Elemental and structured spaces

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Abstract

The word space is used in many ways and most of these applications give this word a different meaning. This makes the notion of space very obscure. Already in the common life of humans takes the word space many different uses. Especially philosophers, mathematicians, and physicists have attributed a huge number of interpretations of the noun "space". This has led to a huge number of different forms of space. Humans live in an environment that is characterized by space and time. This paper focuses on the most elemental meanings that mathematicians and physicists attribute to the word "space". Next, the immediate extensions of this elementary space are investigated. Since physicists investigate our physical reality, the paper also investigates how physical reality treats the notion of space.

1 Mathematics versus reality

Mathematicians are humans and therefore they need names or symbols and extensive descriptions and recipes of the notions that they use. Without these linguistic extensions, for humans, mathematics would be unworkable. Physical reality does not require these additions. Reality does not use manuals or handbooks. Reality just applies the bare concepts. Still, it must obey the rules that are set by the structures and recipes. Physical reality does not intelligently obey rules. Probably, reality uses the trial-and-error approach. But that means that this approach must be efficient enough. The structures and mechanisms that reality applies must guide their usage automatically. Simple structures must automatically emerge into more complicated structures that offer restrictions that guide their usage. Mechanisms must limit the ways that they can be accessed.

In mathematics, spaces exist in many forms, and in combination with mechanisms they constitute dynamic systems. We will investigate these spaces and mechanisms to explain how these bare ingredients can successfully constitute dynamic systems.

The elemental spaces must emerge into more complicated spaces and the capabilities of these extensions must become automatically accessible.

The restrictions that go together with the extension of the model limit the structures and mechanisms that reality applies. This limits the part of mathematics that is suitable for comprehending the lower levels of the structure and behavior of physical reality. This does not imply that the current state of humanly developed mathematics covers all aspects of these lower levels. The lower levels of the structure and behavior of physical reality still contain incomprehensible mysteries. One of them is formed by the origin of the stochastic processes that control part of the dynamics of physical reality.

2 Vector space

In human mathematics, space is not a well-defined concept. A vector space is considered as a quite elemental form of space. It is a set of points, vectors, and scalars. The points can be connected by vectors. The scalars can be added and multiplied with other scalars. The vectors can be scaled via multiplication with a scalar. The vectors own a direction and a length. A parallel shift of the vector does not change the vector. Vectors can be added by shifting them such that the endpoint of the first vector coincides with the start point of the other vector. The start point of the first vector and the endpoint of the second vector form the sum vector. Another name for vector space is linear space. In mathematics, many notions of space are extensions of an elemental vector space.

3 Real numbers

Simple scalars can be used to count objects. Together the counts form the ***natural numbers***. If no objects exist in a set, then the count gets a special symbol that humans call ***zero***. A shortage can be indicated by a negative count. Together with zero the negative counts and the natural numbers form the ***integer numbers***.

The counting also introduces the possibility to sequence scalars. Scalars can be attached to points. This enables the division of a vector into parts. The lengths of these fractions introduce the ***rational*** scalars. In this way, any location on a vector direction line can be approached arbitrarily close. The direction line can also give place to scalars that cannot be interpreted as a fraction. Humans call these scalars ***irrational***. The combination of the rational scalars and the irrational scalars form the set of what humans call the ***real numbers***. The set of irrational numbers cannot be counted. The real scalars form a vector space in which only one vector has a unit length.

In the real number space, division by a scalar is defined for all scalars that are not identical to zero.

4 Complex numbers

In the real number space, the product of a scalar with itself is defined for all scalars. However, this procedure cannot be reversed for negative scalars. To enable the reverse procedure a second vector must be introduced that will carry the square roots of the negative scalars. The two normed vectors are independent and join at the location of point zero. The resulting number space contains what humans call complex numbers. Humans call the square roots of negative numbers the imaginary part of the complex number. Some people consider this qualification confusing because 'imaginary' also has other meanings. The complex number space is constituted of two different vector spaces. In one of the two vector spaces, arithmetic is defined that also holds for real numbers. In the other vector space, the product of the vectors is given by different arithmetic. In number theory, the qualification 'imaginary' indicates that the number obeys different arithmetic than real numbers do.

5 Arithmetic for multiple dimensions

In mathematics, a field is a set on which addition, subtraction, multiplication, and division are defined and behave mostly as the corresponding operations on rational and real numbers do. Division rings differ from fields only in that their multiplication is not required to be commutative.

Number systems can contain several independent vectors. Along the direction of one of these vectors, the real numbers are arranged. The other dimensions obey a different kind of arithmetic. This invites to divide the numbers into two parts, a real scalar part, and an imaginary spatial part. We will use boldface to indicate the spatial part and we will indicate the scalar part with suffix $_r$.

Thus, the number a will be represented by the sum $a=a_r+\mathbf{a}$. This means that the product $c=a\,b$ of two numbers a and b will split into several terms

$$c = c_r+\mathbf{c} = a\,b = (a_r+\mathbf{a})\,(b_r+\mathbf{b}) = a_r\,b_r + a_r\,\mathbf{b} + \mathbf{a}\,b_r + \mathbf{a}\,\mathbf{b}$$

The product d of two spatial numbers **a** and **b** results in a real scalar part and a new spatial part

$$d = d_r+\mathbf{d} = \mathbf{a}\,\mathbf{b}$$

$d_r = -\langle \mathbf{a},\mathbf{b} \rangle$ is the inner product of **a** and **b**

$\mathbf{d} = \mathbf{a} \times \mathbf{b}$ is the outer product of **a** and **b**

The spatial vector $\mathbf{d}$ is independent of $\mathbf{a}$ and independent of $\mathbf{b}$. This means that $\langle \mathbf{a},\mathbf{d} \rangle=0$, and $\langle \mathbf{b},\mathbf{d} \rangle=0$

For the inner product and the norm, $\|\mathbf{a}\|$ holds $\langle \mathbf{a},\mathbf{a} \rangle = \|\mathbf{a}\|^2$

Only three mutually independent spatial number parts can be involved in the outer product.

These formulas still do not determine the sign of the outer product. Apart from that sign, the outer product is fixed.

The product of multidimensional numbers will split into five terms.

$$c = c_r + \mathbf{c} = a\, b \equiv (a_r + \mathbf{a})\,(b_r + \mathbf{b}) = a_r\, b_r - \langle \mathbf{a},\mathbf{b} \rangle + \mathbf{a}\, b_r + a_r\, \mathbf{b} \pm \mathbf{a}\times\mathbf{b}$$

Before these formulas are used, the sign of the outer product must be selected.

All number systems that are associative division rings are either real numbers, complex numbers, or ***quaternions***. The irrational numbers also obey the arithmetic that is shown above. Especially in multidimensional number systems, irrational numbers add stickiness to the number system.

The conjugate a^* of number $a= a_r+\mathbf{a}$ is defined by $a^*=a_r-\mathbf{a}$

$(ab)^* =b^*a^*$

The norm $\|a\|$ of a is defined by

$a\, a^*= a^*a = \|a\|^2= a_r^{\,2}+\|\mathbf{a}\|^2$

5.1 Coordinate systems

Multidimensional number systems exist in many versions that distinguish by the Cartesian and polar coordinate systems that sequence their elements. This sequencing determines the geometric symmetry and the geometric center of this version of the number system.

In three spatial dimensions, the polar coordinates are called spherical coordinates. Cartesian coordinates can be converted into polar or spherical coordinates. Also, the reverse conversion exists.

The number system is mapped onto the underlying elemental vector space. The map offers much freedom to the corresponding sequencing procedure. For example, the enumeration of the Cartesian coordinates along the separate axes can be reversed on each of the axes. Also, the geometrical center of the coordinate system can be selected. The axes must be mutually independent, but further, the directions of the spatial axes can be selected freely. The final choice determines the geometric symmetry and the geometric center of the number system.

Vector spaces that are equipped with an inner product are called Euclidean spaces. The existence of a uniquely defined outer product is not required for Euclidean space.

6 Map of vector space

If two vector spaces have the same number of mutually independent vectors, then they have the same dimension. This enables constructing a map of the first vector space onto the second vector space. This map introduces relations between the original vectors and their maps. It is possible to map a vector space onto itself. In that case, one of the relations is called the inner product, and the vector space is called the inner product space. This naming is confusing because this inner product differs considerably from the inner product that exists between spatial parts in number systems.

The resulting inner product space features the astonishing capability that its maps can archive the numbers that are delivered by the inner product of vectors that map onto themselves. For that reason, the maps are also called operators. The archived numbers are called eigenvalues and the involved vectors are called eigenvectors. The operators manage the archived numbers in their eigenspaces. The inner product space is a direct extension of the underlying elemental vector space.

This investigation passes the interesting question of why vector spaces exist that can map onto other vector spaces or themselves and what activates these spaces to construct that map. This paper leaves that question open. A century ago a group of mathematicians discovered the existence of such vector spaces.

6.1 Hilbert space

At the beginning of the last century David Hilbert and others discovered this behavior of inner product spaces. John von Neumann, the assistant of David Hilbert introduced the name Hilbert space for inner product spaces that are complete. The most important aspect of Hilbert spaces is their capability to archive sets of numbers inside the eigenspaces of operators. The eigenvalues of all operators of a Hilbert space must be a member of a selected version of an associative division ring[2]. This selected version supplies the Hilbert space with a private parameter space that determines the geometric symmetry and the geometric center of the Hilbert space. This private parameter space is the natural parameter space of the Hilbert space. It is the parameter space of functions for which the target values populate the eigenspaces of a class of natural operators. Other operators can exist in a Hilbert space that manages a different parameter space of a function in their eigenspace. These are not natural operators.

6.2 Bra's and ket's

Paul Dirac introduced a handy notation for the relationship that exists between an original vector and its map. This relation applies to a bra and a ket [1]. This section treats the case that the inner product space applies quaternions to specify the values of its inner products.

The bra $\left\langle \vec{f} \right|$ is a covariant vector, and the ket $\left| \vec{g} \right\rangle$ is a contravariant vector. The inner product $\left\langle \vec{f} \mid \vec{g} \right\rangle$ acts as a metric. It has a quaternionic value. Since the product of quaternions is not commutative, care must be taken with the format of the formulas.

6.2.1 Ket vectors

The addition of ket vectors is commutative and associative.

$$\left|\vec{f}\right\rangle + \left|\vec{g}\right\rangle = \left|\vec{g}\right\rangle + \left|\vec{f}\right\rangle = \left|\vec{f} + \vec{g}\right\rangle \tag{6.2.1}$$

$$\left(\left|\vec{f} + \vec{g}\right\rangle\right) + \left|\vec{h}\right\rangle = \left|\vec{f}\right\rangle + \left(\left|\vec{g} + \vec{h}\right\rangle\right) = \left|\vec{f} + \vec{g} + \vec{h}\right\rangle \tag{6.2.2}$$

Together with quaternions, a set of ket vectors forms a ket vector space. Ket vectors are covariant vectors.

A quaternion α can be used to construct a covariant linear combination with the ket vector $\left|\vec{f}\right\rangle$

$$\left|\alpha \vec{f}\right\rangle = \left|\vec{f}\right\rangle \alpha \tag{6.2.3}$$

6.2.2 Bra vectors

For bra vectors hold

$$\left\langle\vec{f}\right| + \left\langle\vec{g}\right| = \left\langle\vec{g}\right| + \left\langle\vec{f}\right| = \left\langle\vec{f} + \vec{g}\right| \tag{6.2.4}$$

$$\left(\left\langle\vec{f} + \vec{g}\right|\right) + \left\langle\vec{h}\right| = \left\langle\vec{f}\right| + \left(\left\langle\vec{g} + \vec{h}\right|\right) = \left\langle\vec{f} + \vec{g} + \vec{h}\right| \tag{6.2.5}$$

Bra vectors are contravariant vectors.

$$\left\langle\alpha \vec{f}\right| = \alpha^* \left\langle\vec{f}\right| \tag{6.2.6}$$

Quaternions can constitute linear combinations with bra vectors.

A set of bra vectors form the vector space that is adjunct to the vector space of ket vectors that are the origins of these maps. If the map images the adjunct space onto the original vector space, then the bra vectors may be mapped onto the same ket vector.

6.2.3 Inner products

For the inner product holds

$$\left\langle \vec{f} \mid \vec{g} \right\rangle = \left\langle \vec{g} \mid \vec{f} \right\rangle^* \qquad (6.2.7)$$

For quaternionic numbers α and β hold

$$\left\langle \alpha \vec{f} \mid \vec{g} \right\rangle = \left\langle \vec{g} \mid \alpha \vec{f} \right\rangle^* = \left(\left\langle \vec{g} \mid \vec{f} \right\rangle \alpha \right)^* = \alpha^* \left\langle \vec{f} \mid \vec{g} \right\rangle \qquad (6.2.8)$$

$$\left\langle \vec{f} \mid \beta \vec{g} \right\rangle = \left\langle \vec{f} \mid \vec{g} \right\rangle \beta \qquad (6.2.9)$$

$$\begin{aligned} \left\langle (\alpha + \beta) \vec{f} \mid \vec{g} \right\rangle &= \alpha^* \left\langle \vec{f} \mid \vec{g} \right\rangle + \beta^* \left\langle \vec{f} \mid \vec{g} \right\rangle \\ &= (\alpha + \beta)^* \left\langle \vec{f} \mid \vec{g} \right\rangle \end{aligned} \qquad (6.2.10)$$

This corresponds with (6.2.3) and (6.2.6)

$$\left\langle \alpha \vec{f} \right| = \alpha^* \left\langle \vec{f} \right| \qquad (6.2.11)$$

$$\left| \alpha \vec{g} \right\rangle = \left| \vec{g} \right\rangle \alpha \qquad (6.2.12)$$

We made a choice. Another possibility would be $\left\langle \alpha \vec{f} \right| = \alpha \left\langle \vec{f} \right|$ and $\left| \alpha \vec{g} \right\rangle = \alpha^* \left| \vec{g} \right\rangle$

6.2.4 Operator construction

$\left| \vec{f} \right\rangle \left\langle \vec{g} \right|$ is a constructed operator.

$$\left| \vec{g} \right\rangle \left\langle \vec{f} \right| = \left(\left| \vec{f} \right\rangle \left\langle \vec{g} \right| \right)^\dagger \qquad (6.2.13)$$

The superfix † indicates the adjoint version of the operator.

For the orthonormal base $\{ | \vec{q}_i \rangle \}$ consisting of eigenvectors of the reference operator, holds

$$\langle \vec{q}_n \mid \vec{q}_m \rangle = \delta_{nm} \tag{6.2.14}$$

The ***reverse bra-ket method*** enables the definition of new operators that are defined by quaternionic functions.

$$\langle \vec{g} \mid F \mid \vec{h} \rangle = \sum_{i=1}^{N} \left\{ \langle \vec{g} | \vec{q}_i \rangle F(q_i) \langle \vec{q}_i | \vec{h} \rangle \right\} \tag{6.2.15}$$

The symbol F is used both for the operator F and the quaternionic function $F(q)$. This enables the shorthand

$$F \equiv |\vec{q}_i\rangle F(q_i) \langle \vec{q}_i| \tag{6.2.16}$$

for operator F. It is evident that for the adjoint operator

$$F^\dagger \equiv |\vec{q}_i\rangle F^*(q_i) \langle \vec{q}_i| \tag{6.2.17}$$

For ***reference operator*** $\mathfrak{R}$ holds

$$\mathfrak{R} = |\vec{q}_i\rangle q_i \langle \vec{q}_i| \tag{6.2.18}$$

If $\{q_i\}$ consists of all rational values of the version of the quaternionic number system that Hilbert space $\mathfrak{H}$ applies then the eigenspace of $\mathfrak{R}$ represents the natural parameter space of the separable Hilbert space $\mathfrak{H}$. It is also the parameter space of the function $F(q)$ that defines the operator F in the formula (6.2.16).

6.2.5 Operator types

I is used to indicate the identity operator.

For normal operator N holds $NN^\dagger = NN^\dagger$.

For unitary operator U holds $UU^\dagger = U^\dagger U = I$

For Hermitian operator H holds $H = H^\dagger$

A normal operator N has a Hermitian part $\frac{N+N^{\dagger}}{2}$ and an anti-Hermitian part $\frac{N-N^{\dagger}}{2}$

For anti-Hermitian operator A holds $A=-A^{\dagger}$

6.3 Separable space

In mathematics a topological space is called separable if it contains a countable dense subset; that is, there exists a sequence $\left\{\left|\vec{f}_i\right\rangle\right\}_{i=\infty}^{i=0}$ of elements of the space such that every nonempty open subset of the space contains at least one element of the sequence.

Its values on this countable dense subset determine every continuous function on the separable inner product space.

The Hilbert space $\mathfrak{H}$ is separable. That means that a countable row of elements $\left\{\left|\vec{f}_n\right\rangle\right\}$ exists that spans the whole space. In this Hilbert space, the quaternions are treated as a mathematical field.

If $\left\langle \vec{f}_m \mid \vec{f}_n \right\rangle = \delta(m,n)$ [1 if n=m; otherwise 0], then $\left\{\left|\vec{f}_n\right\rangle\right\}$ is an orthonormal base of Hilbert space $\mathfrak{H}$.

A ket base $\left\{\left|\vec{k}\right\rangle\right\}$ of $\mathfrak{H}$ is a minimal set of ket vectors $\left|\vec{k}\right\rangle$ that span the full Hilbert space $\mathfrak{H}$.

Any ket vector $\left|\vec{f}\right\rangle$ in $\mathfrak{H}$ can be written as a linear combination of elements of $\left\{\left|\vec{k}\right\rangle\right\}$.

$$\left|\vec{f}\right\rangle = \sum_k \left|\vec{k}\right\rangle \left\langle \vec{k} \mid \vec{f}\right\rangle \qquad (6.3.1)$$

A bra base $\left\{\left\langle \vec{b} \right|\right\}$ of $\mathfrak{H}^{\dagger}$ is a minimal set of bra vectors $\left\langle \vec{b} \right|$ that span the full Hilbert space $\mathfrak{H}^{\dagger}$.

Any bra vector $\left\langle \vec{f} \right|$ in $\mathfrak{H}^{\dagger}$ can be written as a linear combination of elements of $\left\{\left\langle \vec{b} \right|\right\}$.

$$\left\langle \vec{f} \right| = \sum_{b} \left\langle \vec{f} \mid \vec{b} \right\rangle \left\langle \vec{b} \right| \tag{6.3.2}$$

Usually, a base selects vectors such that their norm equals 1. Such a base is called an orthonormal base.

Separable Hilbert spaces do not support closed sets of irrational numbers. The eigenspaces of their operators are countable.

6.4 Non-separable Hilbert space

Every infinite-dimensional separable Hilbert space owns a unique non-separable companion Hilbert space that embeds its separable partner. The non-separable Hilbert space allows operators that maintain eigenspaces that contain closed sets of irrational eigenvalues. These eigenspaces behave as dynamic sticky continuums.

Gelfand triple and ***Rigged Hilbert space*** are other names for the general non-separable Hilbert spaces.

In the non-separable Hilbert space, for operators with continuum eigenspaces, the reverse bra-ket method turns from a summation into an integration.

$$\left\langle \vec{g} \mid F \mid \vec{h} \right\rangle \equiv \int\iiint \left\{ \left\langle \vec{g} \middle| \vec{q} \right\rangle F(q) \left\langle \vec{q} \middle| \vec{h} \right\rangle \right\} dV d\tau \tag{6.4.1}$$

Here we omitted the enumerating subscripts that were used in the countable base of the separable Hilbert space.

The shorthand for the operator F is now

$$F \equiv |\vec{q}\rangle F(q) \langle\vec{q}| \tag{6.4.2}$$

For eigenvectors $|q\rangle$, the function $F(q)$ defines as

$$F(q) = \langle\vec{q} \mid F\vec{q}\rangle = \int\iiint \{\langle\vec{q} \mid \vec{q}\,'\rangle F(q') \langle\vec{q}\,' \mid \vec{q}\rangle\} dV' d\tau' \tag{6.4.3}$$

The reference operator $\mathcal{R}$ that provides the continuum natural parameter space as its eigenspace follows from

$$\left\langle \vec{g} \mid \mathcal{R}\vec{h} \right\rangle \equiv \int\iiint \left\{ \left\langle \vec{g} \middle| \vec{q} \right\rangle q \left\langle \vec{q} \middle| \vec{h} \right\rangle \right\} dV d\tau \tag{6.4.4}$$

The corresponding shorthand is

$$\mathcal{R} \equiv |\vec{q}\rangle q \langle\vec{q}| \tag{6.4.5}$$

The reference operator is a special kind of defined operator. Via the quaternionic functions that specify defined operators, the claim becomes clear that every infinite-dimensional separable Hilbert space owns a unique non-separable companion Hilbert space that can be considered to embed its separable companion.

The reverse bracket method combines Hilbert space operator technology with quaternionic function theory and indirectly with quaternionic differential and integral technology.

6.5 Quaternionic function space

Each quaternionic separable Hilbert space owns a reference operator that manages an eigenspace that is formed by the version of the quaternionic number system that this Hilbert space applies to specify the values of the inner product of its vector pairs. This eigenspace is the natural eigenspace of this Hilbert space.

The eigenvectors of the reference operator constitute an orthonormal base of the Hilbert space. The reference operator is a natural operator. A category of normal operators can be defined that share the eigenvectors of the reference operator and use the target values that belong to the original eigenvalues as the new eigenvalues of the defined operator. These operators are natural operators. According to this reasoning is every quaternionic separable Hilbert space a quaternionic function space. In that function space, the eigenvectors of the reference operator represent Dirac delta distributions.

6.5.1 Position space and change space

If the members of the real axis are interpreted as instants of time, then the spatial parts of the quaternions form spatial positions in a dynamic ***position space***. The dynamic position space corresponds to the eigenspace of the natural reference operator. Thus, another name of the natural reference operator is position operator.

Another orthonormal base of the Hilbert space forms another function space. An orthonormal base exists in which each member can be written as a linear combination of all base vectors of the position space such that all superposition coefficients have the same norm. We call the resulting space a ***change space***. The eigenvectors of the change operator correspond to the parameter space of the change space. This is not a natural parameter space and the change operator is not a natural operator. Any dynamic function that is defined in the position space corresponds with a function in the change space. That function is the Fourier transform of the original function that is defined in the dynamic position space.

Integrating in position space in a selected spatial direction results in the full compression of that dimension in change space.

6.5.2 Fourier transform

Fourier transforms are easier described in a complex-number-based Hilbert space. The complex-number-based Hilbert space results from selecting all base vectors that belong to the same spatial direction in the dynamic position space of the quaternionic Hilbert space and construct a new complex-number-based Hilbert space from the selected orthonormal base.

The Fourier transform in this complex-number-based Hilbert space is given by the relation between $f(x)$ and $\tilde{f}\left(\xi_n\right)$ in the sum

$$f(x)=\sum_{n=-\infty}^{\infty}\left\{\tilde{f}\left(\xi_n\right)e^{2\pi i\xi_n x}\left(\xi_{n+1}-\xi_n\right)\right\} \qquad (6.5.1)$$

In the limit where $\Delta\xi=\left(\xi_{n+1}-\xi_n\right)\rightarrow 0$ the sum becomes an integral

$$f(x)=\int_{-\infty}^{\infty}\left\{\tilde{f}\left(\xi\right)e^{2\pi i\xi x}\right\}d\xi \qquad (6.5.2)$$

In these formulas, the symbol i represents a normalized spatial number part of a complex number.

The function $e^{2\pi ipx}$ is an eigenfunction of the operator $i\frac{\partial}{\partial x}$.

$$i\frac{\partial}{\partial x}e^{2\pi ipx}=2\pi pe^{2\pi ipx} \qquad (6.5.3)$$

The eigenvalue p represents the eigenfunction and the eigenvector p in the change space. In the same sense, the Dirac delta $\delta(x)$ is an eigenfunction of the position operator and corresponds with the eigenvalue x of the position operator $-i\frac{\partial}{\partial p}$.

$$-i\frac{\partial}{\partial p}e^{2\pi ipx}=2\pi xe^{-2\pi ipx} \qquad (6.5.4)$$

$$\delta(x-a)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ip(x-a)}dp \qquad (6.5.5)$$

$$e^{2\pi ipa} = \int_{-\infty}^{\infty} \delta(x-a) e^{2\pi ipx} dx \qquad (6.5.6)$$

6.5.3 Suggestion for quaternionic Hilbert space

In the quaternionic Hilbert space the exponential poses problems. We suggest splitting that term into two inner products $\langle \vec{q} \mid \vec{\xi} \rangle = \langle \vec{\xi} \mid \vec{q} \rangle^*$ with a constant norm $\left\| \langle \vec{q} \mid \vec{\xi} \rangle \right\|$

$$f(q) = \langle \vec{q} \mid F\vec{q} \rangle = \int\iiint \left\{ \langle \vec{q} \mid \vec{\xi} \rangle \, \tilde{f}(\xi) \langle \vec{\xi} \mid \vec{q} \rangle \right\} dV' d\tau' \qquad (6.5.7)$$

The operator F manages the base vectors of the position space as its eigenvectors and its eigenspace is described by function $f(q)$

$$F \equiv |\vec{q}\rangle \, f(q) \langle \vec{q}| \qquad (6.5.8)$$

The two inner products rotate part of the value of $\tilde{f}(\xi)$.

$$\tilde{f}(\xi) = \langle \vec{\xi} \mid \tilde{F}\vec{\xi} \rangle = \int\iiint \left\{ \langle \vec{\xi} \mid \vec{q} \rangle \, f(q) \langle \vec{q} \mid \vec{\xi} \rangle \right\} dV d\tau \qquad (6.5.9)$$

The operator $\tilde{F}$ manages the base vectors of the change space as its eigenvectors and its eigenspace is described by function $\tilde{f}(\xi)$

$$\tilde{F} \equiv |\vec{\xi}\rangle \, \tilde{f}(\xi) \langle \vec{\xi}| \qquad (6.5.10)$$

The two inner products rotate part of the value of $f(q)$.

In physics the change operator $\nabla = \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}$ is often called the momentum operator and the change space is called ***momentum space***.

These names are confusing because the momentum operator that represents the product of mass and velocity differs from the change operator.

The *quaternionic momentum operator* is treated later in more detail.

7 Field equations

Field equations are quaternionic functions or quaternionic differential and integral equations that describe the behavior of the continuum part of fields.

7.1 Quaternions

We will use a vector cap to indicate the spatial part and we will indicate the scalar part with suffix $_r$. This differs from the earlier notation that uses boldface for the spatial part of the quaternion.

Thus, the number a will be represented by the sum $a = a_r + \vec{a}$. This means that the product $c = ab$ of two numbers a and b will split into several terms

$$c = c_r + \vec{c} = ab = \left(a_r + \vec{a}\right)\left(b_r + \vec{b}\right) = a_r b_r + a_r \vec{b} + \vec{a} b_r + \vec{a}\vec{b} \quad (7.1.1)$$

The product d of two spatial numbers $\vec{a}$ and $\vec{b}$ results in a real scalar part and a new spatial part

$$d = d_r + \vec{d} = \vec{a}\vec{b} \quad (7.1.2)$$

$d_r = -\left\langle \vec{a}, \vec{b} \right\rangle$ is the inner product of $\vec{a}$ and $\vec{b}$

$\vec{d} = \vec{a} \times \vec{b}$ is the outer product of $\vec{a}$ and $\vec{b}$

The spatial vector $\vec{d}$ is independent of $\vec{a}$ and independent of $\vec{b}$. This means that $\left\langle \vec{a}, \vec{d} \right\rangle = 0$ and $\left\langle \vec{b}, \vec{d} \right\rangle = 0$

For the inner product and the norm $\|\vec{a}\|$ holds $\left\langle \vec{a}, \vec{a} \right\rangle = \|\vec{a}\|^2$

Only three mutually independent spatial number parts can be involved in the outer product.

These formulas still do not determine the sign of the outer product. Apart from that sign, the outer product is fixed.

Quaternionic multiplication obeys the equation

$$\begin{aligned} c = c_r + \vec{c} = ab = \left(a_r + \vec{a}\right)\left(b_r + \vec{b}\right) \\ = a_r b_r - \left\langle \vec{a}, \vec{b} \right\rangle + a_r \vec{b} + \vec{a} b_r \pm \vec{a} \times \vec{b} \end{aligned} \quad (7.1.3)$$

The $\pm$ sign indicates the freedom of choice of the handedness of the product rule that exists when selecting a version of the quaternionic number system. The version must be selected before it can be used in calculations.

Two quaternions that are each other's inverse can rotate the spatial part of another quaternion.

$$c = ab / a \quad (7.1.4)$$

The construct rotates the spatial part of b that is perpendicular to $\vec{a}$ over an angle that is twice the angular phase θ of $a = \|a\| e^{\vec{i}\theta}$ where $\vec{i} = \vec{a} / \|\vec{a}\|$.

Cartesian quaternionic functions apply a quaternionic parameter space that is sequenced by a Cartesian coordinate system. In the parameter space, the real scalar parts of quaternions are often interpreted as instances of (proper) time, and the spatial parts are often interpreted as spatial locations. The real scalar parts of quaternionic functions represent dynamic scalar fields. The spatial parts of quaternionic functions represent dynamic vector fields.

7.2 Quaternionic differential calculus

The differential change can be expressed in terms of a linear combination of partial differentials. Now the total differential change df of field f equals

$$df = \frac{\partial f}{\partial \tau} d\tau + \frac{\partial f}{\partial x} \vec{i} dx + \frac{\partial f}{\partial y} \vec{j} dy + \frac{\partial f}{\partial z} \vec{k} dz \qquad (7.2.1)$$

In this equation, the partial differentials $\frac{\partial f}{\partial \tau}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial y}$ behave like quaternionic differential operators.

The quaternionic nabla ∇ assumes the ***special condition*** that partial differentials direct along the axes of the Cartesian coordinate system in a natural parameter space of a non-separable Hilbert space. Thus,

$$\nabla = \sum_{i=0}^{4} \vec{e}_i \frac{\partial}{\partial x_i} = \frac{\partial}{\partial \tau} + \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \qquad (7.2.2)$$

This will be applied in the next section by splitting both the quaternionic nabla and the function in a scalar part and a vector part.

The first-order partial differential equations divide the first-order change of a quaternionic field into five different parts that each represent a new field. We will represent the quaternionic field change operator by a quaternionic nabla operator. This operator behaves like a quaternionic multiplier.

The first order partial differential follows from

$$\nabla = \left\{ \frac{\partial}{\partial \tau}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\} = \nabla_r + \vec{\nabla} \qquad (7.2.3)$$

The spatial nabla $\vec{\nabla}$ is well-known as the del operator and is treated in detail in Wikipedia [5].

$$\phi = \nabla \psi = \left(\frac{\partial}{\partial \tau} + \vec{\nabla} \right) (\psi_r + \vec{\psi})$$
$$= \nabla_r \psi_r - \left\langle \vec{\nabla}, \vec{\psi} \right\rangle + \nabla_r \vec{\psi} + \vec{\nabla} \psi_r \pm \vec{\nabla} \times \vec{\psi} \tag{7.2.4}$$

In a selected version of the quaternionic number system, only the corresponding version of the quaternionic nabla is active. In a selected Hilbert space, this version is always and everywhere the same.

The differential $\nabla \psi$ describes the change of field ψ. The five separate terms in the first-order partial differential have a separate physical meaning. All basic fields feature this decomposition. The terms may represent new fields.

$$\phi_r = \nabla_r \psi_r - \left\langle \vec{\nabla}, \vec{\psi} \right\rangle \tag{7.2.5}$$

$$\vec{\phi} = \nabla_r \vec{\psi} + \vec{\nabla} \psi_r \pm \vec{\nabla} \times \vec{\psi} \tag{7.2.6}$$

$\vec{\nabla} f$ is the gradient of f.

$\left\langle \vec{\nabla}, \vec{f} \right\rangle$ is the divergence of $\vec{f}$.

$\vec{\nabla} \times \vec{f}$ is the curl of $\vec{f}$.

$$\left(\vec{\nabla}, \vec{\nabla} \right) \psi = \Delta \psi = \nabla^2 \psi \tag{7.2.7}$$

$$\left(\vec{\nabla}, \vec{\nabla} \times \vec{\psi} \right) = 0 \tag{7.2.8}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \psi_r \right) = 0 \tag{7.2.9}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{\psi} \right) = \vec{\nabla} \left(\vec{\nabla}, \vec{\psi} \right) - \left(\vec{\nabla}, \vec{\nabla} \right) \vec{\psi} \tag{7.2.10}$$

Sometimes parts of the change get new symbols

$$\vec{E} = -\nabla_r \vec{\psi} - \vec{\nabla} \psi_r \tag{7.2.11}$$

$$\vec{B} = \vec{\nabla} \times \vec{\psi} \qquad (7.2.12)$$

The formula (7.2.4) does not leave room for gauges. In Maxwell equations, the equation (7.2.5) is a gauge.

$$\left(\vec{\nabla}, \vec{B}\right) = 0 \qquad (7.2.13)$$

$$\vec{\nabla} \times \vec{E} = -\nabla_r \vec{\nabla} \times \vec{\psi} - \vec{\nabla} \times \vec{\nabla} \psi_r = -\nabla_r \vec{B} \qquad (7.2.14)$$

$$\begin{aligned} &\left(\vec{\nabla}, \vec{E}\right) = -\nabla_r \left(\vec{\nabla}, \vec{\psi}\right) - \left(\vec{\nabla}, \vec{\nabla}\right) \psi_r \\ &\left\{\nabla_r \nabla_r + \left(\vec{\nabla}, \vec{\nabla}\right)\right\} \psi_r = x \end{aligned} \qquad (7.2.15)$$

The conjugate of the quaternionic nabla operator defines another type of field change.

$$\nabla^* = \nabla_r - \vec{\nabla} \qquad (7.2.16)$$

$$\begin{aligned} \zeta = \nabla^* \phi &= \left(\frac{\partial}{\partial \tau} - \vec{\nabla}\right)\left(\phi_r + \vec{\phi}\right) \\ &= \nabla_r \phi_r + \left\langle \vec{\nabla}, \vec{\phi} \right\rangle + \nabla_r \vec{\phi} - \vec{\nabla} \phi_r \mp \vec{\nabla} \times \vec{\phi} \end{aligned} \qquad (7.2.17)$$

All dynamic quaternionic fields obey the same first-order partial differential equations (7.2.4) and (7.2.17).

$$\nabla^\dagger = \nabla^* = \nabla_r - \vec{\nabla} = \nabla_r + \vec{\nabla}^\dagger = \nabla_r + \vec{\nabla}^* \qquad (7.2.18)$$

In the Hilbert space, the quaternionic nabla is a normal operator.

$$\nabla^\dagger \nabla = \nabla \nabla^\dagger = \nabla^* \nabla = \nabla \nabla^* = \nabla_r \nabla_r + \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle \qquad (7.2.19)$$

Are normal operators who are also Hermitian.

The separate operators $\nabla_r \nabla_r$ and $\left\langle \vec{\nabla}, \vec{\nabla} \right\rangle$ are also Hermitian operators. They can also be combined as $\square = \nabla_r \nabla_r - \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle$. This is the d'Alembert operator. The solutions of $\nabla_r \nabla_r + \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle = 0$ and $\nabla_r \nabla_r + \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle = 0$ differ. They differentiate between the behavior of the field. The equations describe the behavior of the embedding field that physicists call their universe. This dynamic field exists everywhere in the reach of the parameter space of the function. The homogeneous d'Alembert equation is known as the wave equation and offers waves and wave packages as its solutions. Both equations offer shock fronts as solutions but only the operators in (7.2.19) deliver shock fronts that feature a spin or polarization vector. Integration over the time domain turns both equations in the Poisson equation and removes the spin or polarization vector. Shock fronts require a corresponding actuator and occur only in odd numbers of participating dimensions. Spherical shock fronts require an isotropic actuator.

7.3 Continuity equations

Continuity equations are partial quaternionic differential equations.

7.3.1 Field excitations

The dynamic changes of the field are then interpreted as field excitations or as field deformations or field expansions.

Field excitations are solutions of second-order partial differential equations.

One of the second-order partial differential equations results from combining the two first-order partial differential equations $\phi = \nabla \psi$ and $\zeta = \nabla^* \phi$.

$$\zeta = \nabla^* \varphi = \nabla^* \nabla \psi = \nabla \nabla^* \psi = \left(\nabla_r + \vec{\nabla}\right)\left(\nabla_r - \vec{\nabla}\right)\left(\psi_r + \vec{\psi}\right)$$
$$= \left(\nabla_r \nabla_r + \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle\right)\psi \qquad (7.3.1)$$

Integration over the time domain results in the Poisson equation

$$\rho = \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle \psi \qquad (7.3.2)$$

Under isotropic conditions, a very special solution of the Poisson equation is the Green's function $\frac{1}{4\pi\left|\vec{q} - \vec{q'}\right|}$ of the affected field [33].

This solution is the spatial Dirac $\delta\left(\vec{q}\right)$ pulse response of the field under strict isotropic conditions.

$$\nabla \frac{1}{\left|\vec{q} - \vec{q'}\right|} = -\frac{\left(\vec{q} - \vec{q'}\right)}{\left|\vec{q} - \vec{q'}\right|^3} \qquad (7.3.3)$$

$$\left\langle \vec{\nabla}, \vec{\nabla} \right\rangle \frac{1}{\left|\vec{q} - \vec{q'}\right|} \equiv \left\langle \vec{\nabla}, \vec{\nabla} \frac{1}{\left|\vec{q} - \vec{q'}\right|} \right\rangle$$
$$= -\left\langle \vec{\nabla}, \frac{\left(\vec{q} - \vec{q'}\right)}{\left|\vec{q} - \vec{q'}\right|^3} \right\rangle = 4\pi\delta\left(\vec{q} - \vec{q'}\right) \qquad (7.3.4)$$

This solution corresponds with an ongoing source or sink that exists in the field.

Change can take place in one dimension or combined in two or three dimensions.

Under isotropic conditions, the dynamic spherical pulse response of the field is a solution of a special form of the equation (7.3.1)

$$\left(\nabla_r\nabla_r + \left\langle\vec{\nabla},\vec{\nabla}\right\rangle\right)\psi = 4\pi\delta\left(\vec{q}-\overrightarrow{q'}\right)\theta\left(\tau\pm\tau'\right) \qquad (7.3.5)$$

Here $\theta(\tau)$ is a step function and $\delta(\vec{q})$ is a Dirac pulse response. For the spherical pulse response, the pulse must be isotropic.

After the instant τ', the equation turns into a homogeneous equation.

A remarkably simple solution is the shock front in one dimension along the line $\vec{q}-\overrightarrow{q'}$.

$$\psi = f\left(\left|\vec{q}-\overrightarrow{q'}\right| \pm c\left(\tau-\tau'\right)\vec{n}\right) \qquad (7.3.6)$$

Here $\vec{n}$ is a normed spatial quaternion. This spatial quaternion has an arbitrary direction that does not vary in time. Here, the normalized vector $\vec{n}$ can be interpreted as the polarization of the solution [41].

In isotropic conditions, we better switch to polar coordinates. Then the equation gets the form

$$\begin{aligned}&\left(\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial r^2}+2\frac{\partial}{r\partial r}\right)\psi \\ &=\left(\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial r^2}\right)(\psi r)=0\end{aligned} \qquad (7.3.7)$$

The second line describes the second-order change of ψr in one dimension along the radius r. That solution is described above. A solution of this line is

$$\psi r = f\left(r \pm c\tau\vec{n}\right) \qquad (7.3.8)$$

The solution of (7.3.7) is described by

$$\psi = \frac{f\left(\left|\vec{q}-\overrightarrow{q'}\right| \pm c(\tau - \tau')\vec{n}\right)}{\left|\vec{q}-\overrightarrow{q'}\right|} \tag{7.3.9}$$

The normalized vector $\vec{n}$ can be interpreted as the spin of the solution. The spherical pulse response acts either as an expanding or as a contracting spherical shock front. Over time this pulse response integrates into the Green's function. This means that the isotropic pulse injects the volume of the Green's function into the field. Subsequently, the front spreads this volume over the field. The contracting shock front collects the volume of the Green's function and sucks it out of the field. The $\pm$ sign in the equation (7.3.5) selects between injection and subtraction.

Shock fronts only occur in one and three dimensions. A pulse response can also occur in two dimensions, but in that case, the pulse response is a complicated vibration that looks like the result of a throw of a stone in the middle of a pond.

Equations (7.3.1) and (7.3.2) show that the operators $\frac{\partial^2}{\partial \tau^2}$ and $\langle \vec{\nabla}, \vec{\nabla} \rangle$ are valid second-order partial differential operators. These operators combine in the quaternionic equivalent of the wave equation [6].

$$\varphi = \left(\frac{\partial^2}{\partial \tau^2} - \langle \vec{\nabla}, \vec{\nabla} \rangle \right) \psi = \Box \psi \tag{7.3.10}$$

This equation also offers one-dimensional and three-dimensional shock fronts as its solutions.

$$\psi = \frac{f\left(\left|\vec{q}-\overrightarrow{q'}\right| \pm c(\tau - \tau')\right)}{\left|\vec{q}-\overrightarrow{q'}\right|} \tag{7.3.11}$$

$$\psi = f\left(\left|\vec{q} - \vec{q'}\right| \pm c\left(\tau - \tau'\right)\right) \tag{7.3.12}$$

These pulse responses do not contain the normed vector $\vec{n}$. Apart from pulse responses, the wave equation offers waves as its solutions.

If locally the field can be split into a time-dependent part $T(\tau)$ and a location-dependent part $A(\vec{q})$, then the homogeneous version of the wave equation can be transformed into the Helmholtz equation [7].

$$\frac{\partial^2 \psi}{\partial \tau^2} = \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle \psi = -\omega^2 \psi \tag{7.3.13}$$

$$\psi(\vec{q}, \tau) = A(\vec{q}) T(\tau) \tag{7.3.14}$$

$$\frac{1}{T} \frac{\partial^2 T}{\partial \tau^2} = \frac{1}{A} \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle A = -\omega^2 \tag{7.3.15}$$

$$\left\langle \vec{\nabla}, \vec{\nabla} \right\rangle A + \omega^2 A = 0 \tag{7.3.16}$$

$$\frac{\partial^2 T}{\partial \tau^2} + \omega^2 T = 0 \tag{7.3.17}$$

The time-dependent part $T(\tau)$ depends on initial conditions, or it indicates the switch of the oscillation mode. The switch of the oscillation mode means that temporarily the oscillation is stopped and instead an object is emitted or absorbed that compensates the difference in potential energy. The location-dependent part of the field $A(\vec{q})$ describes the possible oscillation modes of the field and depends on boundary conditions. The oscillations have a binding effect. They keep moving objects within a bounded region.

For three-dimensional isotropic spherical conditions, the solutions have the form

$$A(r,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left\{ \left(a_{lm} j_l(kr) \right) + b_{lm} Y_l^m(\theta,\varphi) \right\} \tag{7.3.18}$$

Here j_l and y_l are the spherical Bessel functions, and Y_l^m are the spherical harmonics [13][14]. These solutions play a role in the spectra of atomic modules.

Planar and spherical waves are the simpler wave solutions of the equation (7.3.13)

$$\psi(\vec{q},\tau) = \exp\left\{ \vec{n} \left(\left\langle \vec{k}, \vec{q} - \vec{q}_0 \right\rangle - \omega\tau + \varphi \right) \right\} \tag{7.3.19}$$

$$\psi(\vec{q},\tau) = \frac{\exp\left\{ \vec{n} \left(\left\langle \vec{k}, \vec{q} - \vec{q}_0 \right\rangle - \omega\tau + \varphi \right) \right\}}{\left| \vec{q} - \vec{q}_0 \right|} \tag{7.3.20}$$

A more general solution is a superposition of these basic types.

Two quite similar homogeneous second-order partial differential equations exist. They are the homogeneous versions of equations (7.3.5) and (7.3.10). The equation (7.3.5) has spherical shock front solutions with a spin vector that behaves like the spin of elementary particles. Obviously, the field only reacts dynamically when it gets triggered by corresponding actuators. Pulses may cause shock fronts that after the trigger keeps traveling. Oscillations must be triggered by periodic mechanisms.

The inhomogeneous pulse activated equations are

$$\left(\nabla_r \nabla_r \pm \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle \right) \psi = 4\pi \delta\left(\vec{q} - \vec{q'} \right) \theta\left(\tau \pm \tau' \right) \tag{7.3.21}$$

7.4 Enclosure balance equations

Enclosure balance equations are quaternionic integral equations that describe the balance between the inside, the border, and the outside of an enclosure.

These integral balance equations base on replacing the del operator $\vec{\nabla}$ with a normed vector $\vec{n}$. The vector $\vec{n}$ is oriented outward and perpendicular to a local part of the closed boundary of the enclosed region.

$$\vec{\nabla}\psi \Leftrightarrow \vec{n}\psi \tag{7.4.1}$$

This approach turns part of the differential continuity equation into a corresponding integral balance equation.

$$\iiint \vec{\nabla}\psi dV = \oiint \vec{n}\psi dS \tag{7.4.2}$$

$\vec{n}\ dS$ plays the role of a differential surface. $\vec{n}$ is perpendicular to that surface.

This result separates into three parts

$$\begin{aligned}\vec{\nabla}\psi &= -\left\langle \vec{\nabla},\vec{\psi}\right\rangle + \vec{\nabla}\psi_r \pm \vec{\nabla}\times\vec{\psi} \Leftrightarrow \vec{n}\psi \\ &= -\left\langle \vec{n},\vec{\psi}\right\rangle + \vec{n}\psi_r \pm \vec{n}\times\vec{\psi}\end{aligned} \tag{7.4.3}$$

The first part concerns the gradient of the scalar part of the field

$$\vec{\nabla}\psi_r \Leftrightarrow \vec{n}\psi_r \tag{7.4.4}$$

$$\iiint \vec{\nabla}\psi_r dV = \oiint \vec{n}\psi_r dS \tag{7.4.5}$$

The divergence is treated in an integral balance equation that is known as the Gauss theorem. It is also known as the divergence theorem [15].

$$\left\langle \vec{\nabla}, \vec{\psi} \right\rangle \Leftrightarrow \left\langle \vec{n}, \vec{\psi} \right\rangle \tag{7.4.6}$$

$$\iiint \left\langle \vec{\nabla}, \vec{\psi} \right\rangle dV = \oiint \left\langle \vec{n}, \vec{\psi} \right\rangle dS \tag{7.4.7}$$

The curl is treated in a corresponding integrated balance equation

$$\vec{\nabla} \times \vec{\psi} \Leftrightarrow \vec{n} \times \vec{\psi} \tag{7.4.8}$$

$$\iiint \vec{\nabla} \times \vec{\psi} dV = \oiint \vec{n} \times \vec{\psi} dS \tag{7.4.9}$$

Equation (7.4.7) and equation (7.4.9) can be combined in the extended theorem

$$\iiint \vec{\nabla} \vec{\psi} dV = \oiint \vec{n} \ \vec{\psi} dS \tag{7.4.10}$$

The method also applies to other partial differential equations. For example

$$\begin{aligned} \vec{\nabla} \times \left(\vec{\nabla} \times \vec{\psi} \right) = \vec{\nabla} \left\langle \vec{\nabla}, \vec{\psi} \right\rangle - \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle \vec{\psi} \Leftrightarrow \vec{\nabla} \times \left(\vec{\nabla} \times \vec{\psi} \right) \\ = \vec{n} \left\langle \vec{n}, \vec{\psi} \right\rangle - \left\langle \vec{n}, \vec{n} \right\rangle \vec{\psi} \end{aligned} \tag{7.4.11}$$

$$\iiint_V \left\{ \vec{\nabla} \times \left(\vec{\nabla} \times \vec{\psi} \right) \right\} dV = \oiint_S \left\{ \vec{\nabla} \left\langle \vec{\nabla}, \vec{\psi} \right\rangle \right\} dS - \oiint_S \left\{ \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle \vec{\psi} \right\} dS \tag{7.4.12}$$

One dimension less, a similar relation exists.

$$\iint_S \left(\left\langle \vec{\nabla} \times \vec{a}, \vec{n} \right\rangle \right) dS = \oint_C \left\langle \vec{a}, d\vec{l} \right\rangle \tag{7.4.13}$$

This is known as the Stokes theorem[16]

The curl can be presented as a line integral

$$\left\langle \vec{\nabla} \times \vec{\psi}, \vec{n} \right\rangle \equiv \lim_{A \to 0} \left(\frac{1}{A} \oint_C \left\langle \vec{\psi}, d\vec{r} \right\rangle \right) \tag{7.4.14}$$

7.5 Derivation of physical laws

The quaternionic equivalents of Ampère's law are [19]

$$\vec{J} \equiv \vec{\nabla} \times \vec{B} = \nabla_r \vec{E} \Leftrightarrow \vec{J} \equiv \vec{n} \times \vec{B} = \nabla_r \vec{E} \tag{7.5.1}$$

$$\iint_S \left\langle \vec{\nabla} \times \vec{B}, \vec{n} \right\rangle dS = \oint_C \left\langle \vec{B}, d\vec{l} \right\rangle = \iint_S \left\langle \vec{J} + \nabla_r \vec{E}, \vec{n} \right\rangle dS \tag{7.5.2}$$

The quaternionic equivalents of Faraday's law are [20]:

$$\nabla_r \vec{B} = \vec{\nabla} \times (\nabla_r \vec{\psi}) = -\vec{\nabla} \times \vec{E} \Leftrightarrow \nabla_r \vec{B} = \vec{n} \times (\nabla_r \vec{\psi}) = -\vec{\nabla} \times \vec{E} \tag{7.5.3}$$

$$\oint_c \left\langle \vec{E}, d\vec{l} \right\rangle = \iint_S \left\langle \vec{\nabla} \times \vec{E}, \vec{n} \right\rangle dS = -\iint_S \left\langle \nabla_r \vec{B}, \vec{n} \right\rangle dS \tag{7.5.4}$$

$$\vec{J} = \vec{\nabla} \times \left(\vec{B} - \vec{E} \right) = \vec{\nabla} \times \vec{\varphi} - \nabla_r \vec{\varphi} = \vec{v} \rho \tag{7.5.5}$$

$$\iint_S \left\langle \vec{\nabla} \times \vec{\varphi}, \vec{n} \right\rangle dS = \oint_C \left(\left\langle \vec{\varphi}, d\vec{l} \right\rangle \right) = \iint_S \left\langle \vec{v} \rho + \nabla_r \vec{\varphi}, \vec{n} \right\rangle dS \tag{7.5.6}$$

The equations (7.5.4) and (7.5.6) enable the derivation of the ***Lorentz force*** [21].

$$\vec{\nabla} \times \vec{E} = -\nabla_r \vec{B} \tag{7.5.7}$$

$$\frac{d}{d\tau} \iint_S \left\langle \vec{B}, \vec{n} \right\rangle dS = \iint_{S(\tau_0)} \left\langle \dot{\vec{B}}(\tau_0), \vec{n} \right\rangle ds + \frac{d}{d\tau} \iint_{S(\tau)} \left\langle \vec{B}(\tau_0), \vec{n} \right\rangle ds \tag{7.5.8}$$

The Leibniz integral equation states [22]

$$\frac{d}{dt} \iint_{S(\tau)} \left\langle \vec{X}(\tau_0), \vec{n} \right\rangle dS$$

$$= \iint_{S(\tau_0)} \left\langle \dot{\vec{X}}(\tau_0) + \left\langle \vec{\nabla}, \vec{X}(\tau_0) \right\rangle \vec{v}(\tau_0), \vec{n} \right\rangle dS - \oint_{C(\tau_0)} \left\langle \vec{v}(\tau_0) \times \vec{X}(\tau_0), d\vec{l} \right\rangle$$

(7.5.9)

With $\vec{X} = \vec{B}$ and $\left\langle \vec{\nabla}, \vec{B} \right\rangle = 0$ follows

$$\frac{d\Phi_B}{d\tau} =$$

$$\frac{d}{d\tau} \iint_{S(\tau)} \left\langle \dot{\vec{B}}(\tau), \vec{n} \right\rangle dS = \iint_{S(\tau_0)} \left\langle \vec{B}(\tau_0), \vec{n} \right\rangle dS - \oint_{C(\tau_0)} \left\langle \vec{v}(\tau_0) \times \vec{B}(\tau_0), d\vec{l} \right\rangle$$

$$= - \oint_{C(\tau_0)} \left\langle E(\tau_0), d\vec{l} \right\rangle - \oint_{C(\tau_0)} \left\langle \vec{v}(\tau_0) \times \vec{B}(\tau_0), d\vec{l} \right\rangle$$

(7.5.10)

The electromotive force (EMF) ε equals [23]

$$\varepsilon = \oint_{C(\tau_0)} \left\langle \frac{\vec{F}(\tau_0)}{q}, d\vec{l} \right\rangle = - \left. \frac{d\Phi_B}{d\tau} \right|_{\tau=\tau_0}$$

$$= \oint_{C(\tau_0)} \left\langle \vec{E}(\tau_0), d\vec{l} \right\rangle + \oint_{C(\tau_0)} \left\langle \vec{v}(\tau_0) \times \vec{B}(\tau_0), d\vec{l} \right\rangle \quad (7.5.11)$$

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} \quad (7.5.12)$$

8 Systems of Hilbert spaces

Only a subtle difference exists between an elemental vector space and the Hilbert space that maps this vector space onto itself. The inner product is the most important difference. The fact that the Hilbert space is complete is another difference. Several properties of Hilbert spaces are the consequence of this difference. An important property is the private natural parameter space of the Hilbert space that provides its geometric symmetry and its geometric center. An important restriction is that Hilbert spaces can only cope with number systems that are associative division rings. This excludes octonions and biquaternions. Each Hilbert space selects a version of an associative division ring that is determined by the coordinate systems, which sequence the elements of the chosen number system. These choices determine the geometric symmetry and the geometric center of the Hilbert space.

These restrictions still leave the possibility that in a system of Hilbert spaces all members share the same underlying elemental vector space. In this system, one of the members acts as the background platform. All other members float with their geometric center over the parameter space of the background platform. If the background platform features infinite dimensions, then its non-separable companion also becomes part of the background platform. The resulting system of Hilbert spaces will be called the ***Hilbert repository***.

8.1 Hilbert repository

Sharing the same underlying vector space imposes new restrictions and enables new capabilities. The restrictions enforce that not all possible Hilbert spaces can be a member of the Hilbert repository. The coordinate systems of the selected versions of the number systems must have their Cartesian coordinate axes in parallel. This limits the allowed symmetries to a small set.

This restriction is not obvious and current mathematics does not yet deliver this hard requirement. The existence of this restriction is derived from the Standard Model of particle physics. The Standard Model reflects the knowledge of particle physicists that is derived from measurements. In the Standard Model, the set of elementary fermions show great similarity with the set of separate quaternionic Hilbert systems that populate the Hilbert repository. Elementary fermion types appear to correspond with the differences between the symmetries of the allowed floating separable Hilbert spaces and the symmetry of the background platform.

The differences between the symmetries of the floating platforms and the background platform generate sources and sinks that locate at the geometric centers of the floating platforms. The sources and sinks correspond to symmetry-related charges that may be zero or can have one of a restricted set of values. Non-zero symmetry-related charges generate corresponding symmetry-related fields.

8.2 Embedding in the background platform

The differences in the symmetry between the platforms only become apparent when a floating platform is embedded into the background platform. A special operator in the non-separable Hilbert space of the background platform acts as the embedding field for discrete eigenvalues that originate from the eigenspace of the footprint operators that reside in the floating platform and are mapped in the eigenspace of this special operator. The special operator represents the dynamic universe field. If an isotropic difference between the embedded eigenvalue and the embedding field exists, then the embedding field reacts with a spherical pulse response. The pulse response temporarily deforms the embedding field.

8.3 Footprint

An ongoing embedding of a stream of symmetry-breaking eigenvalues will cause a persistent deformation of the embedding field. The eigenspace of the footprint operator can archive a cord of quaternionic storage bins that contain the timestamps and the embedding locations. After sequencing the timestamps, the archive shows an ongoing hopping path that translates into an ongoing embedding process. This embedding process runs during the running episode of the Hilbert repository and acts as an ***imaging process*** in which the image quality is characterized by an Optical Transfer Function [25][26]. This function is the Fourier transfer of the Point Spread Function. The Point Spread Function can be interpreted as a hop landing location density distribution. Its Fourier transform is the Optical Transfer Function of the embedding of the footprint of the considered object.

8.3.1 Footprint mechanism

The mechanism that generates the content of the eigenspace of the footprint operator did its work in the creation episode of the Hilbert repository. The private natural parameter space of the Hilbert space also existed in this creation episode. The timestamps and the hopping locations of the hopping path were taken from this private parameter space. The footprint mechanism owns a characteristic function that ensured that the hopping path recurrently regenerates a hop landing location swarm that features a stable location density distribution which is the Fourier transform of the characteristic function of the footprint mechanism. The location density distribution equals the mentioned Point Spread Function, and the characteristic function equals the corresponding Optical Transfer Function [26].

The hopping path, the hop landing location swarm, the location density distribution, and the Point Spread Function reside in the position space.

The location density distribution equals the Point Spread Function and describes the hop landing location swarm.

The Optical Transfer Function equals the characteristic function of the footprint mechanism and both reside in the change space.

Nothing is said about the distribution of the timestamps. In imaging processes, the distribution of discrete objects in the imaging beam can often be characterized as the result of a combination of a Poisson process and a binomial process, where the binomial process is implemented by a spatial point spread function. In that case, the Poisson process handles the distribution of the timestamps.

8.3.2 Footprint characteristics

The footprint generates a nearly constant stream of potential point-like actuators in the form of a swarm that features a constant location density distribution. The actuators that originate from the same floating separable Hilbert space have a constant symmetry. Some of these actuator symmetries can break the symmetry of the embedding field and therefore they can generate pulse responses that at least temporarily deform this field. A sufficiently constant and sufficiently dense and coherent stream of actuators can generate a persistent deformation.

8.4 Stickiness

The first order partial differential equation indicates what happens when a field resists change.

In that case, the terms in the equation try to compensate each other.

$$\phi = \phi_r + \vec{\phi} = \nabla \psi = \nabla_r \psi_r - \langle \vec{\nabla}, \vec{\psi} \rangle + \nabla_r \vec{\psi} + \vec{\nabla} \psi_r \pm \vec{\nabla} \times \vec{\psi} = 0? \quad (8.4.1)$$

The scalar part and the vector part are treated separately.

$$\phi_r = \nabla_r \psi_r - \langle \vec{\nabla}, \vec{\psi} \rangle = 0? \quad (8.4.2)$$

$$\vec{\phi} = \nabla_r \vec{\psi} + \vec{\nabla} \psi_r \pm \vec{\nabla} \times \vec{\psi} = 0? \tag{8.4.3}$$

For example, if the curl equals zero, then

$$\nabla_r \vec{\psi} = -\vec{\nabla} \psi_r \tag{8.4.4}$$

will set the vector part of the change to zero. In this way, vector change parts can compensate for scalar change parts.

The Green’s function, the shock fronts, and the oscillations also demonstrate the stickiness of dynamic quaternionic fields. Discrete sets of quaternions do not show this stickiness.

8.4.1 Potential

In physics, potential energy is the energy held by an object because of its position relative to other objects.

The gravitational potential at a location is equal to the work (energy transferred) per unit mass that would be needed to move an object to that location from a fixed reference location [29][30][31][32][34].

The spherical shock fronts integrate over time into the Green’s function of the field. Thus, the shock front injects the content of the Green’s function into the affected field. All spherical shock fronts spread the contents of the front over the full field.

We consider the gravitational potential to be zero at infinity. Thus, if infinity is selected as a reference location, then the gravitational potential at a considered location is equal to the work (energy transferred) per unit mass that would be needed to move an object from infinity to that location. Thus, the potential represents the reverse action of the combined spherical shock fronts.

8.4.2 Center of deformation

The deformation potential $V(r)$ describes the effect of a local response to an isotropic point-like actuator and reflects the work that must be done by an agent to bring a unit amount of the injected stuff from infinity back to the considered location.

$$V(r) = m_p G / r \qquad (8.4.5)$$

Here m_p represents the mass that corresponds to the full pulse response. G takes care for adaptation to physical units. r is the distance to the location of the pulse.

A stream of footprint actuators recurrently regenerates a coherent swarm of embedding locations in the dynamic universe field. That swarm generates a potential

$$V(r) = MG / r \qquad (8.4.6)$$

Here M represents the mass that corresponds to the considered swarm of pulse responses. r is the distance to the center of the deformation. This formula is valid at sufficiently large values of r such that the whole swarm can be considered as a point-like object.

In a coherent swarm of massive objects $p_i, i = 1,2,3,...n$, each with static mass m_i at locations r_i, the center of mass $\vec{R}$ follows from [28]

$$\sum_{i=1}^{n} m_i \left(\vec{r}_i - \vec{R} \right) = \vec{0} \qquad (8.4.7)$$

Thus

$$\vec{R} = \frac{1}{M} \sum_{i=1}^{n} m_i \vec{r}_i \qquad (8.4.8)$$

Where

$$M = \sum_{i=1}^{n} m_i \qquad (8.4.9)$$

In the following, we will consider an ensemble of massive objects that own a center of mass $\vec{R}$ and a fixed combined mass M as a single massive object that locates at $\vec{R}$. The separate masses m_i may differ because, at the instant of summation, the corresponding deformation might have partly faded away.

$\vec{R}$ can be a dynamic location. In that case, the ensemble must move as one unit. The problem with the treatise in this paragraph is that in physical reality, point-like objects that possess a static mass do not exist. Only pulse responses that temporarily deform the field exist. Except for black holes, these pulse responses constitute all massive objects that exist in the universe.

8.5 Pulse location density distribution

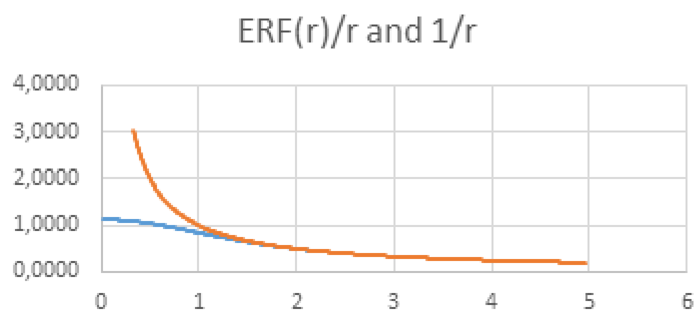
It is false to treat a pulse location density distribution as a set of point-like masses as is done in formulas (8.4.7) and (8.4.8). Instead, the gravitational potential follows from the convolution of the location density distribution and the Green's function. This calculation is still not correct, because the exact result depends on the fact that the deformation that is due to a pulse response quickly fades away and the result also depends on the density of the distribution. If these effects can be ignored, then the resulting gravitational potential of a Gaussian density distribution would be given by [35]

$$g(r) \approx GM \frac{ERF(r)}{r} \qquad (8.5.1)$$

Where $ERF(r)$ is the well-known error function. Here the gravitational potential is a perfectly smooth function that at some distance from the center equals the approximated gravitational potential that was described above in the equation (8.4.6). As indicated above, the convolution only offers an approximation because this computation does not account for the influence of the density of the swarm and it does not compensate for the fact that the deformation by the individual pulse responses quickly fades away. Thus, the exact result depends on the duration of the recurrence cycle of the swarm.

In the example, we apply a normalized location density distribution, but the actual location density distribution might have a higher amplitude.

This might explain why some elementary module types exist in three generations. These generations appear to have their own mass.



This might also explain why different first-generation elementary particle types show different masses. Due to the convolution, and the coherence of the location density distribution, the blue curve does not show any sign of the singularity that is contained in the red curve, which shows the Green's function.

In physical reality, no point-like static mass object exists. The most important lesson of this investigation is that far from the gravitational center of the distribution the deformation of the field is characterized by the here shown simplified form of the gravitation potential

$$\phi(r) \approx \frac{GM}{r} \tag{8.5.2}$$

Warning: This simplified form shares its shape with the Green's function of the deformed field. This does not mean that the Green's function owns a mass that equals $M_G = \frac{1}{G}$. The functions only share the form of their tail.

8.6 Rest mass

The weakness in the definition of the gravitation potential is the definition of the unit of mass and the fact that shock fronts move with a fixed finite speed. Thus, the definition of the gravitation potential only works properly if the geometric center location of the swarm of injected spherical pulses is at rest in the affected embedding field. The consequence is that the mass that follows from the definition of the gravitation potential is the ***rest mass*** of the considered swarm. We will call the mass that is corrected for the motion of the observer relative to the observed scene the ***inertial mass***.

8.7 Observer

The inspected location is the location of a hypothetical test object that owns an amount of mass. It can represent an elementary particle or a conglomerate of such particles. This location is the target location in the embedding field. The embedding field is supposed to be deformed by the embedded objects.

Observers can access information that is retrieved from storage locations that for them have a historic timestamp. That information is transferred to them via the dynamic universe field. This dynamic field embeds both the observer and the observed event. The dynamic geometric data of point-like objects are archived in Euclidean format as a combination of a timestamp and a three-dimensional spatial location. The embedding field affects the format of the transferred information. The observers perceive in spacetime format. A hyperbolic Lorentz transform converts the Euclidean coordinates of the background parameter space into the spacetime coordinates that are perceived by the observer.

8.7.1 Lorentz transform

In dynamic fields, shock fronts move with speed c. In the quaternionic setting, this speed is unity.

$$x^2 + y^2 + z^2 = c^2\tau^2 \quad (8.7.1)$$

In flat dynamic fields, swarms of triggers of spherical pulse responses move with lower speed v.

For the geometric centers of these swarms still holds:

$$x^2 + y^2 + z^2 - c^2\tau^2 = x'^2 + y'^2 + z'^2 - c^2\tau'^2 \quad (8.7.2)$$

If the locations $\{x, y, z\}$ and $\{x', y', z'\}$ move with uniform relative speed v, then

$$ct' = ct\cosh(\omega) - x\sinh(\omega) \quad (8.7.3)$$

$$x' = x\cosh(\omega) - ct\sinh(\omega) \quad (8.7.4)$$

$$\cosh(\omega) = \frac{\exp(\omega) + \exp(-\omega)}{2} = \frac{c}{\sqrt{c^2 - v^2}} \quad (8.7.5)$$

$$\sinh(\omega) = \frac{\exp(\omega) - \exp(-\omega)}{2} = \frac{v}{\sqrt{c^2 - v^2}} \quad (8.7.6)$$

$$\cosh(\omega)^2 - \sinh(\omega)^2 = 1 \quad (8.7.7)$$

This is a hyperbolic transformation that relates two coordinate systems, which is known as a Lorentz boost [8].

This transformation can concern two platforms P and P' on which swarms reside and that move with uniform relative speed.

However, it can also concern the storage location P that contains a timestamp t and spatial location $\{x,y,z\}$ and platform P' that has coordinate time t and location $\{x',y',z'\}$.

In this way, the hyperbolic transform relates two platforms that move with uniform relative speed. One of them may be a floating Hilbert space on which the observer resides. Or it may be a cluster of such platforms that cling together and move as one unit. The other may be the background platform on which the embedding process produces the image of the footprint.

The Lorentz transform converts a Euclidean coordinate system consisting of a location $\{x,y,z\}$ and proper timestamps τ into the perceived coordinate system that consists of the spacetime coordinates $\{x',y',z',ct'\}$ in which t' plays the role of coordinate time. The uniform velocity v causes time dilation $\Delta t' = \frac{\Delta\tau}{\sqrt{1-\frac{v^2}{c^2}}}$ and length contraction

$$\Delta L' = \Delta L\sqrt{1-\frac{v^2}{c^2}}$$

8.7.2 Minkowski metric

Spacetime is ruled by the Minkowski metric [9].

In flat field conditions, proper time τ is defined by

$$\tau = \pm\frac{\sqrt{c^2t^2 - x^2 - y^2 - z^2}}{c} \tag{8.7.8}$$

And in deformed fields, still

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{8.7.9}$$

Here ds is the spacetime interval and $d\tau$ is the proper time interval. dt is the coordinate time interval

8.7.3 Schwarzschild metric

Polar coordinates convert the Minkowski metric to the Schwarzschild metric [10]. The proper time interval $d\tau$ obeys

$$c^2 d\tau^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2 d\varphi^2\right) \quad (8.7.10)$$

Under pure isotropic conditions, the last term on the right side vanishes.

According to mainstream physics, in the environment of a black hole, the symbol r_s stands for the Schwarzschild radius [11].

$$r_s = \frac{2GM}{c^2} \quad (8.7.11)$$

The variable r equals the distance to the center of mass of the massive object with mass M .

The Hilbert Book model finds a different value for the boundary of a spherical black hole. That radius is a factor of two smaller.

8.8 Inertial mass

The Lorentz transform also gives the transform of the rest mass to the mass that is relevant when the embedding field moves relative to the floating platform of the observed object with uniform speed $\vec{v}$.

In that case, the inertial mass M relates to the test mass M0 as

$$M = \gamma M_0 = \frac{M_0}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (8.8.1)$$

This indicates that the formula (8.4.6) for the gravitational potential at distance r must be changed to

$$V(r) = \frac{M_0 G}{r\sqrt{1 - \frac{v^2}{c^2}}} \tag{8.8.2}$$

8.9 Inertia

The relation between inertia and mass is complicated [36][37]. We apply an artificial field that resists its changing. The condition that for each type of massive object, the gravitational potential is a static function, and the condition that in free space, the massive object moves uniformly, establish that inertia rules the dynamics of the situation. These conditions define an artificial quaternionic field that resists change. The scalar part of the artificial field is represented by the gravitational potential, and the uniform speed of the massive object represents the imaginary (vector) part of the field.

The first-order change of the quaternionic field can be divided into five separate partial changes. Some of these parts can compensate each other.

Mathematically, the statement that in the first approximation nothing in the field ξ changes indicates that locally, the first-order partial differential $\nabla\xi$ will be equal to zero.

$$\zeta = \nabla\xi = \nabla_r \xi_r - \left\langle \vec{\nabla}, \vec{\xi} \right\rangle + \vec{\nabla}\xi_r + \nabla_r \vec{\xi} \pm \vec{\nabla} \times \vec{\xi} = 0 \tag{8.9.1}$$

Thus

$$\zeta_r = \nabla_r \xi_r - \left\langle \vec{\nabla}, \vec{\xi} \right\rangle = 0 \tag{8.9.2}$$

$$\vec{\zeta} = \vec{\nabla}\xi_r + \nabla_r \vec{\xi} \pm \vec{\nabla} \times \vec{\xi} = 0 \tag{8.9.3}$$

These formulas can be interpreted independently. For example, according to the equation (8.9.2), the variation in time of ξ_r can compensate the divergence of $\vec{\xi}$. The terms that are still eligible for change must together be equal to zero. For our purpose, the curl $\vec{\nabla}\times\vec{\xi}$ of the vector field $\vec{\xi}$ is expected to be zero. The resulting terms of the equation (8.9.3) are

$$\nabla_r\vec{\xi}+\vec{\nabla}\xi_r=0 \tag{8.9.4}$$

In the following text plays $\vec{\xi}$ the role of the vector field and ξ_r plays the role of the scalar gravitational potential of the considered object. For elementary modules, this special field concerns the effect of the hop landing location swarm that resides on the floating platform on its image in the embedding field. It reflects the activity of the stochastic process and the uniform movement in the free space of the floating platform over the background platform. It is characterized by a mass value and by the uniform velocity of the platform concerning the background platform. The real part conforms to the deformation that the stochastic process causes. The imaginary part conforms to the speed of movement of the floating platform. The main characteristic of this field is that it tries to keep its overall change zero. The author calls ξ the ***conservation field***.

At a large distance r, we approximate this potential by using the formula

$$\zeta_r(r)\approx\frac{GM}{r} \tag{8.9.5}$$

Here M is the inertial mass of the object that causes the deformation. The new artificial field $\xi=\left\{\frac{GM}{r},\vec{v}\right\}$ considers a uniformly moving mass as a normal situation. It is a combination of scalar potential $\frac{GM}{r}$ and uniform speed $\vec{v}$.

If this object accelerates, then the new field $\left\{\frac{GM}{r},\vec{v}\right\}$ tries to counteract the change $\dot{\vec{v}}$ of the vector field $\vec{v}$ by compensating this with an equivalent change of the scalar part $\frac{GM}{r}$ of the new field ξ. According to the equation (8.9.4), this equivalent change is the gradient of the real part of the field.

$$\vec{a}=\dot{\vec{v}}=-\vec{\nabla}\left(\frac{GM}{r}\right)=\frac{GM\,\vec{r}}{\left|\vec{r}\right|^{3}} \tag{8.9.6}$$

This generated vector field acts on masses that appear in its realm.

Thus, if two uniformly moving masses m and M exist in each other's neighborhood, then any disturbance of the situation will cause the gravitational force

$$\vec{F}\left(\vec{r}_1-\vec{r}_2\right)=m_0\vec{a}=\frac{Gm_0M\left(\vec{r}_1-\vec{r}_2\right)}{\left|\vec{r}_1-\vec{r}_2\right|^{3}}=\gamma\frac{Gm_0M_0\left(\vec{r}_1-\vec{r}_2\right)}{\left|\vec{r}_1-\vec{r}_2\right|^{3}} \tag{8.9.7}$$

Here $M=\gamma M_0$ is the inertial mass of the object that causes the deformation. m_0 is the rest mass of the observer.

The inertial mass M relates to its rest mass M_0 as

$$M=\gamma M_0=\frac{M_0}{\sqrt{1-\frac{v^2}{c^2}}} \tag{8.9.8}$$

This formula holds for all elementary particles except for quarks.

The problem with quarks is that these particles do not provide an isotropic symmetry difference. They must first combine into hadrons to be able to generate an isotropic symmetry difference. This phenomenon is known as ***color confinement***.

8.10 Momentum field

In the formula (8.9.7) that relates mass to force the factor γ that corrects for the relative speed can be attached to m_0 or to M_0

$$\vec{F}\left(\vec{r}_1-\vec{r}_2\right)=\gamma\frac{Gm_0M_0\left(\vec{r}_1-\vec{r}_2\right)}{\left|\vec{r}_1-\vec{r}_2\right|^3} \tag{8.10.1}$$

The force relates to the temporal change of the momentum $\vec{P}$ of the observer

$$\vec{F}=\dot{\vec{P}}=\frac{d\vec{P}}{dt} \tag{8.10.2}$$

The momentum vector $\vec{P}$ is part of a quaternionic field P. The momentum depends on the relative speed of the moving object that causes the deformation which defines the mass. The speed is determined relative to the field that embeds the object and that gets deformed by the investigated object.

$$P=P_r+\vec{P} \tag{8.10.3}$$

$$\|P\|^2=P_r^2+\|\vec{P}\|^2 \tag{8.10.4}$$

$$\vec{P}=\gamma m_0\vec{v} \tag{8.10.5}$$

$$\|\vec{P}\|^2=\gamma^2m_0^2\|\vec{v}\|^2 \tag{8.10.6}$$

$$\|P\|^2=\gamma^2m_0^2c^2=P_r^2+\gamma^2m_0^2\|\vec{v}\|^2 \tag{8.10.7}$$

$$\|P\|=\gamma m_0c=E/c \tag{8.10.8}$$

$$E=\gamma m_0c^2 \tag{8.10.9}$$

$$P_r^2 = \gamma^2 m_0^2 c^2 - \gamma^2 m_0^2 \|\vec{v}\|^2$$
$$= \gamma^2 m_0^2 \left(c^2 - \|\vec{v}\|^2\right) = \gamma^2 m_0^2 c^2 \left(1 - \left\|\frac{\vec{v}}{c}\right\|^2\right) = m_0^2 c^2 \qquad (8.10.10)$$

$$P_r = m_0 c = \frac{E}{\gamma c} \qquad (8.10.11)$$

$$\|\vec{P}\| = \gamma m_0 \|\vec{v}\| \qquad (8.10.12)$$

$$P = P_r + \vec{P} = m_0 c + \gamma m_0 \vec{v} = \frac{E}{\gamma c} + \gamma m_0 \vec{v} \qquad (8.10.13)$$

If $\vec{v} = \vec{0}$ then $\vec{P} = \vec{0}$ and $\|P\| = P = P_r = m_0 c$

Here Einstein’s famous mass-energy equivalence is involved.

$$E = \gamma m_0 c^2 = mc^2 \qquad (8.10.14)$$

The disturbance by the ongoing expansion of the embedding field suffices to put the gravitational force into action. The description also holds when the field ξ describes a conglomerate of platforms and M_2 represents the mass of the conglomerate.

The artificial field ξ represents the habits of the underlying model that ensures the constancy of the gravitational potential and the uniform floating of the considered massive objects in free space.

Inertia ensures that the third-order differential (the third-order change) of the deformed field is minimized. It does that by varying the speed of the platforms on which the massive objects reside.

Inertia bases mainly on the definition of mass that applies to the region outside the sphere where the gravitational potential behaves like the Green’s function of the field. There, the formula $\xi_r = \frac{GM}{r}$ applies.

Further, it bases on the intention of modules to keep the gravitational potential inside the mentioned sphere constant. At least that holds when this potential is averaged over the regeneration period. In that case, the overall change $\nabla\xi$ in the conservation field ξ equals zero. Next, the definition of the conservation field supposes that the swarm which causes the deformation moves as one unit. Further, the fact is used that the solutions of the homogeneous second-order partial differential equation can superpose in new solutions of that same equation.

The popular sketch in which the deformation of our living space is presented by smooth dips is obviously false. The story that is represented in this paper shows the deformations as local extensions of the field, which represents the universe. In both sketches, the deformations elongate the information path, but none of the sketches explain why two masses attract each other. The above explanation founds on the habit of the stochastic process to recurrently regenerate the same time average of the gravitational potential, even when that averaged potential moves uniformly. Without the described habit of the stochastic processes, inertia would not exist.

The applied artificial field also explains the gravitational attraction by black holes.

The artificial field that implements mass inertia also plays a role in other fields. Similar tricks can be used to explain the electrical force from the fact that the electrical field is produced by sources and sinks that can be described with the Green's function.

8.10.1 Momentum operator

The momentum function applies the natural parameter space and together with the reference operator they define the momentum operator. Thus, the momentum operator applies the eigenvectors of the reference operator. This operator has no direct relation to the change space. An indirect relation runs via the Fourier transform.

The momentum operator shows how to repair the defect of the definition of mass via the gravitational potential in the condition that the observer and the observed event move uniformly concerning each other. This operator links inertial mass to rest mass via the Lorentz boost.

8.10.2 Forces

In the Hilbert repository, all symmetry-related charges are located at the geometric center of an elementary particle and all these particles own a footprint that for isotropic symmetry differences can deform the embedding field. In that case, the particle features mass and forces might be coupled to acceleration via

$$F = m\vec{a} \tag{8.10.15}$$

Or to momentum via

$$F = \dot{\vec{P}} \tag{8.10.16}$$

9 Symmetry restrictions

9.1 Using volume integrals to determine the symmetry-related charges

In its simplest form in which no discontinuities occur in the integration domain Ω, the generalized Stokes theorem runs as

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega = \oint_{\Omega} \omega \tag{9.1.1}$$

We separate all point-like discontinuities from the domain Ω by encapsulating them in an extra boundary. Symmetry centers represent spherically shaped or cube-shaped closed parameter space regions H_n^x that float on a background parameter space $\Re$. The boundaries ∂H_n^x separate the regions from the domain H_n^x. The regions

H_n^x are platforms for local discontinuities in basic fields. These fields are continuous in the domain $\Omega - H$.

$$H = \bigcup_{n=1}^{N} H_n^x \tag{9.1.2}$$

The symmetry centers $\mathfrak{S}_n^x$ are encapsulated in regions H_n^x, and the encapsulating boundary ∂H_n^x is not part of the disconnected boundary, which encapsulates all continuous parts of the quaternionic manifold ω that exists in the quaternionic model.

$$\int_{\Omega - H} d\omega = \int_{\partial\Omega \cup \partial H} \omega = \int_{\partial\Omega} \omega - \sum_{k=1}^{N} \int_{\partial H_n^x} \omega \tag{9.1.3}$$

In fact, it is sufficient that ∂H_n^x surrounds the current location of the elementary module. We will select a boundary, which has the shape of a small cube of which the sides run through a region of the parameter spaces where the manifolds are continuous.

If we take everywhere on the boundary the unit normal to point outward, then this reverses the direction of the normal on ∂H_n^x which negates the integral. Thus, in this formula, the contributions of boundaries $\left\{\partial H_n^x\right\}$ are subtracted from the contributions of the boundary $\partial\Omega$. This means that $\partial\Omega$ also surrounds the regions $\left\{\partial H_n^x\right\}$

This fact renders the integration sensitive to the ordering of the participating domains.

Domain Ω corresponds to part of the background parameter space $\mathfrak{R}$. As mentioned before the symmetry centers $\mathfrak{S}_n^x$ represent encapsulated regions $\left\{\partial H_n^x\right\}$ that float on the background parameter space $\mathfrak{R}$. The

Cartesian axes of $\mathfrak{S}_n^x$ are parallel to the Cartesian axes of background parameter space $\mathfrak{R}$. Only the orderings along these axes may differ.

Further, the geometric center of the symmetry center $\mathfrak{S}_n^x$ is represented by a floating location on parameter space $\mathfrak{R}$.

The symmetry center $\mathfrak{S}_n^x$ is characterized by a private symmetry flavor. That symmetry flavor relates to the Cartesian ordering of this parameter space. With the orientation of the coordinate axes fixed, eight independent Cartesian orderings are possible.

The consequence of the differences in the symmetry flavor on the subtraction can best be comprehended when the encapsulation ∂H_n^x is performed by a ***cubic space form*** that is aligned along the Cartesian axes that act in the background parameter space. Now the six sides of the cube contribute differently to the effects of the encapsulation when the ordering of H_n^x differs from the Cartesian ordering of the reference parameter space $\mathfrak{R}$. Each discrepant axis ordering corresponds to one-third of the surface of the cube. This effect is represented by the ***geometric symmetry-related charge***, which includes the ***color charge*** of the symmetry center. It is easily comprehensible related to the algorithm which below is introduced for the computation of the geometric symmetry-related charge. Also, the relation to the color charge will be clear. ***Thus, this effect couples the ordering of the local parameter spaces to the geometric symmetry-related charge of the encapsulated elementary module.*** The differences with the ordering of the surrounding parameter space determine the value of the geometric symmetry-related charge of the object that resides inside the encapsulation!

9.2 Symmetry flavor

The Cartesian ordering of its private parameter space determines the symmetry flavor of the platform [17]. For that reason, this symmetry is compared with the reference symmetry, which is the symmetry of the background parameter space. Four arrows indicate the symmetry of the platform. The background is represented by:

⬆⬆⬆⬆

Now the geometric symmetry-related charge follows in two steps.

1. Count the difference of the spatial part of the geometric symmetry of the platform with the spatial part of the geometric symmetry of the background parameter space.
2. Switch the sign of the result for anti-particles.

Symmetrieversion					
Ordering x y z τ	Sequence	Handedness Right/Left	Color charge	Electric charge * 3	Symmetry type.
⬆⬆⬆⬆	⓪	R	N	+0	neutrino
⬇⬆⬆⬆	①	L	R	− 1	down quark
⬆⬇⬆⬆	②	L	G	− 1	down quark
⬇⬇⬆⬆	③	R	B	+2	up quark
⬆⬆⬇⬆	④	L	B	−1	down quark
⬇⬆⬇⬆	⑤	R	G	+2	up quark
⬆⬇⬇⬆	⑥	R	R	+2	up quark
⬇⬇⬇⬆	⑦	L	N	− 3	electron
⬆⬆⬆⬇	⑧	R	N	+3	positron
⬇⬆⬆⬇	⑨	L	R	− 2	anti-up quark
⬆⬇⬆⬇	⑩	L	G	− 2	anti-up quark
⬇⬇⬆⬇	⑪	R	B	+1	anti-down quark
⬆⬆⬇⬇	⑫	L	B	− 2	anti-up quark
⬇⬆⬇⬇	⑬	R	G	+1	anti-down quark
⬆⬇⬇⬇	⑭	R	R	+1	anti-down quark
⬇⬇⬇⬇	⑮	L	N	− 0	anti-neutrino

Probably, the neutrino and the antineutrino own an abnormal handedness.

The suggested particle names that indicate the symmetry type are borrowed from the Standard Model. In the table, compared to the standard model, some differences exist with the selection of the anti-predicate. All considered particles are elementary fermions. The freedom of choice in the polar coordinate system might determine the spin [18]. The azimuth range is 2π radians, and the polar angle range is π radians. Symmetry breaking means a difference between the platform symmetry and the symmetry of the background. Neutrinos do not break the symmetry. Instead, they probably may cause conflicts with the handedness of the multiplication rule.

In the Hilbert repository, only point-like charges occur that represent sources or sinks. These charges move

9.3 Potential of the electric field

The potential of an electromagnetic field is a quaternionic function.

$$\phi(r) = \vec{\phi}(r) + \vec{\phi}(r) \tag{9.3.1}$$

The corresponding force is the Lorentz force.

$$\begin{aligned} \vec{F}(r) &= Q\left[-\vec{\nabla}\phi_r - \nabla_r\vec{\phi} + \vec{v}\times\left(\vec{\nabla}\times\vec{\phi}\right)\right] \\ &= Q\left[\vec{E} + \vec{v}\times\vec{B}\right] \end{aligned} \tag{9.3.2}$$

A stream of symmetry-related actuators that is represented by a source or sink and is characterized by a symmetry-related charge Q generates a scalar potential

$$\phi_r(r) = \frac{Q}{4\pi\varepsilon_0 r} \tag{9.3.3}$$

This means that its observation is affected by inertia in a way that is like the way that the observation of the gravitational potential is affected. This becomes noticeable in the electric force between two charges.

9.3.1 Coulomb force

The electric charge is coupled to the geometric center of a massive object.

Another electric charge is coupled to another massive object. The charges repel or attract the charges that are located at the other geometric center. Thus, a relative speed of the two geometric centers is changed into an acceleration.

With electromagnetic potentials the force from the Lorentz force. If the magnetic potential $\vec{\xi}$ equals zero, then only part of the electric field results.

$$E = -\vec{\nabla}\xi_r = \frac{Q_1\left(\vec{r}_1 - \vec{r}_2\right)}{4\pi\varepsilon_0\left|\vec{r}_1 - \vec{r}_2\right|^3} \tag{9.3.4}$$

Thus, if two uniformly moving charges Q_1 and Q_2 exist in each other's neighborhood, then any disturbance of the situation will cause the electrical force

$$\vec{F}\left(\vec{r}_1 - \vec{r}_2\right) = Q_2 E = \frac{\varepsilon Q_1 Q_2\left(\vec{r}_1 - \vec{r}_2\right)}{\left|\vec{r}_1 - \vec{r}_2\right|^3} \tag{9.3.5}$$

The force repels for two sources or two sinks and attracts for the combination of a source and a sink.

These formulas hold for all elementary particles including quarks.

10 Basic fields

10.1 Coupling of basic fields

Besides the fact that the geometric center of the elementary particles also forms the geometric center of the symmetry-related field of this particle the coupling of the symmetry of the particle to the Cartesian coordinate system of the particle couples the basic fields of the particle to the background field that acts as our universe. It tries to keep the Cartesian coordinate systems in parallel. This couples the curl of the particle's geometric symmetry-related field to the curl of the embedding background field. A non-zero curl might even couple to the otherwise undetermined direction of the spin vector in the spherical shock fronts. This couples the direction of spin to a non-zero magnetic field.

11 Conglomerates

The Hilbert repository suggests that apart from the quarks all elementary fermions are constituted by excitations of the dynamic field that represents our universe. These excitations are spherical pulse responses that act as spherical shock fronts that locally and temporarily deform this embedding field. The quarks can combine into hadrons and are then also capable of generating spherical shock fronts.

Further, the spherical shock fronts appear to constitute all discrete massive objects that exist in the universe. The exception to this rule is encapsulated regions that contain countable sets of objects and therefore do not form a compact continuum. We call these regions black holes because no field excitations exist in these regions and no field excitations can enter or leave these regions. Still, the region can and will deform its continuous surround.

The above statement suggests that elementary fermions can constitute higher generations of fermions and can generate bosons. The notorious exception is formed by photons. Photons are constituted by chains of equidistant one-dimensional shock fronts.

The reason for this suggestion is that the footprint of elementary fermions is generated by stochastic processes that own a characteristic function, which is controlled from and specified in change space.

This opens the possibility to also define the conglomerates of elementary particles in change space. Each conglomerate is defined by a private stochastic process that owns a characteristic function, which is a dynamic superposition of the characteristic functions of the components of the conglomerate. The superposition coefficients act as displacement generators. In this way, these coefficients specify the internal positions of the components. These dynamic coefficients define internal oscillations.

In change space, the location in the configuration space has no significance. Thus, components of a composite can locate far from each other in configuration space. This is the reason that entanglement exists. Entanglement becomes noticeable when components obey exclusion principles.

11.1 Modular system

The definition of these conglomerates causes that apart from black holes and photons, the discrete objects that exist in our universe and embed in the dynamic universe field form an extensive modular system with the elementary fermions as the elementary modules and individual modular systems at the top of the hierarchies.

11.2 Module types

Module types form type communities. These communities have a much longer lifespan than individual modules. In the competition between module communities, the community that takes the best care for its members and that also takes care of the module communities on which it relies have the best chance of survival. This fact contrasts Darwin's statement about the survival of the fittest individual.

11.3 Atoms

Compound modules are composite modules for which the geometric centers of the platforms of the components coincide. The charges of the platforms of the elementary modules establish the binding of the corresponding platforms. Physicists and chemists call these compound modules atoms or atomic ions.

In free compound modules, the geometric symmetry-related charges do not take part in the oscillations. The targets of the private stochastic processes of the elementary modules oscillate. This means that the hopping path of the elementary module folds around the oscillation path and the hop landing location swarm gets smeared along the oscillation path. The oscillation path is a solution to the Helmholtz equation. Each fermion must use a different oscillation mode. A change of the oscillation mode goes together with the emission or the absorption of a photon. The center of emission coincides with the geometrical center of the compound module. During the emission or absorption, the oscillation mode, and the hopping path halt, such that the emitted photon does not lose its integrity. Since all photons share the same emission duration, that duration must coincide with the regeneration cycle of the hop landing location swarm. Absorption cannot be interpreted so easily. In fact, it can only be comprehended as a time-reversed emission act. Otherwise, the absorption would require an incredible aiming precision for the photon.

The type of stochastic process that controls the binding of components appears to be responsible for the absorption and emission of photons and the change of oscillation modes. If photons arrive with too low energy, then the energy is spent on the kinetic energy of the common platform. If photons arrive with too high energy, then the energy is distributed over the available oscillation modes, and the rest is spent on the kinetic energy of the common platform, or it escapes into free space. The process must somehow archive the modes of the components. It can apply the private platform of the components for that purpose. Most probably, the current value of the dynamic superposition coefficient is stored in the eigenspace of a special superposition operator.

11.4 Molecules

Molecules are conglomerates of compound modules that each keep their private geometrical center. However, electron oscillations are shared among the compound modules. Together with the geometric symmetry-related charges, this binds the compound modules into the molecule.

12 Two episodes

The footprint operator is already present at the time of the creation of the Hilbert repository and determines the behavior of the elementary particle throughout its existence. This fact is a great mystery. The humanly derived math does not yet offer an explanation. However, the existence of the footprint operator makes it possible to divide the model of physical reality into a preparatory episode in which there is no flowing time and an ongoing episode in which a continuing step-by-step embedding of the hop landing locations mimics the activities of the stochastic processes. The embedding process uses the stored and ordered time stamps to realize the corresponding hop landings. The range of running time is equal to the range of the archived time stamps. At the beginning of the running time, the field that represents our universe is still virginal and corresponds to the background parameter space. After the first footprints are completed, the relevant elementary particles can start to form composite objects.

12.1 In the beginning

Before the embedding processes that mimic the activity of the stochastic processes started their action, the content of the universe was empty. It was represented by a flat field that in its spatial part, was equal to the parameter space of the background platform. At the beginning instant, a huge number of these mimicked stochastic processes started their triggering of the dynamic field that represents the universe. The triggers may cause spherical pulse responses that act as spherical shock fronts. These spherical shock fronts temporarily deform the universe field. In that case, they will also persistently expand the universe. Thus, from that moment on, the universe started expanding. This did not happen at a single point. Instead, it happened at a huge number of locations that were distributed all over the spatial part of the parameter space of the quaternionic function that describes the dynamic universe field.

Close to the beginning of time, all distances were equal to the distances in the flat parameter space. Soon, these islands were uplifted with volume that was emitted at nearby locations. This flooding created growing distances between used locations. After some time, all parameter space locations were reached by the generated shock fronts. From that moment on the universe started acting as an everywhere expanded continuum that contained deformations which in advance were exceedingly small. Where these deformations grew, the distances grew faster than in the environment. A more uniform expansion appears the rule and local deformations form the exception. Deformations make the information path longer and give the idea that time ticks slower in the deformed and expanded regions. This corresponds with the gravitational redshift of photons.

Composed modules only started to be generated after the presence of enough elementary modules. The generation of photons that reflected the signatures of atoms only started after the presence of these compound modules. However, the spurious one-dimensional shock fronts could be generated from the beginning.

This picture differs considerably from the popular scene of the big bang that started at a single location [12].

The expansion is the fastest in areas where spherical pulse responses are generated. For that reason, it is not surprising that the measured Hubble constant differs from place to place.

12.2 RTOS

The archival of dynamic geometric data that takes place in the creation episode is determining the life story of the elementary particles. The activity of the stochastic processes is mimicked by the ongoing embedding process that implements the dynamic geometric data as an ongoing hopping path that recurrently regenerates a coherent hop landing location swarm that has a stable location density distribution. This location density distribution is the Fourier transform of the characteristic function of the stochastic process that filled the eigenspace of the footprint operator that resides at the private platform of the elementary particle. This activity acts as a Real-Time Operating System. The recurrent regeneration of the hop landing location swarm implements an effective guard against deadlocks and race conditions.

13 Dark objects

Mainstream physics suggests the existence of two types of dark objects [39][40]. These are dark energy and dark matter. In contrast to mainstream physics, the Hilbert Book Model presents these two types of dark objects as field excitations that act as shock fronts. Together these special field excitations constitute, except for black holes, all discrete objects that exist in the universe.

Dark matter objects are spherical pulse responses that behave as spherical shock fronts. They constitute the footprints of elementary particles. Further, they populate as a veiling glare the universe in the neighborhood of large assemblies of conglomerates of elementary particles.

Dark energy objects are one-dimensional pulse responses and behave as one-dimensional shock fronts. They appear spread over the universe, but more specifically divided equidistantly in chains that constitute photons. Photons obey the Planck-Einstein relation $E = h\nu$ [24]. This means that the emission duration of photons is fixed and since all shock fronts move with speed c, at the instant of emission, all photons must feature the same length.

13.1 Black holes

We introduce a ***discontinuum*** as the antonym of a continuum. The universe is a mixed field. It can contain a set of enclosed spatial regions that encapsulate a discontinuum. A discontinuum is a dense discrete set. A discontinuum is countable. In physics, the equivalent of a discontinuum is a black hole. The enclosing surface is a continuum with a lower dimension than the enclosed region. No field excitations exist inside the discontinuum. Thus, no field excitations can pass the enclosing surface. Since a discontinuum deforms the surrounding continuum, this enclosed region owns an amount of mass. Together with the spherical shock fronts and the elementary modules, the discontinuums are the only objects in the universe that own mass. The mass of spherical shock fronts is volatile. Only when gathered in coherent and dense ensembles these shock fronts can cause a persistent amount of mass. That happens in the footprint of elementary modules. It also happens in the halos of galaxies. So, black holes can only be perceived by their gravitational potential. However, outside the border of the black hole, many phenomena can occur that are caused by the activity of massive objects that are attracted by the enormous gravitation that the black hole generates. Elementary particles that hover with their platform over the encapsulated region can drop part of their footprint actuators into the black hole. In this way, black holes can steadily grow. This paper does not consider the join of black holes and it does not consider the birth of a black hole by squeezing one or more neutron stars.

14 Conclusions

The structure and the behavior of the Hilbert repository show an astonishing similarity with the structure and behavior of the set of elementary fermions in the Standard Model of particle physics.

The universe is a dynamic field that is archived in the background platform of the Hilbert repository. This dynamic field can be described by a quaternionic function. Quaternionic differential calculus describes the dynamics of this field. Apart from the wave equation exists another second-order partial differential equation.

Electric charges only appear at the geometric centers of the floating platforms on which elementary fermions reside.

The shortlist of electric charges and color charges in the Standard Model conforms with the shortlist of symmetry-related charges in the Hilbert repository.

Sources and sinks represent the symmetry-related charges.

Elementary fermions behave as elementary modules. Except for black holes they constitute all massive objects that exist in the universe.

Stochastic processes that own a characteristic function and can be considered as a combination of a Poisson process and a binomial process implement the wavefunction of elementary fermions. These processes produce an ongoing hopping path. An ongoing embedding process images the hop landing locations on the dynamic universe field.

Dark objects play an essential role in the dynamics of the universe field.

Dark matter objects are spherical pulse responses that behave as spherical shock fronts and integrate over time in the Green's function of the field.

Dark matter objects constitute the footprints of elementary fermions.

Dark matter objects explain the origin of gravity.

Dark energy objects are one-dimensional pulse responses and behave as one-dimensional shock fronts. They appear spread over the universe, but more specifically they constitute photons divided equidistantly in chains. Photons obey the Planck-Einstein relation.

Black holes are considered as encapsulated discontinuous regions that exist in a continuous surround. They become noticeable by their gravitational potential and by the phenomena that occur at their border.

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15 End