"Relativistic Ring" Simulation - New Approach resolves apparent Paradoxes

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February 15, 2021

Abstract

Recent analysis of the "Relativistic Ring" problem [1]. revealed that its angular momentum features a "paradox" maximum at circumferential velocity $\approx 0,24$ c declining to near zero with increasing velocity. This apparent "paradox" can be resolved by a new approach based on simulated high external forces up to the "Weak Energy Limit" in a low velocity regime $0 < v << c$. Simulations comprise a "Relativitic Rod" in uniform translational motion subjected to a pair of mutually opposed external forces and a pressurized "Relativistic Ring". If the external forces simulate centrifugal force on a "Relativitic Rod" its canonical momentum features a maximum at velocity $\hat{v} = \sqrt{\frac{2}{3}}$ c being analogue to the maximum canonical angular momentum of a RR. Remarkably, rotational velocity of a pressurized "Relativistic Ring" can be modulated by variation of pressure - at constant canonical angular momentum.

1 Introduction

The "Relativistic Ring" (RR) paradox is a simplified version of famous "Ehrenfest Paradox" (EP) [2]. The EP was to demonstrate that the classical notion of a rigid-body and Euclidean metric were untenable in relativistic mechanics thus attracting substantial attention in the early days of relativity, presumably inspiring Einstein in his formulation General Relativity. Initially, the RR problem focused on the change of the radius of a rotating RR with increasing rotational velocity. [2]. A frequent source of confusion is that a rotating RR is subjected to the combined effects of various critical issues like centrifugal and Coriolis-forces, Lorentz-factors, non-Euclidian metrics, a non-inertial frame of reference and speculative hypothesis of rigidness. It will be revealed that a key issue of the RR angular momentum paradox is the decline of moment of inertia with increasing velocity finally vanishing at the Weak- Energy Limit (WEL).

2 Methodology

The new approach focuses on determination, isolation and simulation of crucial factors and their particular effects on canonical angular momentum and moment of inertia of a RR, unbiased by their combined effects and interactions.
As centrifugal force is a key issue of apparent RR paradoxes the analysis simulates the effects of centrifugal force by two methods:

a) A "Relativistic Rod" in slow uniform motion subjected to a pair of mutually opposed external forces $\pm F$ in proportion to the square of velocity acting on its leading and trailing faces.
b) Frictionless isostatic pressure acting on a slowly rotating RR.

For the sake of brevity, clarity and instructiveness most of the analysis will be restricted to a low velocity domain ($v/c_0 \approx 0$, $dt'/dt \approx 1$ and Lorentz-factors ≈ 1) without introducing any significant error. It will be shown why the relevant entity is canonical angular momentum, instead of kinetic angular momentum.

3 Hypothesis of rigidness

It is clear from the outset that elasticity, compressibility, mass-density and other material characteristics are crucial for the characteristics of a RR. For the sake of simplicity and instructiveness the "Highest Relativistically Compatible Rigidness" (HRCR) conjectured in [2]. [3]. will be used.
HRCR corresponds to a linear-elastic (Hooke's) law in accordance with Young's modulus $\lambda = \rho_0 c^2$ where ρ_0 is mass density [2]..
HRCR satisfies the following essential requirements:

3.1 Weak Energy Limit

The "Weak Energy Limit" (WEL) reflects the ultimate relativistically permitted force-density or stress transmittable by an object of mass-density ρ_0.
Consequently for a rigid body

$$\hat{w} = \rho_0 c^2 = T_{00} c^2 = -T_{11} \tag{1}$$

where $\hat{w} = \rho_0 c^2$ is peak energy density, $\rho_0 = T_{00}$ mass-density, T_{11} a main-axis (normal) component of the stress-energy-momentum 4-tensor T_{ik}.

3.1.1 Velocity of sound

In a linear-elastic material with Young's modulus λ the velocity of sound is $c = \sqrt{\lambda/\rho_0}$. For HRCR any disturbance must propagate with c thus $\lambda = \rho_0 c^2$.

3.1.2 Extensibility

The WEL can also be met by any material of Young's modulus $\lambda < \rho_0 c^2$ provided its extensibility allows for an expansion factor ϵ sufficiently large to satisfy the condition $\lambda\epsilon \geq \rho_0 c^2$.

3.2 Inertial frames of reference

S will denote a laboratory inertial frame with coordinates x,t, and S' an inertial frame moving with $|v_x| = |v|$ throgh S along the x-axis with coordinates x', t' where $x' \parallel x$.
Generally, index "0" will denote an initial state of an entity, index "1" a subsequent changed state of that entity, like velocity, length, radius, mass or force.

4 Relativistic Rod

4.1 General

A "Relativistic Rod" is a highly instructive analogue model of a RR differing from a RR in that it is in slow uniform translational motion v devoid of acceleration and inertial forces. Thus it enables to circumvent the analytical complexities related to relativistic formulation of centrifugal and Coriolis forces as well as a non-inertial frame of reference.
Instead, the effect of centrifugal force in a RR is simulated by a pair of mutually opposed external drawing forces $\pm F \propto v^2$ each of them acting on the leading and trailing faces of a "Relativistic Rod" - their magnitude increasing with the square of velocity like centrifugal force.
For the sake of clarity and instructiveness, the issues of elastic expansion and expansion energy-mass-inertia will only be considered where a substantial effect would alter a result.

4.2 Lagrangian of a Relativistic rod

For a relaxed "Relativistic Rod" ($\pm F = 0$) and $v << c$: Proper-length $l_0 \approx l$ and proper-mass $M_0 \approx M$.

The Lagrangian $\mathcal{L} = w_k - w_p$ of an object moving with velocity v through S is the difference among its kinetic w_k and potential energy w_p. For a rod of length l and mass M subjected to a pair of external forces $\pm F$:

$$\mathcal{L} \approx \frac{Mv^2}{2} - \frac{Flv^2}{2c^2} \tag{2}$$

4.3 Canonical Momentum of a Relativistic Rod

Canonical Momentum of a.m. "Relativistic Rod" results from an Euler-Lagrange transformation of (2):

$$P \approx v(M - \frac{Fl}{c^2}) \tag{3}$$

The term Flv/c^2 in (4) reflects the momentum assignable to the passive mechanical energy flux Fv corresponding relativistic momentum flux Fv/c^2 from the leading (A) to the trailing face (B) of the rod. (Note that the term in brackets in (3) can be interpreted as the inertia of a "Relativistic Rod" subjected to $\pm F$.)

4.3.1 Canonical Momentum of a "Relativistic Dipole"

A "Relativistic Rod" can most instructively be imagined as an electric dipole $p = ql$ comprising a pair of charges $\pm q$ kept at distance l moving through an electrostatic potential field $\Phi(x)$. Such dipole is subjected to a pair of forces $\pm F = \pm q\nabla\Phi$ corresponding to a potential difference $p\nabla\Phi = Fl$ among its leading and trailing charges$\pm q$. [5].

The Lagrangian of an electric dipole is

$$\mathcal{L}_q \approx \frac{Mv^2}{2} - \frac{ql\nabla\Phi v^2}{2c^2} \tag{4}$$

Canonical momentum P_q of a dipole results from an Euler-Lagrange transformation of (4):

$$P_q \approx v(M - \frac{p\nabla\Phi}{c^2}) \tag{5}$$

If such dipole was in uniform motion with $\vec{v} \parallel \vec{\nabla}\Phi$ its leading charge would permanently absorb external field energy flowing backwards through the rod to be released into the potential field throgh the trailing charge at distance l. In result relativistic field energy/mass of Φ is permanently offset backwards through the moving dipole corresponding to relativistic momentum-flux $g = -\frac{p\nabla\Phi v}{c^2}$ through the rod.

4.3.2 Variation of external force at constant canonical momentum

This is to determine how variable of external forces $\pm F$ would affect the velocity v of a "Relativistic Rod" if its initial canonical momentum $\mathbb{P}_0$ remained conserved. Imagine a rod of proper length $l_0 \approx l$ aligned parallel to the $x-$ axis, in uniform motion through S with velocity $v << c$ along the $x-$ axis. At a given instant in S a pair of mutually opposed forces $\pm F$ begins to act simultaneously in S at (A) and (B). The task is to determine how the simultaneous incidence of forces $\pm F$ in S ($\Delta t = 0$) transforms into non-simultaneous incidence in S' by a time-interval $\Delta t' \neq 0$ given by a Lorentz-transformation:

$$\Delta t' \approx \frac{vl_0}{c^2} \tag{6}$$

As the pair of forces $\pm F$ is initiated simultaneously ($\Delta t = 0$) in S no retroactive momentum can be transferred from S' to S during this event i.e. $\pm F\Delta t = 0$. However in S' force $+F$ acting on (A) begins to act by time shift $\Delta t'$ before the force $-F$ at (B) begins to act in opposed direction establishing a new equilibrium of forces in the rod.

In result the non-simultaneous incidence of $\pm F$ in S' on the rod changes its kinetic momentum by

$$\Delta p' = F\Delta t' = \frac{Fvl}{c^2} \tag{7}$$

Note that $\Delta p'$ is identical with the relativistic momentum assignable to the passive energy-momentum-flux from A to B $-Fv/c^2$ - opposed to v - which reflects conservation of canonical momentum in S and S'. Recall that forces acting parallel to $\vec{v}$ i.e. $\pm\vec{F} \parallel \vec{v}$ are identical in S and S' thus $F'_x = F_x$.
For $0 < v << c$ and $dt'/dt \approx 1$ the approximation $\Delta p' \approx \Delta p \approx Fvl_0/c^2$ can be used without introducing any substantial error.

4.4 Conservation of canonical momentum

$P_0 = M_0v_0$ is the initial (constant) canonical momentum of a relaxed "Relativistic Rod" ($\pm F = 0$) of mass $M \approx M_0$ moving with $v_0 << c$ through S, and P_1 its canonical momentum after being subjected to external forces $\pm F$. Then conservation of canonical momentum requires $P_0 = P_1 = \text{const}$. Accordingly the right side of (3) $\rightarrow (Mv - Flv/c^2)$ must remain unchanged for any incremental change $\Delta p = Flv/c^2$. Consequently an incremental change of kinetic momentum $M\Delta v = M(v_1 - v_0)$ is required to counterbalance Δp i.e. $M\Delta v = -Flv/c^2$. This implies that both incremental changes are interlaced by a reactive principle. The task thus is to determine velocity v_1 as a function of v_0 after the incidence of $\pm F$ on the rod.

Conservation of canonical momentum requires $P_1 = P_0 = Mv_0$:

$$P_1 = Mv_1 - \frac{Flv_1}{c^2} = Mv_0 \tag{8}$$

yielding

$$v_1 = \frac{v_0}{1 - \frac{F}{mc^2}} \tag{9}$$

where $m = M/l$. Note that for $F = mc^2$ in (9) a singularity would exist. (9) also implies an upper limit of expansion $\hat{\epsilon} = \hat{\Delta} l/l = 1$ demarked by the WEL.

It can further be concluded that relativistic inertia $(Mv - Flv/c^2)$ of a "Relativistic Rod" declines with increasing forces $\pm F$ and would vanish at the WEL $\hat{F} = mc^2$.

Consideration of expansion energy-mass in (9) - based on Hooke's law with Young's modulus $\hat{k} = mc^2$- would lead to a relativistic mass-increment $\Delta M \leq 0,5M_0$. However at the same instance the length l_0 of the "Relativistic Rod" would expand to $l_1 = 2l_0$ counterbalancing any effect of mass-increase ΔM on v_1.

4.5 Simulation of Centrifugal Force

Let a "Relativistic Rod" of mass M_0 and length l_0 be steadily accelerated up to velocity $\hat{v}$, while at the same time an external pair of forces acts on the "Relativistic Rod" with magnitude $\pm F_1 \propto v^2$ thus simulating centrifugal force. The task is to determine the velocity $\hat{v}_1$ at which peak canonical momentum $\hat{P}_1$ is achieved.

In this section: Lorentz-factor $\gamma = (1 - v^2/c^2)^{-1/2} \approx 1 + \frac{v^2}{2c^2}$ will apply.

$$\pm F_1 = \pm \alpha v_1^2 \tag{10}$$

$$M_1 = \gamma_1 M_0 \tag{11}$$

Following (8)

$$P_1 == \gamma_1 M_0 v_1 - \frac{\alpha v_1^3 l_0 \gamma_1^{-1}}{c^2} \tag{12}$$

A relative maximum $\hat{P}_1$ at $\hat{v}_1$ in (12) is determined by $dP_1/dv_1 = 0$:

$$\frac{dP_1}{dv_1} = \gamma_1 M_0 - \frac{3\alpha l_0 v_1^2}{c^2} = 0 \tag{13}$$

$$\hat{v_1}^2 \approx c^2 \frac{\gamma_1 M_0}{3\alpha l_0} \tag{14}$$

For $\alpha = M_0/l_0$

$$\hat{v_1}^2 \approx c^2 \frac{\gamma_1}{3} \tag{15}$$

For $\gamma_1 \approx 1 + \frac{v_1^2}{2c^2}$ and $\frac{d\gamma_1}{dv} \approx 0$

$$\hat{v_1} \approx \sqrt{\frac{2}{3}} c \tag{16}$$

The force $\pm F$ corresponding to $\hat{v}_1$ would be $\frac{2}{3} m_0 c^2$ thus 2/3 of the WEL - which would be reached at $v_1 = c$.

5 Pressurized "Relativistic Ring"

This is to demonstrate why and how a key issue of RR paradoxes is the decline of its relativistic moment of inertia with increasing centrifugal force. The issue of centrifugal force and angular momentum of a RR in a high velocity regime is a frequent source of complexities and confusion which however can be circumvented in a low velocity regime $v << c$ by substitution of centrifugal force with equivalent isostatic (non-rotating) frictionless (perfect) fluid pressure like in a rotating pressure-vessel. Recall that isostatic pressure is Lorentz-invariant and always acts perpendicular to the local (tangential) velocity of a RR section, like centrifugal force.

5.1 Relativistic Moment of Inertia of a pressurized RR

5.1.1 Pressurization at rest

This is to demonstrate that if centrifugal force is substituted by isostatic pressure any desired variation of its moment of inertia can be induced by variation of pressure/force without affecting its canonical angular momentum, at any velocity $0 \leq v << c$. The first task thus will be to determine a relationship among pressure/force and moment of inertia of a RR at rest. The analysis is restricted to the axis of rotation (z-axis in the centroid, perpendicular to the plane of the RR).

Let: $M_0 \approx M_1 \approx M \rightarrow$ mass, $m_0, m_1 \rightarrow$ mass per unit length, ${\Theta_0}^{Cl} = \Theta_0 \rightarrow$ classical moment of inertia of a relaxed RR at rest and $\theta_1 \rightarrow$ moment of inertia of a RR at rest expanded to radius r_1 corresponding to expansion-factor $\epsilon = r_1 - r_0/r_0$ i.e. $r_1 = r_0(1+\epsilon)$.

$$\theta_0 = M_0 {r_0}^2 = 2\pi m_0 {r_0}^3 \tag{17}$$

$$\theta_1 = M_1 {r_1}^2 (1 - \frac{F_1 2\pi r_1}{M_1 c^2}) = M_1 {r_1}^2 (1 - \frac{F_1}{m_1 c^2}) \tag{18}$$

where $F_1 \rightarrow$ pressure-induced force, $\mathrm{m}_1 = M_1/2\pi r_1$.

Hence

$$\Theta_1 = M_1 {r_1}^2 (1-\epsilon) = 4\theta_0(1-\epsilon) \tag{18.a}$$

Classically (for $k << mc^2$) and $r_1 = 2r_0 \rightarrow {r_1}^2 = 4{r_0}^2$ the moment of inertia of an expanded ring would be ${\Theta_1}^{Cl} = 4{\Theta_0}^{Cl}$. For the highest relativistically compatible Young-modulus $\hat{k} = mc^2$ at expansion-factor $\epsilon = 1$ the ultimate force $\hat{F_1} = \hat{k}\hat{\epsilon} = m_1 c^2$ corresponds to the WEL thus in (18, 18.a) $\Theta_1 = 0$. Consequently a RR subjected to $\hat{F_1}$ can't embody angular momentum and would acquire characteristics analogue to those of a toroidal magnetostatic field.

5.1.2 Mass-equivalent of elastic expansion-energy

If a linear-elastic law of force during expansion of a RR with Young's modulus $\hat{k} = mc^2$ was hypothesized the expansion-energy-mass equivalent at $\epsilon = 1$ would be $M' = 0,5M_0$ thus its total inertial mass $3/2M_0$. If elastic expansion-energy-mass equivalent was considered in (18, 18.a) with $M' = 3/2M_0$

$$\Theta_1{}' = \frac{3}{2}M_0{r_1}^2(1-\epsilon) = 6\theta_0(1-\epsilon) \tag{18.b}$$

Nonetheless Θ_1 would vanish during pressurization at $\epsilon = 1$.

5.2 Reactive Acceleration of a RR

This is to determine the counteracting effects of reactive acceleration and Coriolis deceleration on velocity during elastic expansion of a RR of HRCR, if an initially given canonical angular momentum L_{C0} remains conserved, with consideration of the decline of its moment of inertia.

Definitions and Abbreviations in this section:

$L_{C0} = \Theta_0\omega_0$ Initial angular momentum of a relaxed RR
$L_{C1} = \Theta_1\omega_1$ Angular momentum of an expanded RR
Θ_0 = Moment of inertia of a relaxed RR
Θ_1 = Moment of inertia of an expanded RR
$\Theta_1{}'$ = Moment of inertia of an expanded RR including expansion-energy-mass
ω_0 = Angular velocity of a relaxed RR
ω_1 = Angular velocity of an expanded RR
r_0 = Radius of a relaxed RR
r_1 = Radius of an expanded RR
$v_0 = \omega_0 r_0$ = Circumferential velocity of a relaxed RR
$v_1 = \omega_1 r_1$ = Circumferential velocity of an expanded RR
M_0 = inertial mass of a relaxed RR
M_{el} = Mass-equivalent of elastic expansion-energy
$M_1 = M_1 + M_{el}$ inertial mass of an expanded RR
$m_0 = M_0/2\pi r_0$ = peripheral mass density of a relaxed RR at r_0
$m_1 = M_1/2\pi r_1$ = peripheral mass density of an expanded RR at r_1

Imagine an initially relaxed RR of radius r_0 and mass M_0 freely rotating in its own plane with initial tangential velocity $v_0 = \omega_0 r_0 << c$ and initial canonical angular momentum $L_{C0} = \Theta_0\omega_0$.

Classically, if that ring was elastically expanded from r_0 to $r_1 = 2r_0$ at constant angular momentum its circumferential velocity would decelerate to $v_1 = 0,5v_0$, due to angular momentum conservation or the Coriolis effect.

This is to determine the change of rotational velocity from $v_0 = \omega_0 r_0$ to $v_1 = \omega_1 r_1$ caused by the decline of moment of inertia from θ_0 to θ_1 during elastic expansion as a function of pressure-induced force F_1, for constant canonical angular momentum, i.e. L_{C0} =L$_{C1}$.

Let $L_{C0} = M_0 r_0 \times v_0 \rightarrow$ initial canonical angular momentum of a relaxed RR of radius r_0 and $v_0 = \omega_0 r_0 \rightarrow$ initial tangential velocity. During elastic expansion a force F_1 emerges in the RR and causes a decline of its moment of inertia in accordance with (18a).

Canonical angular momentum L_{C1} of a RR expanded from radius r_0 to radius r_1 and periphery $l_1 = 2\pi r_1$ subjected to force F_1

$$L_{C1} = M_1(r_1 \times v_1)(1 - \frac{F_1 l_1}{M_1 c^2}) = L_{C0} \tag{19}$$

Substitution in (19): $l_1 = 2\pi r_1$, $M_1 = 2\pi r_1 m_1$ yields

$$L_{C1} = 2\pi r_1 m_1 (r_1 \times v_1)(1 - \frac{F_1}{m_1 c^2}) = L_{C0} = M_0 r_0 v_0 \tag{19.a}$$

$$v_1 = v_0 \frac{r_0}{r_1}(1 - \frac{F_1}{m_1 c^2})^{-1} \tag{20}$$

$$v_1 = v_0 \frac{r_0}{r_1}(1 - \epsilon_1)^{-1} \tag{20.a}$$

Note in (18) that for $F_1 = 0$ and $\lambda = 0$ the classical expression of Coriolis-effect results.

For HRCR $\rightarrow \hat{F_1} = \hat{k}\epsilon_1 = m_1 c^2 \epsilon_1$ and for $\acute{\epsilon_1} = 1 \rightarrow \hat{F_1} = m_1 c^2$ represents the WEL.

As (20.a) is analogue to (9) a potential singularity at the WEL: $\epsilon_1 = 1$ does exist.

6 Summary and Conclusions

The new approach to the RR paradox is a first draft characterized by restriction to a low-velocity ($v << c$) - high external force regime, including standstill. It is revealed that apparent RR paradoxes result from a decline of moment of inertia with increasing force/expansion, up to the WEL. The analysis starts with a "Relativistic Rod" in uniform motion at constant canonical momentum, subjected to a pair of mutually opposed external forces. If the magnitude of such pair of forces would increase with the square of velocity - like centrifugal force - its canonical momentum featured a maximum at velocity $\hat{v} = \frac{\sqrt{3}}{2}c$- vanishing at the WEL.

For the sake of simplicity and instructiveness a linear-elastic material characteristic with Young's modulus $k = mc^2$ - satisfying the criteria for highest relativistically compatible rigidness (HRCR) - is hypothesized.

Centrifugal force is analytically simulated by frictionless isostatic pressure acting on a RR without affecting its canonical angular momentum. Analysis reveals a steady decline of relativistic moment of inertia with increasing pressure/force - vanishing at the WEL.

If a slowly rotating RR ($v << c$) of HRCR is elastically expanded by pressurization at constant canonical angular momentum its velocity increases due to the decline of its moment of inertia with increasing pressure, outpacing Coriolis-deceleration up to the WEL.

It can be concluded that the crucial issue of apparent RR paradoxes is the decline of its relativistic moment of inertia with increasing centrifugal force - being caused by passive energy-momentum circulation increasing with the 3rd power of velocity.
Any hypothetical material of HRCR would however be unrealistic as long as no adequate quantum field theory exists.

The issue of energy conservation will be subject of a follow-up article.

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