A PROOF OF LEMOINE'S CONJECTURE BY CIRCLES OF PARTITION

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Abstract. In this paper we use a new method to study problems in additive number theory. We leverage this method to prove the Lemoine conjecture, a closely related problem to the binary Goldbach conjecture. In particular, we show by using the notion of circles of partition that for all odd numbers $n \geq 9$ holds

$$n = p + 2q \text{ for not necessarily different primes } p, q.$$

1. Introduction

Let $\mathbb{P}$ and $2\mathbb{P}$ denotes primes numbers and their doubles, respectively. Then Lemoine's conjecture, roughly speaking, purports all odd numbers can be partition into the set of all prime numbers and their doubles. More formally the conjecture states

Conjecture 1.1. *The equation $2n + 1 = p + 2q$ always has a solution in primes p and q (not necessarily distinct) for $n > 2$.*

The conjecture was first formulated and posed by Emile Lemoine in 1895 but was wrongly attributed to Hyman Levy in the 1960 (see [1]), which is why it is sometimes referred to as Levy's conjecture. The Lemoine conjecture has not gained much popularity as does the binary Goldbach conjecture but is closely related to and certainly implies the ternary Goldbach conjecture. There has been an amazing computational work in verifying the conjecture, and it is now known that the conjecture holds upto 10^{10} [2].
In this paper we apply a method developed in [3] to study the conjecture; In particular, we show that the conjecture holds for all odd numbers $n \geq 9$.

2. The Circle of Partition

Here we repeat the base results of the method of circles of partition developed in [3].

Definition 2.1. Let $n \in \mathbb{N}$ and $\mathbb{M} \subseteq \mathbb{N}$. We denote with

$$\mathcal{C}(n, \mathbb{M}) = \{[x] \mid x, y \in \mathbb{M}, n = x + y\}$$

the **Circle of Partition** generated by n with respect to the subset $\mathbb{M}$. We will abbreviate this in the further text as CoP. We call members of $\mathcal{C}(n, \mathbb{M})$ as points and denote them by $[x]$. For the special case $\mathbb{M} = \mathbb{N}$ we denote the CoP shortly as

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$\mathcal{C}(n)$. We denote with $\|[x]\| := x$ the **weight** of the point $[x]$ and correspondingly the weight set of points in the CoP $\mathcal{C}(n, \mathbb{M})$ as $\|\mathcal{C}(n, \mathbb{M})\|$. Obviously holds

$$\|\mathcal{C}(n)\| = \{1, 2, \ldots, n-1\}.$$

Definition 2.2. We denote the line $\mathbb{L}_{[x],[y]}$ joining the point $[x]$ and $[y]$ as an axis of the CoP $\mathcal{C}(n, \mathbb{M})$ if and only if $x + y = n$. We say the axis point $[y]$ is an axis partner of the axis point $[x]$ and vice versa. We do not distinguish between $\mathbb{L}_{[x],[y]}$ and $\mathbb{L}_{[y],[x]}$, since it is essentially the the same axis. The point $[x] \in \mathcal{C}(n, \mathbb{M})$ such that $2x = n$ is the **center** of the CoP. If it exists then we call it as a **degenerated axis** $\mathbb{L}_{[x]}$ in comparison to the **real axes** $\mathbb{L}_{[x],[y]}$. We denote the assignment of an axis $\mathbb{L}_{[x],[y]}$ to a CoP $\mathcal{C}(n, \mathbb{M})$ as

$$\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n, \mathbb{M}) \text{ which means } [x], [y] \in \mathcal{C}(n, \mathbb{M}) \text{ with } x + y = n.$$

Remark 2.3. In the following we consider only real axes. Therefore we abstain from the attribute *real* in the sequel.

Proposition 2.4. *Each axis is uniquely determined by points* $[x] \in \mathcal{C}(n, \mathbb{M})$.

Proof. Let $\mathbb{L}_{[x],[y]}$ be an axis of the CoP $\mathcal{C}(n, \mathbb{M})$. Suppose as well that $\mathbb{L}_{[x],[z]}$ is also an axis with $z \neq y$. Then it follows by Definition 2.2 that we must have $n = x + y = x + z$ and therefore $y = z$. This cannot be and the claim follows immediately. □

Corollary 2.5. *Each point of a CoP* $\mathcal{C}(n, \mathbb{M})$ *except its center has exactly one axis partner.*

Proof. Let $[x] \in \mathcal{C}(n, \mathbb{M})$ be a point without an axis partner being not the center of the CoP. Then holds for every point $[y] \neq [x]$ except the center

$$x + y \neq n.$$

This is a contradiction to the Definition 2.1. Due to Proposition 2.4 the case of more than one axis partners is impossible. This completes the proof. □

Notations. We denote by

$$\mathbb{N}_n = \{m \in \mathbb{N} \mid \ m \leq n\} \tag{2.1}$$

the **sequence** of the first n natural numbers.

Now we add here a lemma which not is contained in [3] but is needed in the next section.

Lemma 2.6. *We denote an odd number* $n = p + 2q \mid p, q \in \mathbb{P}$ *as Lemoine number and* $\mathcal{Q}$ *as the set of all Lemoine numbers. Then holds*

$$\mathcal{Q} = \{n \in 2\mathbb{N} + 1 \mid \mathcal{C}(n, \mathbb{P} \cup 2\mathbb{P}) \neq \emptyset\}.$$

Proof. In virtue of Definition 2.1 holds

$$\mathcal{C}(n, \mathbb{P} \cup 2\mathbb{P}) = \{[x] \mid x, n - x \in \mathbb{P} \cup 2\mathbb{P}\}.$$

Since n is odd only sums of an odd prime with an even double of a prime are possible. Therefore x must be a prime and $n-x$ is a double of a prime or reversely. Then is n a Lemoine number. This means that n is a Lemoine number if and only if $\mathcal{C}(n, \mathbb{P} \cup 2\mathbb{P})$ is not empty. $\square$

3. **The Fundamental Theorem and its Conclusions**

Theorem 3.1 (Fundamental)**.** *Let $n, r \in \mathbb{N}$, $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}(n, \mathbb{M})$ be a non-empty CoP with an axis $\mathbb{L}_{[x],[n-x]} \hat{\in} \mathcal{C}(n, \mathbb{M})$ [1] . If holds $x + r \in \mathbb{M}$ then $\mathcal{C}(n + r, \mathbb{M})$ is a non-empty CoP too.*

Proof. Since $\mathbb{L}_{[x],[n-x]} \hat{\in} \mathcal{C}(n, \mathbb{M})$, x and $n - x$ are members of $\mathbb{M}$. And due to the premise also $x + r \in \mathbb{M}$. Then holds

$$n + r - (x + r) = n - x \in \mathbb{M}.$$

Ergo there is an axis $\mathbb{L}_{[x+r],[n+r-(x+r)]} \hat{\in} \mathcal{C}(n + r, \mathbb{M})$ and $\mathcal{C}(n + r, \mathbb{M})$ is non-empty. $\square$

Corollary 3.2. *Let the requirements of Theorem 3.1 be fulfilled. If the base set $\mathbb{M}$ is an infinite set and there exists a non-empty CoP $\mathcal{C}(n_o, \mathbb{M})$ then there exist infinitely many positive integers $n > n_o$ with non-empty CoPs $\mathcal{C}(n, \mathbb{M})$.*

Proof. Let $\mathbb{L}_{[x],[n-x]}$ be an axis of $\mathcal{C}(n_o, \mathbb{M})$. Then is due to Theorem 3.1 also $\mathcal{C}(n_o + r_1, \mathbb{M})$ non-empty with $r_1 > 0$ and if $x + r_1 \in \mathbb{M}$. From this CoP we can continue this process with $r_2 > 0$ to the non-empty CoP $\mathcal{C}(n_o + r_1 + r_2, \mathbb{M})$. Since the base set is an infinite set this process can be repeated infinitely many. $\square$

Lemma 3.3. *It is possible to construct all CoPs $\mathcal{C}(n, \mathbb{P} \cup 2\mathbb{P})$ containing a certain member $[x_o]$ with $n \geq x_o + 6$ for $n \in 2\mathbb{N} + 1$ with $n \geq 9$.*

Proof. We start with the least generator $n_o = x_o + 6$ of a CoP containing the axis $\mathbb{L}_{[6],[x_o]}$ and $y_o = n_o - x_o$. Now we consider the axis $\mathbb{L}_{[y_o],[n_o-y_o]} = \mathbb{L}_{[y_o],[x_o]} \hat{\in} \mathcal{C}(n_o, \mathbb{P} \cup 2\mathbb{P})$. In virtue of Theorem 3.1 holds also

$$\mathbb{L}_{[y_1],[n_1-y_1]} = \mathbb{L}_{[y_o+d_o],[n_o-y_o]} \hat{\in} \mathcal{C}(n_o + d_o, \mathbb{P} \cup 2\mathbb{P})$$

and therefore

$$\mathbb{L}_{[y_o+d_o],[n_o-y_o]} = \mathbb{L}_{[y_1],[x_o]} \hat{\in} \mathcal{C}(n_1, \mathbb{P} \cup 2\mathbb{P})$$

with $y_1 = y_o + d_o$ and $n_1 = n_o + d_o$, if d_o is the distance to the immediately subsequent prime after y_o. Thus we have found with $\mathcal{C}(n_o, \mathbb{P} \cup 2\mathbb{P})$ and $\mathcal{C}(n_1, \mathbb{P} \cup 2\mathbb{P})$ two CoPs both containing $[x_o]$. Since y_1 is the immediately subsequent prime after y_o there is no CoP $\mathcal{C}(n, \mathbb{P} \cup 2\mathbb{P})$ with $n_o < n < n_1$ containing $[x_o]$ because there is no axis $\mathbb{L}_{[x_o],[y]} \hat{\in} \mathcal{C}(n, \mathbb{P} \cup 2\mathbb{P})$ between

$$\mathbb{L}_{[x_o],[y_o]} \hat{\in} \mathcal{C}(n_o, \mathbb{P} \cup 2\mathbb{P}) \text{ and } \mathbb{L}_{[x_o],[y_1]} \hat{\in} \mathcal{C}(n_1, \mathbb{P} \cup 2\mathbb{P}).$$

[1]The axis can also be a degenerated axis with $x = n - x = \frac{n}{2}$ if it exists.

By virtue of Corollary 3.2 we can repeat this procedure with $y_1, y_2, \ldots, d_1, d_2, \ldots$ and $n_1, n_2, \ldots$ infinitely many often and obtain a chain of axes

$$\mathbb{L}_{[y_o],[x_o]}, \mathbb{L}_{[y_1],[x_o]}, \mathbb{L}_{[y_2],[x_o]}, \ldots \mathbb{L}_{[y_s],[x_o]}, \ldots$$

of the chain of all CoPs

$$\mathcal{C}(n_o, \mathbb{P} \cup 2\mathbb{P}), \mathcal{C}(n_1, \mathbb{P} \cup 2\mathbb{P}), \mathcal{C}(n_2, \mathbb{P} \cup 2\mathbb{P}), \ldots \mathcal{C}(n_s, \mathbb{P} \cup 2\mathbb{P}), \ldots$$

containing all the fixed point $[x_o]$. □

Theorem 3.4 (Infinite Lemoine)**.** *There are infinitely many odd numbers $n \geq 9$ having at least one representation as sum of a prime and double of a prime.*

Proof. As an equivalent to the claim we prove that there exist non-empty CoPs for infinitely many odd generators $n \geq 9$. We construct by virtue of Lemma 3.3 all chains of non-empty CoPs starting with generators $n_o = p + 6$ for all $p \geq 3$. These are infinitely many chains with infinite lengths of each one. □

Even if the statement of this theorem seems trivial, it illustrates in a beauty fashion the application of the method of circles of partition on issues in additive number theory.

Whether each odd number ≥ 9 generates a CoP in this pool remains still open, but it will be solved in the sequel. The argument employed can be thought of as an inclusion law, that once the truth of the conjecture holds for some odd number then by necessity it must hold for the next subsequent odd number. This trick is then exploited to cover all odd numbers, but before then we introduce and study the following sets.

By virtue of Lemma 3.3 let

$$\mathbb{H}_x := \{n \in 2\mathbb{N} + 1 \mid [x] \in \mathcal{C}(n, \mathbb{P} \cup 2\mathbb{P})\}, x \in \mathbb{P} \quad \textbf{or} \quad x \in 2\mathbb{P} \tag{3.1}$$

be the set of odd generators of all CoPs containing the point $[x]$ and

$$\mathbb{H}_x(n) := \{m \in \mathbb{H}_x \mid m \leq n\} \tag{3.2}$$

the set of such generators not greater than n. Further let be

$$\mathbb{H}(n) := \bigcup_{\substack{x \in \mathbb{P} \cup 2\mathbb{P} \\ 3 \leq x \leq n-3}} \mathbb{H}_x(n). \tag{3.3}$$

Proposition 3.5.

$$\#\{p \leq n \mid 2p \leq n\} = \pi(\frac{n}{2}).$$

Proof. We can write

$$\#\{p \leq n \mid 2p \leq n\} = \sum_{\substack{p \leq n \\ 2p \leq n}} 1 = \sum_{p \leq \frac{n}{2}} 1 = \pi(\frac{n}{2})$$

□

Proposition 3.5 does indicates that the weight of the co-axis point with a prime weight of the CoP $\mathcal{C}(n, \mathbb{P} \cup 2\mathbb{P}) \neq \emptyset$ must not exceed the threshold $\frac{n}{2}$. That is to say, for any axis $\mathbb{L}_{[p],[2q]} \hat{\in} \mathcal{C}(n, \mathbb{P} \cup 2\mathbb{P})$ such that $p, q \in \mathbb{P}$ then it is required that $q \leq \frac{n}{2}$ for the CoP to have any chance of being non-empty.

Proposition 3.6. *For all $p \in \mathbb{P} \mid p \geq 3$ holds*

$$\mathbb{H}_p = 2\mathbb{P} + \{p\} \text{ and } \mathbb{H}_p(n) = 2\mathbb{P}_{\frac{n}{2}} + \{p\}$$

where $\mathbb{P}_n = \mathbb{P} \cap \mathbb{N}_n$ and for $n \in (2\mathbb{N} + 1)$.

Proof. Since $[p] \in \mathcal{C}(n, \mathbb{P} \cup 2\mathbb{P})$ also holds $[n-p] \in \mathcal{C}(n, \mathbb{P} \cup 2\mathbb{P})$ and hence $n - p \in 2\mathbb{P}$ and for $p \geq 3$

$$\begin{aligned}\mathbb{H}_p &= \{n \in 2\mathbb{N} \mid n - p \in 2\mathbb{P},\ n - p \geq 6\}\\ &= \{q + p \mid q \in 2\mathbb{P},\ q \geq 6\}\\ &= 2\mathbb{P} + \{p\}.\end{aligned}$$

It follows obviously that $\mathbb{H}_p(n) = 2\mathbb{P}_{\frac{n}{2}} + \{p\}$. □

Proposition 3.7. *For all $2p \in 2\mathbb{P}$ for $p \geq 3$ holds*

$$\mathbb{H}_{2p} = \mathbb{P} + \{2p\} \quad \textbf{and} \quad \mathbb{H}_{2p}(n) = \mathbb{P}_n + \{2p\}$$

where $\mathbb{P}_n = \mathbb{P} \cap \mathbb{N}_n$ and for $n \in (2\mathbb{N} + 1)$.

Proof. Since $[p] \in \mathcal{C}(n, \mathbb{P} \cup 2\mathbb{P})$ also holds $[n-p] \in \mathcal{C}(n, \mathbb{P} \cup 2\mathbb{P})$ and hence $n - 2p \in \mathbb{P}$ and for $p \geq 3$

$$\begin{aligned}\mathbb{H}_p &= \{n \in 2\mathbb{N} \mid n - 2p \in \mathbb{P},\ n - 2p \geq 3\}\\ &= \{q + 2p \mid q \in \mathbb{P},\ q \geq 3\}\\ &= \mathbb{P} + \{2p\}.\end{aligned}$$

It follows obviously that $\mathbb{H}_p(n) = \mathbb{P}_n + \{2p\}$. □

Remark 3.8. It is crucially important to recognize that CoPs with odd number generators with base set $\mathbb{P} \cup 2\mathbb{P}$ do not have a center. The upshot is that any such non-empty CoP $\mathcal{C}(n, \mathbb{P} \cup 2\mathbb{P})$ must contain only real axis.

Lemma 3.9 (Main Lemma)**.** *Let $\mathbb{H}_x(2n + 1)$ by virtue of* (3.2) *be the generator set of CoPs containing the point $[x]$ such that their generators are not greater than $2n + 1$ by $n \in \mathbb{N} \mid n \geq 3$. Then contains $\mathbb{H}(2n + 1)$ as defined in* (3.3) *all odd numbers between 9 and $2n + 1$ inclusive.*

Proof. At first we prove that the following statement is equivalent to the claim

$$\forall n \in (2\mathbb{N} + 1) \mid 9 \leq n \leq 2n + 1 \text{ holds } \mathcal{C}(n, \mathbb{P} \cup 2\mathbb{P}) \neq \emptyset. \tag{3.4}$$

Let be

$$\eta(n, p) = \begin{cases} 0 & \text{for } p > n - 3 \vee n - p \notin 2\mathbb{P} \\ 1 & \text{for } n - p \in 2\mathbb{P}. \end{cases} \tag{3.5}$$

Then is obviously

$$\|\mathcal{C}(n, \mathbb{P} \cup 2\mathbb{P})\| = \{p \in \mathbb{P} \mid 3 \leq p \leq n - 3,\ \eta(n, p) > 0\}$$

$$\text{and for } n \in (2\mathbb{N} + 1)$$

$$\mathbb{H}_p(2n + 1) = \{m \in (2\mathbb{N} + 1) \mid 9 \leq m \leq 2n + 1,\ \eta(m, p) > 0\}.$$

It follows that if $\mathcal{C}(2n+1,\mathbb{P}\cup 2\mathbb{P})=\emptyset$ then holds

$$\eta(2n+1,p)=0 \text{ for } 3\leq p\leq 2n-3$$

and reversely. And this means that the sets $\mathbb{H}_p(2n+1)$ contain for no p the generator $2n+1$ and reversely that if $\mathcal{C}(2n+1,\mathbb{P}\cup 2\mathbb{P})\neq\emptyset$ then $2n+1$ belongs to at least one set $\mathbb{H}_p$. The equivalence between (3.4) and the claim of this lemma is demonstrated.

Now we assume that for the odd number $2n_o+1$ holds that $\mathbb{H}(2n_o+1)$ contains all odd numbers between 9 and $2n_o-1$ inclusive except $2n_o+1$. This would mean that it holds $\mathcal{C}(2n_o+1,\mathbb{P}\cup 2\mathbb{P})=\emptyset$ and $\mathcal{C}(n,\mathbb{P}\cup 2\mathbb{P})\neq\emptyset$ for $9\leq n\leq 2n_o-1$. **The validity of this claim needs to be reconstructed, so this result is somewhat conditional.** Because of Lemma **??** there is always a prime p_o and an even number d_o such that p_o-d_o is also prime and it holds

$$[p_o-d_o]\in\mathcal{C}(2n_o+1-d_o,\mathbb{P}\cup 2\mathbb{P}).$$

in the case the case the lower axis point of the CoP has a prime number weight. Then holds in virtue of Theorem 3.1 that there is an axis

$$\mathbb{L}_{[p_o],[2n_o+1-p_o]}\ \hat{\in}\ \mathcal{C}(2n_o+1,\mathbb{P}\cup 2\mathbb{P}).$$

But this contradicts the assumption that $\mathcal{C}(2n_o+1,\mathbb{P}\cup 2\mathbb{P})=\emptyset$.
Also in the case the lower axis point of the CoP belongs to the set of doubles of primes $2\mathbb{P}$, then by appealing to Lemma **??** there exists some $2p_o\in 2\mathbb{P}$ and some even number d such that $2p_o-d\in 2\mathbb{P}$ and it holds

$$[2p_o-d]\in\mathcal{C}(2n_o+1-d,\mathbb{P}\cup 2\mathbb{P}).$$

Then holds in virtue of Theorem 3.1 that there is an axis

$$\mathbb{L}_{[2p_o],[2n_o+1-2p_o]}\ \hat{\in}\ \mathcal{C}(2n_o+1,\mathbb{P}\cup 2\mathbb{P}).$$

This also contradicts the assumption that the CoP $\mathcal{C}(2n_o+1,\mathbb{P}\cup 2\mathbb{P})=\emptyset$. Hence $2n_o+1$ is member of

$$\mathbb{H}(2n_o+1)=\{9,\ldots,2n_o-1,2n_o+1\}=(2\mathbb{N}_{n_o}+1)\setminus\{1,3,5,7\}$$

and the CoP $\mathcal{C}(2n_o+1,\mathbb{P}\cup 2\mathbb{P})$ is non-empty. $\square$

Since we did not put any restriction on the odd number $2n_o+1$ this statement is valid for all odd numbers.

Corollary 3.10. *From Lemma 3.9 follows by $n\to\infty$*

$$\mathbb{H}(n)\longrightarrow(2\mathbb{N}+1)\setminus\{1,3,5,7\}.$$

This means that there are no empty CoPs with the base set $\mathbb{P}\cup 2\mathbb{P}$ for all odd generators ≥ 9 and proves the following Theorem.

Theorem 3.11 (Lemoine's conjecture)**.** *For all odd numbers ≥ 7 there exists at least one representation as sum of a prime and double a prime.*

References

1. Weisstein, Eric W *Goldbach conjecture*, https://mathworld. wolfram. com/, Wolfram Research, Inc., 2002.
2. Drăgoi, Andrei-Lucian *The Vertical Generalization of the Binary Goldbachs Conjecture as Applied on Iterative Primes with (Recursive) Prime Indexes (i-primeths)*, Infinite Study, 2017.
3. Agama, Theophilus and Gensel, Berndt *Studies in Additive Number Theory by Circles of Partition*, arXiv preprint arXiv:2012.01329, 2020.

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