ALGEBRAIC CONVERSION BETWEEN RECTANGULAR AND POLAR COORDINATES

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Abstract. Algebraic equations are derived to approximate the relationship between the rectangular coordinates and the polar coordinates of a vector. These equations can then be used without recourse to imaginary numbers, transcendental functions, or infinite sums.

1. Method

θ in radians is traditionally presented as $\frac{arc\,length}{radius}$. The angle is dependent upon the curved length of the arc traced. Since this length is constantly changing direction as the arc is traced, calculations involving the angle must be done using transcendental functions.

This paper will introduce a new approach to angular measure that is based upon the total rectangular distance traced over the unit circle (summing both the x and y dimensions), rather than the traditional arc length.

Place a vector of magnitude 1 with its initial point at the origin, and its terminal point at $(1,0)$, rotating counter-clockwise such that a positive angle θ is formed between the vector and the x-axis. As the terminal point of the vector completes a unit circle around the origin, three different distances are summed:

1: The distance traced along the unit circle will be $D_c = 2\pi$.
2: The total distance traced in the x dimension, from $x = 1$, to $x = -1$, and back again, will be $D_x = 4$.
3: Likewise, the total distance traced in the y dimension,from $y = 0$, to $y = 1$, to $y = -1$, and then back to 0, will be $D_y = 4$.

$\int_0^\theta |\cos\theta| + |-\sin\theta|\,d\theta = \sin\theta - \cos\theta + 1$ for $0 \le \theta \le \frac{\pi}{2}$ will deliver the sum $D_y + D_x$ as a function of θ in the first quadrant, for the unit circle.

2. Defining ϟ as a new type of angle measured in quaterns

ϟ is defined to be $D_y + D_x = \sin\theta - \cos\theta + 1 = y - x + 1$ in the first quadrant, for the unit circle. If $r \neq 1$, then in the first quadrant or in right triangles, equations 1, 2, and 3 are valid, where r is the radius of the circle or the hypotenuse of a right triangle:

$$ϟ = \frac{y - x}{r} + 1 \tag{1}$$

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$$x = \frac{r}{2}(\sqrt{-ϟ^2 + 2ϟ + 1} - ϟ + 1) \tag{2}$$

$$y = \frac{r}{2}(\sqrt{-ϟ^2 + 2ϟ + 1} + ϟ - 1) \tag{3}$$

3. General Forms

The major issue in applying these principles to the other three quadrants is determining an algebraic method to govern the changes in the signs of the coefficients with step functions - so as to be continuous while giving the correct sign at the correct position. In prior works this was accomplished using the imaginary plane. The method presented here accomplishes the same thing by using step functions to govern the coefficients in an algebraic way.

3.1. **The helper functions.** The ϟ constant step function, K, adjusts the constant of 1 to the appropriate quadrant.

$$K = |4\frac{y}{|y|} - \frac{x}{|x|}| - 2\frac{y}{|y|} \tag{4}$$

The circular ϟ step function, Q makes the analog sine and cosine functions in ϟ periodic.

$$Q = ϟ - 8\lfloor \frac{ϟ}{8} \rfloor \tag{5}$$

The step function analog of cosine, C, gives the proper sign of x by quadrant.

$$C = -(-1)^{\lfloor \frac{ϟ-2}{4} \rfloor} \tag{6}$$

The step function analog of sine, S, gives the proper sign of y by quadrant.

$$S = (-1)^{\lfloor \frac{ϟ}{4} \rfloor} \tag{7}$$

The step function analog of tangent, T, gives the proper sign of the slope, $\frac{y}{x}$, by quadrant.

$$T = (-1)^{\lfloor \frac{ϟ}{2} \rfloor} \tag{8}$$

The constant step function for ϟ, Γ, gives the proper constant for equations 2 and 3, by quadrant.

$$\Gamma = |4S - C| - 2S \tag{9}$$

3.2. **The General Conversion Functions.** These functions are continuous; they work uniformly, unambiguously, and universally in all four quadrants, and at all distances, for all values of ϟ:

To extract ϟ, given rectangular coordinates:

$$\text{ϟ} = \frac{\frac{xy}{|x|} - \frac{xy}{|y|}}{r} + K \tag{10}$$

To extract x, for a vector of any magnitude r, given ϟ; (When $r = 1$, this is the analogue of the $\cos\theta$ function):

$$x = r\frac{\sqrt{-Q+\sqrt{2}+\Gamma}\sqrt{Q+\sqrt{2}-\Gamma}+T(\Gamma-Q)}{2C} \tag{11}$$

To extract y, for a vector of any magnitude r, given ϟ; (When $r = 1$, this is the analogue of the $\sin\theta$ function):

$$y = r\frac{\sqrt{-Q+\sqrt{2}+\Gamma}\sqrt{Q+\sqrt{2}-\Gamma}+T(Q-\Gamma)}{2S} \tag{12}$$

4. Converting back to radians

Equation 13 will convert ϟ in quaterns back into Θ, measured in radians. This will give Θ with an error less than one-half percent:

$$\Theta = \frac{\pi}{4}\text{ϟ} + \frac{.1324xy(x^2-y^2)}{(x^2+y^2)^2} \tag{13}$$

5. Consistency with a version of Euler's Formula

It is assumed, though not yet proven, that, given x and y from equations 11 and 12:

$$2.19^{i\text{ϟ}} \approx (x+iy) \tag{14}$$

The assumption was made by using technology to graph various exponential functions. A visual comparison of Figure 2 with Figure 3 shows that the approximation is fairly close. Note that 1 quatern $\approx \frac{pi}{4}$ radians, and that 8 quaterns $=$ 1 full rotation of 2π radians.

6. Graphic comparison

7. Conclusion

Using ϟ rather than Θ gives an exact algebraic relationship between polar and rectangular coordinates. This is done without imaginary numbers, transcendental functions, or infinite sums. There also may be an excellent analogue to quaternion rotation, without calculations involving imaginary numbers.

The archaic Greek symbol for Koppa, ϟ, is suggested for use representing quaterns. There are several reasons for this:

1: Quaternions are the best way to describe rotations, so an allusion is made to that, as well as the phrase "quarter turn."
2: ϟ was the archaic Greek letter "q". It also represented the number "90", which recalls 90 degrees. This is appropriate for a variable that is to represent a type of angle.

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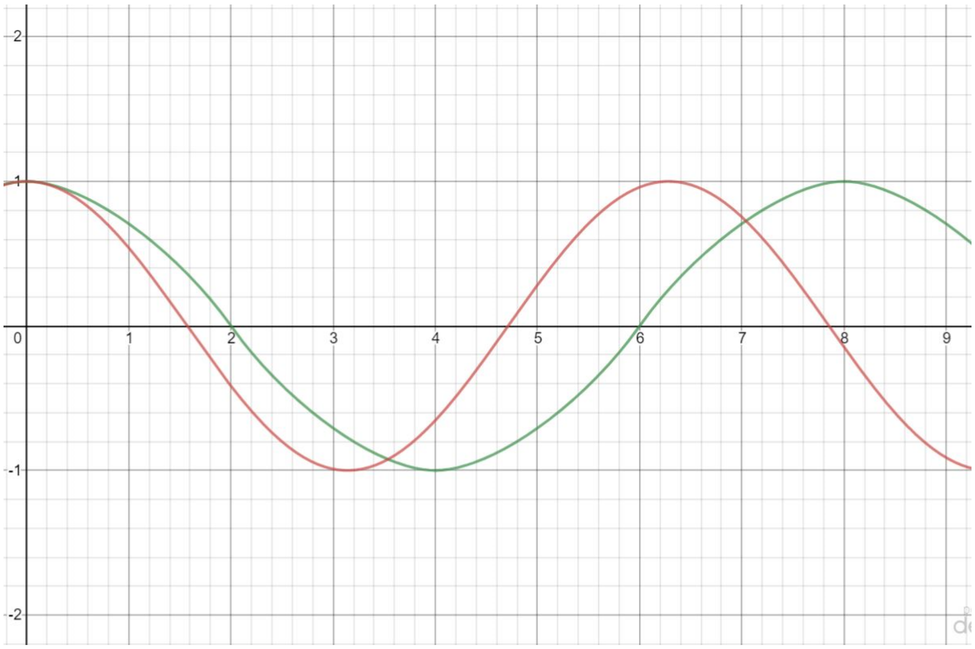

FIGURE 1. $\cos\theta$ in red; analogous function for ϟ is in green, according to eqn. 11; $r = 1$; over the interval $(0 - 9)$

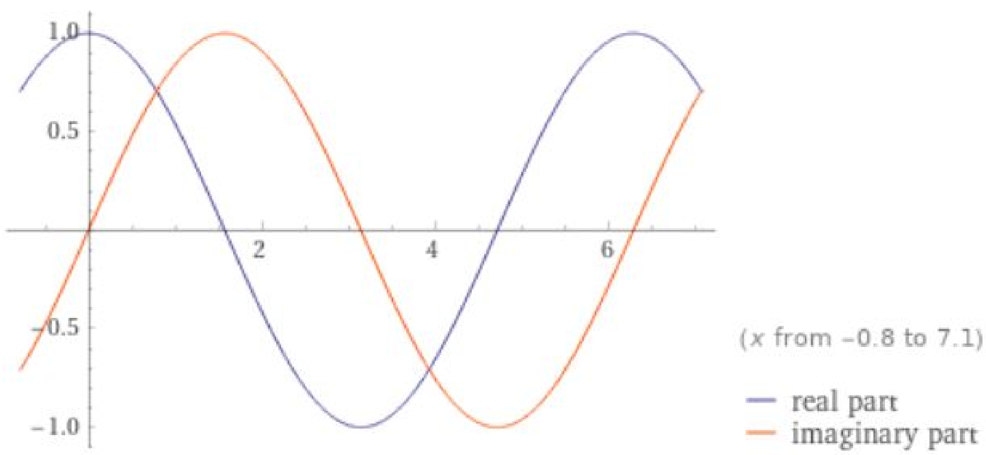

FIGURE 2. Euler's formula for θ. $e^{i\theta} = \cos\theta + i\sin\theta$;
$-\frac{\pi}{4} \leq \theta \leq \frac{9\pi}{4}$

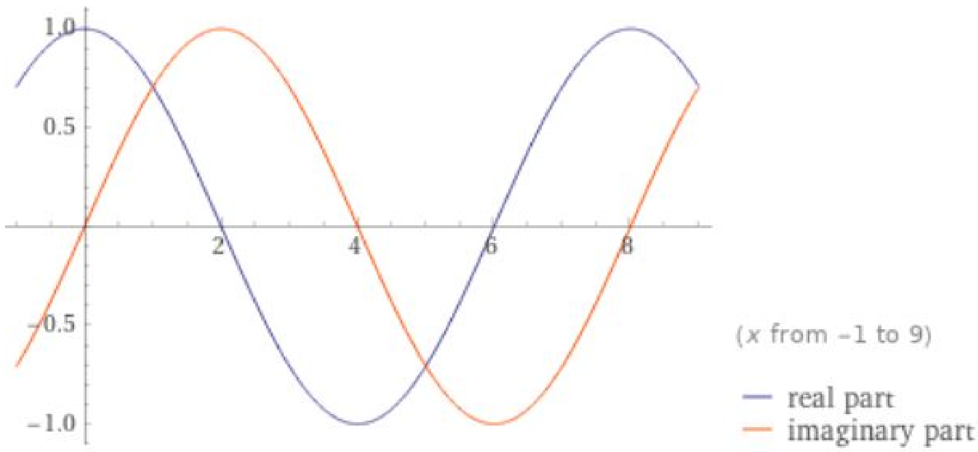

FIGURE 3. Analogous use of Euler's formula for ϟ.
$2.19^{iϟ} \approx (x + iy)$; x and y coming from eqns. 11 and 12; $r = 1$;
$-1 \leq ϟ \leq 9$