Fractional Dynamics and the Cosmological Constant Problem

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Abstract

It is known that the divergence of vacuum energy density (VED) in Quantum Field Theory lies at the core of the cosmological constant (CC) problem. Our brief note suggests that, at least in principle, modeling the quantum vacuum as an ensemble of *fractional oscillators* may regulate the ultraviolet behavior of the VED and evade the CC problem.

Key words: cosmological constant problem, vacuum energy density, fractional dynamics, fractional oscillator.

1. VED and the cosmological constant problem

The traditional computation of VED in Quantum Field Theory starts from the sum [1]

$$\rho_V = \sum_n \left\{ c_n \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \hbar\, \omega_n(k) \right\} \tag{1a}$$

where

$$\omega_n(k) = \sqrt{m_n^2 + k^2} \tag{1b}$$

Here, VED is modeled as a reservoir of free quantum harmonic oscillators in flat spacetime and (1a) represents the integral of the zero-point energy carried over all momenta. For large momenta $k >> m_n$, the oscillator frequency may be approximated as $\omega_n(k) \approx k$, in which case the integral (1) diverges. Inserting an ultraviolet (UV) cutoff Λ in (1a) yields

$$\int_0^\Lambda d^3k \sqrt{m_n^2 + \Lambda^2} = \pi \left\{ \Lambda^4 + m_n^2 \Lambda^2 + \frac{m_n^4}{8} - \frac{1}{2} m_n^4 \ln(2\Lambda / m_n) \right\} + O(\Lambda^{-2}) \tag{2}$$

It is apparent that (2) is quartically divergent as the UV cutoff approaches the Planck region of scales $(\Lambda = O(M_{Pl}) >> m_n)$. To regularize (2), we follow the general renormalization prescription of Quantum Field Theory, according to which one starts with a bare Lagrangian and a cutoff dependent bare VED in the form

$$\rho_b = \rho_b(\Lambda) \tag{3}$$

As a result, the renormalized or effective VED is given by

$$\rho_{eff,\mathrm{v}} = \rho_b(\Lambda) + c\Lambda^4 \tag{4}$$

where c stands for some numerical constant. Astrophysical observations from type I supernovae and from the cosmic microwave background (CMB) radiation show that

$$\rho_{eff,\mathrm{v}}^{1/4} \approx 2 \times 10^{-3}\,\mathrm{eV} \tag{5}$$

Since experiments have confirmed that the Standard Model is valid at least up to an energy scale of $O(1\text{TeV} = 10^{12}\text{eV})$, one may reasonably assume that the UV cutoff can be placed around this scale ($\Lambda = O(10^{12}\text{eV})$). Combined use of (4) and (5) gives

$$(2\times 10^{-3}\text{eV})^4 = \rho_b(\Lambda) + c\,(10^{12}\text{eV})^4 \tag{6}$$

It follows that the bare value of the cosmological constant evaluated at the cutoff must be chosen so that it cancels out a contribution on the order of 10^{48}eV^4 and leaves a contribution on the order of 10^{-12}eV^4. This requires an unnatural fine-tuning of the cosmological constant on the order of 60 decimal places, which lies at the core of the *cosmological constant problem*.

2. Regularization of the VED integral through fractional dynamics

We now conjecture that quantum vacuum may be approximated as an *array of non-relativistic oscillators with long-range interaction*. The dynamics of the array is encoded in the following Hamiltonian [2]

$$H = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2}\dot{\varphi}_n^2 + \frac{1}{2}g_0\sum_{\substack{n=-\infty\\ m\neq n}}^{\infty}\frac{\varphi_n\varphi_m}{|n-m|^{1+\alpha}} + V(\varphi_n)\right] \tag{7}$$

Here, α stands for the oscillator index ($\alpha \neq 0,1,2,\ldots$), g_0 is a coupling constant, $V(\ldots)$ the interaction potential and $|n-m|$ the spatial separation of oscillators located at nodes n and m. According to [2], the equation of motion derived from (7) and applied to the continuous field limit of the array, is given by

$$\frac{\partial^2\varphi(x,t)}{\partial t^2}=\frac{\partial V(\varphi)}{\partial\varphi}+\Delta[g_0,\alpha,\varphi(x,t)] \tag{8}$$

in which

$$\Delta[g_0,\alpha,\varphi(x,t)]=g_0\,a_\alpha D^\alpha\varphi(x,t)-2g_0\sum_{n=1}^{\infty}\frac{\zeta(\alpha+1-2n)}{(2n)!}\frac{\partial^{2n}\varphi(x,t)}{\partial x^{2n}} \tag{9}$$

and

$$a_\alpha=2\Gamma(-\alpha)\cos\left(\pi\alpha/2\right) \tag{10}$$

Here, $D_\alpha=\partial^\alpha/\partial|x|^\alpha$ denotes the fractional derivative operator and $\zeta(\ldots)$ the Riemann zeta-function. A cursory glance at (8) reveals that,

a) the second term in the right-hand side acts as a "*pseudo-force*" induced by fractional dynamics,

b) this force vanishes away when α takes on a range of integer values ($\alpha=0,1,2,\ldots$).

Furthermore, integrating (9) over the field domain generates an additional energy term in the original Hamiltonian, i.e.

$$E_\Delta(g_0,\alpha)\Rightarrow\int\Delta[g_0,\alpha,\varphi(x,t)]\,d\varphi \tag{11}$$

An attractive property of (11) is that, at least in principle, (11) *may offset the quartic divergence* of the VED integral shown in (2).

One arrives at a similar conclusion starting from [3], where the long-range dynamics of nonlinear oscillators is also analyzed in terms of fractional derivatives. In this case, the natural generalization of the dispersion law (1b) may be presented as

$$\omega(k) = \pm\sqrt{m^2 + b^{(\beta-3)/2}|k|^{\beta-1}} \quad ; \quad 2 < \beta < 3 \tag{12}$$

whose low-mass approximation reads

$$\omega(k) \approx b^{(\beta-3)/2}\,|k|^{(\beta-1)/2} \tag{13}$$

Following the underlying principles of fractional dynamics, we now assume that both coefficient b and wavenumber k depend on the dimensionless observation scale μ according to

$$b \sim \mu^{\sigma_b} \tag{14a}$$

$$|k| \sim \mu^{\sigma_k} \tag{14b}$$

By (14), (13) reduces to the power law

$$\omega(\mu) \approx \mu^{\sigma} \tag{15}$$

where

$$\sigma = \frac{1}{2}[\sigma_b(\beta-3) + \sigma_k(\beta-1)] \tag{16}$$

Replacing (15)-(16) in (1a) leads to the conclusion that, for any given index β, (1a) stays *convergent* under a suitable choice of exponents σ_b and σ_k.

References

[1] Goldfain, E. On the Evaluation of Vacuum Energy Density in Quantum Field Theory. Preprints 2021, 2021010562 (doi: 10.20944/preprints202101.0562.v1).

[2] https://arxiv.org/pdf/math-ph/0702065.pdf

[3] https://arxiv.org/pdf/hep-ph/9910419.pdf