A Quasi-algebraic Method for Solving Quintic Equation

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Abstract: A general formula with a parameter for the radical solution of a quintic is derived, based on the well-known Cardano-Tartaglia equation. According to the characteristics of the coefficients of the quintic in the standard Bring-Jerrard form, the range of a real root of the equation is determined, and the parameter can be quickly fixed with the required accuracy in the selected range by a trisection iteration, and all the roots of the quintic can be found. The technique has the potential to be extended to higher degree equations.

Keywords: quintic solving, Cardano-Tartaglia equation, Bring-Jerrard form, quasi-algebraic method, trisection iteration

For hundreds of years, people have been interested in the radical solution of the quintic equation. According to the basic theorem of algebra, in the range of complex numbers, the equation of degree n must have n roots, while for the equation with real coefficients, the complex number roots must be conjugate complex number roots. Therefore, the quintic equation with real coefficients must have a real root. It is generally believed that the condition of the radical solution of the quintic equation is that its Galois group is solvable and belongs to one of three groups [1]. Because Galois theory and Abel theorem limit the existence of direct general formula of a radical solution, seeking indirect general formula or fast numerical solution of equation root becomes a reasonable starting point for solving one-variable quintic equation [2]. For the solution of a quartic equation, one obtains the general radical solution indirectly by reducing the order of the equation. However, the quintic equation is a prime number, it is impossible to use the collocation method or the undetermined coefficient method, etc. to find the parameter or the coefficient of a quadratic equation first [3]. When the coefficients of the quintic are rational numbers, the level of radical expression for directly solving must be great than four, and the rational coefficients can not be directly derived in the 4-level radical form through addition, subtraction, multiplication, and division. In the approximate solution, the iterative method is concerned [4] [5], but the construction of the general iterative scheme is a little complex or depends on the high-order resolvent, so the calculation is inconvenient.

In this paper, a fast and simple method for finding the roots of the quintic equation is proposed. By first finding a specific parameter, limiting the range of real roots, reducing the number of layers of radicals, and then substituting it into the root formula, all five roots of the equation can be obtained quickly through the trisection iteration in the selected range of initial values.

1. Formula Derivation and Trisection Iterative Algorithm

Let the general form of the quintic equation of one variable be

$$x^5 + a_4 \cdot x^4 + a_3 \cdot x^3 + a_2 \cdot x^2 + a_1 \cdot x + a_0 = 0 \quad \text{..........} \quad (1)$$

where: $a_0 \neq 0$.

Through two Tschirnhausen transformations, the general form can always be transformed into the "standard" form [6][7][8] :

$$x^5 + p \cdot x + q = 0 \quad \text{..........} \quad (2)$$

where: $p \cdot q \neq 0$.

Thereafter, only this form is discussed below.

Let $x = u + v$, then

$$\begin{aligned}(u+v)^5 &= u^5 + 5u^4 v + 10u^3 v^2 + 10u^2 v^3 + 5uv^4 + v^5 \\ &= \left(u^5 + v^5\right) + 5uv\left((u+v)^2 - uv\right) \cdot (u+v)\end{aligned}$$

Compared with the standard form's coefficients:

$$u^5 + v^5 = -q$$
$$5uv\left((u+v)^2 - uv\right) = -p$$

For the above expression containing p, U = uv is regarded as a variable and the equation of U is solved:

$$5U^2 - 5x^2U + p = 0$$

The solution is:

$$U = \frac{5x^2 - \sqrt{(5x^2)^2 + 4\cdot 5p}}{2\cdot 5}$$

And at the same time, we have

$$\begin{aligned} u^5 - v^5 &= \sqrt{(u^5 - v^5)^2} \\ &= \sqrt{(u^5 + v^5)^2 - 4u^5v^5} \\ &= \sqrt{q^2 - 4\left(\frac{5x^2 - \sqrt{(5x^2)^2 + 4\cdot 5p}}{2\cdot 5}\right)^5} \end{aligned}$$

So we can get,

$$u^5 = \frac{-q + \sqrt{q^2 - 4\left(\frac{5x^2 - \sqrt{(5x^2)^2 + 4\cdot 5p}}{2\cdot 5}\right)^5}}{2}$$

$$v^5 = \frac{-q - \sqrt{q^2 - 4\left(\frac{5x^2 - \sqrt{(5x^2)^2 + 4\cdot 5p}}{2\cdot 5}\right)^5}}{2}$$

Let x in above two expressions be the parameter, a specific value H to be selected:

$$u^5 = \frac{-q + \sqrt{q^2 - 4\left(\frac{5H^2 - \sqrt{25H^4 + 20p}}{10}\right)^5}}{2} \qquad \text{.......................................} \ (3)$$

$$v^5 = \frac{-q-\sqrt{q^2-4\left(\frac{5H^2-\sqrt{25H^4+20p}}{10}\right)^5}}{2} \qquad \text{.. (4)}$$

where u and v are expressed in a symmetric manner. In order to estimate the range of H, the standard form (2) of the original equation is transformed:

Let $x = \sqrt[4]{|p|} \cdot y$, substituting it into the standard form,

$$\left(\sqrt[4]{|p|} \cdot y\right)^5 + p \cdot \left(\sqrt[4]{|p|} \cdot y\right) + q = 0$$

that is

$$y^5 + y + \frac{q}{\left(\sqrt[4]{|p|}\right)^5} = 0$$

It can be found that:

$$-\sqrt[5]{|q|} < H < 0, \qquad p > 0, \left|\frac{q}{\sqrt[4]{|p|}^{\,5}}\right| > 1, q > 0 \qquad \text{........................... (5)}$$

$$0 < H < \sqrt[5]{|q|}, \qquad p > 0, \left|\frac{q}{\sqrt[4]{|p|}^{\,5}}\right| > 1, q < 0 \qquad \text{........................... (6)}$$

$$-\sqrt[4]{|p|} < H < 0, \qquad p > 0, \left|\frac{q}{\sqrt[4]{|p|}^{\,5}}\right| \le 1, q > 0 \qquad \text{........................... (7)}$$

$$0 < H < \sqrt[4]{|p|}, \qquad p > 0, \left|\frac{q}{\sqrt[4]{|p|}^{\,5}}\right| \le 1, q < 0 \qquad \text{........................... (8)}$$

$$-\sqrt[5]{2} \cdot \sqrt[5]{|q|} < H < 0, \qquad p < 0, \left|\frac{q}{\sqrt[4]{|p|}^{\,5}}\right| > 1, q > 0 \qquad \text{.............. (9)}$$

$$0 < H < \sqrt[5]{2}\cdot\sqrt[5]{|q|}, \qquad p < 0, \left|\frac{q}{\sqrt[4]{|p|}^{5}}\right| > 1, q < 0 \quad \text{.............. (10)}$$

$$-\sqrt[5]{2}\cdot\sqrt[4]{|p|} < H < 0, \qquad p < 0, \left|\frac{q}{\sqrt[4]{|p|}^{5}}\right| \le 1, q > 0 \quad \text{................ (11)}$$

$$0 < H < \sqrt[5]{2}\cdot\sqrt[4]{|p|}, \qquad p < 0, \left|\frac{q}{\sqrt[4]{|p|}^{5}}\right| \le 1, q < 0 \quad \text{................. (12)}$$

According to the value range of H, an estimation interval [L, R] is determined, on which an iterative process can be established to solve the problem.

Let:

$$f(x) = x^5 + px + q \quad \text{.......... (13)}$$

It is easy to verify that

$$f(L)\cdot f(R) < 0$$

so there must be a real root on it. In order to reduce the number of iterations and accelerate the convergence, we use the trisection iteration method.

Let *M1* and *M2* be the two points that divide the interval [L, R] into three equal parts, the algorithm is described as follows:

Trisection iterative algorithm:

$L < R;$

$f(L)\cdot f(R) < 0;$

$M_1 = L + \dfrac{R-L}{3};$

$M_2 = \mathrm{R} - \dfrac{R-L}{3};$

solution accuracy ϵ;

[1] According to (5)-(12), select the range of H, assign L and R, and calculate the trisection point M_1, M_2;

[2] Calculate $f(L)$, $f(M_1)$, $f(M_2)$, $f(R)$;

[3] IF $f(L)\cdot f(M_1) > 0$, THEN $L=M_1$;
IF $f(L)\cdot f(M_2) > 0$, THEN $L=M_2$;
IF $f(R)\cdot f(M_2) > 0$, THEN $R=M_2$;
IF $f(R)\cdot f(M_1) > 0$, THEN $R=M_1$;

[4] IF $R-L<\epsilon$, THEN $H=\frac{1}{2}(R-L)$, end the iteration;
ELSE Calculate the new trisection point M_1, M_2, return to [2] to continue;

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2. Numerical Examples

By substituting the H value obtained by the above iteration into the previous formula (3) (4) for finding the root of the quintic equation of one variable, u and v can be solved:

$$u^5 = \frac{-q+\sqrt{q^2-4\left(\frac{5H^2-\sqrt{25H^4+20p}}{10}\right)^5}}{2}$$

$$v^5 = \frac{-q-\sqrt{q^2-4\left(\frac{5H^2-\sqrt{25H^4+20p}}{10}\right)^5}}{2}$$

By substituting the H value into the previous formula (3) (4), u and v can be solved.

Example1. Solving the equation $x^5-8x-219=0$

For $p<0, \left|\frac{q}{\sqrt[4]{|p|}^5}\right| > 1, q<0$, calculate the range of H value using formula (10) :

$0 < H \le 3.375163549$

The iteration is applied on the selected interval [L, R] = [0, 3.375163549], calculated $M_1=1.125054516$, $M_2=2.25019033$. The four function values $f(L)$, $f(M_1)$, $f(M_2)$, $f(R)$ are

-219 , −226.1979670 , −179.3218599, 191.9986917

So the interval of new iteration is [L, R] = [2.25019033, 3.375163549]. Recalculate M_1=2.625127205, M_2=3.000145377, the correspondent function values are

-115.3339445 , 0.0577204.

the interval of next iteration is: [L, R] = [2.625127205, 3.000145377].

Repeat the above operation for 10 times, we get H=3.000088218 ,and the u and v are

u:
2.938250526;
0.9079693463 + 2.794442310 I;
-2.377094609 + 1.727060327 I;
-2.377094609 − 1.727060327 I;
0.9079693463 − 2.794442310 I;

v:
0.06178008506;
0.01909109620 + 0.05875635247 I;
-0.04998113872 + 0.03631342288 I;
-0.04998113872 − 0.03631342288 I;
0.01909109620 − 0.05875635247 I;

Choose the five roots of x as:
3.000030611;
0.8579882076 + 2.830755733 I;
-2.358003513 + 1.668303975 I;
0.8579882076 − 2.830755733 I;
-2.358003513 − 1.668303975 I;

It is easy to check that the original equation has a real root of 3.

Example 2. Solve equation $x^5 + 11x + 1068 = 0$

For $p > 0$, $\left|\frac{q}{\sqrt[4]{|p|}^{5}}\right| > 1$, $q > 0$, using formula (5) leads to

$-4.033798968 \le H < 0$

In the selected interval, the iteration is calculated for 10 times, H= -4.000052476, the u and v are

u:
0, 0, 0, 0, 0

v :
3.244868603 + 2.357535038 I;
-1.239429517 + 3.814571821 I;
-4.010878172;
-1.239429517 − 3.814571821 I;
3.244868603 − 2.357535038 I;

the five roots are:
3.244868603 + 2.357535038 I;
-1.239429517 + 3.814571821 I;
-4.010878172;
-1.239429517 − 3.814571821 I;
3.244868603 − 2.357535038 I;

3. Discussion and Conclusion

In this paper, a quasi-algebraic formula with a parameter for solving the quintic equation is introduced. A trisection iterative scheme is established by using the Bring-Jerrard canonical form of the quintic equation. Its roots can be obtained quickly by using the trisection iterative algorithm. This method does not need to calculate the derivative and can obtain all the roots of the quintic equation at once. The method of decomposing unknown variable x into the sum of two variables u and v comes from the work of Cardano and Tartaglia in the early 16th century, which was used to solve the cubic equation [9]. The result of this paper shows that the technique can also be used to solve higher degree equations.

A larger amount of numerical experiments have shown that the range of H value determined by the above algorithm is accurate enough to meet the requirement of Newton iteration for the initial value. Although momently we could not give proof for the convergence, it seems that the Newton iteration can be directly used to solve the problem and to obtain the result faster. The trisection iteration, by contrast, is slower than that of Newton's, but it is stable and reliable and can obtain the ideal specific value H finally.

Since it is commonly believed that there is no general radical formula for accurate solution of univariate equation of higher degree ($n \geq 5$), the indirect,or quasi-algebraic solution for the quintic proposed in this paper is somewhat equivalent to direct general radical solution, which can be used to solve the problem in reality. When the real root of the equation is rational or simple irrational, the value obtained by substituting the specific value into the equation is 0, and the root is the traditional radical solution; when the real root of the equation is complex irrational, the solution is a approximate. The reason why the solution is not accurate enough seems due to the higher order radical and the large number of radical layers. By finding the general quasi-algebraic formula for the quintic, this paper proveds a simple and direct way to solve the quintic. It shows a potential for solving higher order ($n > 5$) equations.

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5. Main references

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