3+3a^2+2a;$

$a^3+3a^2+2a \neq a+6;\ a^3+3a^2+a \neq 6.$

Thus, equation (3.3) does not have a natural solution if $0 \leq d \leq 3$.

Note that the above cases $d=1, d=2, d=3,$ correspond to natural numbers representable as a product of 5, 6 and 7 natural numbers, respectively.

Case d > 3.

For $d=4$ based on (3.3) we write

(3.8) $a(a+1)(a+2)(a+3)(a+4)(a+5)(a+6)(a+7) = b(b+1)(b+2)(b+3).$

As can be seen from (3.8), the case $d=4$ corresponds to a natural number representable as a product of 8 consecutive natural numbers.

Next, we will group the factors of the left side (3.8) by two numbers in order to represent the left side as a product of four natural numbers. The number of options for grouping factors depends on their number, which can be calculated using the formula for combining objects, but we will consider only one grouping option that allows us to obtain factors with a minimum difference.

To represent a natural number, which is the product of a larger number of consecutive natural numbers, in the form of a smaller number of factors that are minimally different from each other, we must group in pairs all the factors equidistant from the ends of the sequence, i.e. make the largest number a pair of the smallest number, then make the next largest number a pair of the second smaller number, etc., in this case the difference between the factors will be minimal.

After grouping the factors of the left side (3.8) according to the scheme described above, we obtain:

1) $a(a+7)=a^2+7a$;

2) $(a+1)(a+6)=a^2+7a+6$

3) $(a+2)(a+5)=a^2+7a+10$;

4) $(a+3)(a+4)=a^2+7a+12$.

If we take $a^2+7a=b$, then we get a natural number that has no representation as a product of four consecutive natural numbers, namely, it has the form $b(b+6)(b+10)(b+12)$. This means that the product of eight consecutive natural numbers cannot be represented as the product of four consecutive natural numbers.

Thus, we have shown above that the products of 5, 6, 7 and 8 consecutive natural numbers cannot be represented as the product of four consecutive natural numbers. Since as the number of consecutive factors increases, the difference between the factors increases, we will not consider other natural numbers that can be represented as the product of a large number of consecutive natural numbers.

It should be noted that we showed above the impossibility of representing the product of consecutive natural numbers, the number of which is greater than 4 (m>4), in the form of a product of four consecutive natural numbers, if each natural number is considered as one immutable object. And if you factorize all consecutive factors of a natural number into prime factors, then try to form four consecutive natural numbers from these prime factors, then in rare cases this will be possible, this was said in Theorem 3.1.

From inequality (3.7) it follows that if m>4, then the product of m consecutive natural numbers cannot be represented as a product of four consecutive natural numbers, except for cases when the numerator is representable as a product of four consecutive natural numbers, which means it cannot be represented in the form product of two natural numbers that differ by 2. Next, we will prove that the product of three consecutive natural numbers cannot be represented as the product of four consecutive natural numbers.

Let there be a natural number that can be represented as the product of three consecutive natural numbers, which also has a representation as the product of four consecutive natural numbers,

(3.9) $c(c+1)(c+2)=b(b+1)(b+2)(b+3)$.

Then, based on (3.4), we can write

(3.10) $\frac{c(c+1)(c+2)}{a(a+1)(a+2)(a+3)} = \frac{b(b+1)(b+2)(b+3)}{a(a+1)(a+2)(a+3)} = (a+4)(a+5)\cdot\ldots\cdot(a+e).$

As can be seen, equations (3.4) and (3.10) are identical. We proved above that equation (3.4) has a natural solution only if the numerator is represented as a product of four consecutive natural numbers. It follows that the product of three consecutive natural numbers cannot be represented as the product of two natural numbers that differ by 2.

Thus, we can say that if m≠4, then the product of m consecutive natural numbers cannot be represented as the product of two natural numbers differing by 2, except in rare cases when the product of m consecutive natural numbers has a representation in the form of the product of four consecutive natural numbers, i.e. Theorem 3.1 is correct and it has been proven.

From Theorem 3.1 it follows that to prove the absence of solutions to the Brocard equation, with the exception of the known three solutions, it is enough to prove that the factorial is greater than 7! cannot be represented as the product of four consecutive natural numbers.

4. Solution of Brocard equation

In Chapter 2 it was said that all three factorials are 4!, 5! and 7!, which are known solutions to the Brocard equation, are represented as the product of four consecutive natural numbers. In this case, it is easy to establish that the product of four consecutive natural numbers, the first of which is equal to the argument of the factorial, will be less than n! for $n > 7$.

For example, $8\cdot(8+1)\cdot(8+2)\cdot(8+3) = 8\cdot 9\cdot 10\cdot 11 = 7920 < 8! = 40320.$

From the above it follows that any factorial is greater than 7! does not have a representation in the form of a product of four consecutive natural numbers, the first of which is equal to the argument of the factorial. This means that the factorial is greater than 7!, if it is represented as the product of four consecutive natural numbers, can only be expressed as the product of four consecutive natural numbers, the first of which is greater than the argument of the factorial.

Next, we will show the impossibility of representing a factorial as a product of four consecutive natural numbers starting with the number of the larger argument of the factorial.

Theorem 4.1. There is no factorial corresponding to the condition $n! > 7!$ and representable as the product of four increasing consecutive natural numbers, the first of which is greater than the argument of the factorial.

According to Theorem 4.1, for a factorial corresponding to the condition $n! > 7!$ the following equation does not have a natural solution

(4.1) $n! = (n+s+1) \cdot (n+s+2) \cdot (n+s+3) \cdot (n+s+4)$,
where $n, m, s \in \mathbb{N}, s \geq 1$.

Proof Theorem 4.1.

Let the following equation have a natural solution

(4.2) $n! = (n+s+1) \cdot (n+s+2) \cdot (n+s+3) \cdot (n+s+4)$.
Now consider the factorial $(n+s+4)!$, which contains factors of both sides of identity (4.2)

(4.3)

$$n! \cdot (n+1) \cdot \ldots \cdot (n+s) \cdot (n+s+1) \cdot (n+s+2) \cdot (n+s+3) \cdot (n+s+4) = (n+s+4)!$$

Next, taking into account (4.2) and (4.3) we can write

$$(n+s+4)! = [(n+s+1) \cdot (n+s+2) \cdot (n+s+3) \cdot (n+s+4)] \cdot$$

$$[(n+1) \cdot \ldots \cdot (n+s+1) \cdot (n+s+2) \cdot (n+s+3) \cdot (n+s+4)].$$

(4.4)

$$(n+s+4)! = (n+1) \cdot \ldots \cdot (n+s) \cdot [(n+s+1) \cdot (n+s+2) \cdot (n+s+3) \cdot (n+s+4)]^2.$$

Because we are looking for a factorial greater than 7! representable as a product of four consecutive natural numbers, we take $n = 7$, and for s we take its minimum value $s = 1$, then based on (4.4) we have

$$(7+5)! = (7+1) \cdot [(7+2) \cdot (7+3) \cdot (7+4) \cdot (7+5)]^2 .$$

$$12! = 479\ 001\ 600;\ 8 \cdot 141\ 134\ 400 = 1129\ 075\ 200;\ 479\ 001\ 600 < 1129\ 075\ 200.$$

As can be seen from the example, equation (4.4) for $n = 7$ and $s = 1$ does not have a natural solution. It is not difficult to understand that for $n = 7$ and $s > 1$ the right side of (4.4) will be even larger, since its right side grows much faster than its left side.

It follows that Theorem 4.1 is true and it has been proven.

From Theorem 4.1 it follows that there is no factorial greater than 7! representable as the product of four increasing consecutive natural numbers, the first of which is greater than the factorial argument, and also as the product of two natural numbers differing by 2. This means that

Diophantine Brocard–Ramanujan equation has been solved; it has no solution other than the known three solutions.

5. Conclusion

It is proved that the Diophantine equation $n! + 1 = m^2$ can have a solution only if the factorial can be represented as the product of a pair of natural numbers whose difference is 2. It is further shown that the known three factorials 4!, 5! and 7!, which are solutions of Brocard's Diophantine equation, have a representation as the product of a pair of natural numbers whose difference is 2, and also have a representation as the product of four consecutive natural numbers. After this, a theorem has been proven stating that the product of m consecutive natural numbers cannot be represented as the product of two natural numbers differing by 2 if $m \neq 4$, with the exception of isolated cases when the product of m consecutive natural numbers has a representation in the form of the product of four consecutive natural numbers.

Then it is proven that no factorial is greater than 7! does not have a representation as the product of four consecutive natural numbers. This means that Brocard's Diophantine equation has been solved, or we can say that Brocard's conjecture is true and it has been proven.

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