On sinusoidal periodic solution of mixed Lienard type equations

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Abstract

We study in this paper the existence of exact periodic solutions of the mixed Lienard type equations. We show for the first time the conditions to ensure the exact and explicit integrability and to obtain sinusoidal periodic solution. As a result, the equation can be used to describe harmonic and isochronous oscillations of dynamical systems.

Keywords: Mixed Lienard type equations, sinusoidal periodic solution, harmonic and isochronous oscillations.

Introduction

A well studied equation in the literature is the mixed Lienard type differential equation [1-3]

$$\ddot{x} + u(x)\dot{x}^2 + \vartheta(x)\dot{x} + h(x) = 0 \tag{1}$$

where $u(x)$, $\vartheta(x)$ and $h(x)$ are arbitrary function of x, and the overdot designates the differentiation with respect to time. The equation (1) contains several classes of differential equations. When $u(x) = 0$, the equation (1) becomes

$$\ddot{x} + \vartheta(x)\dot{x} + h(x) = 0 \tag{2}$$

which has been for a long time investigated in the literature [4-9]. An important result is that the equation (2) can exhibit not only periodic solution but also isochronous property under some conditions [5, 7-9]. It is also observed that for some functions $\vartheta(x)$ and $h(x)$, the equation (2) has sinusoidal solution as the linear harmonic oscillator [8]. A general sinusoidal solution has been ensured for the equation (2) in [9] for some choice of $\vartheta(x)$ and $h(x)$. It is for the first time these exceptional results are obtained for a dissipative equation of type (2). Putting $u(x) = 0,$ and $\vartheta(x) = 0$, yields the equation

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$$\ddot{x} + h(x) = 0 \tag{3}$$

widely studied in mathematics and physics as a conservative nonlinear system for $h(x)$ a nonlinear function of x. The famous conservative cubic Duffing equation is a special case of the equation (3). The conservative cubic Duffing equation is well known to exhibit the Jacobi elliptic functions as general periodic solutions [10,11]. But in [12,13] the authors have shown that the conservative cubic Duffing equation can also have unbounded periodic solutions. In [14] Monsia has successfully presented an exceptional equation of type (3) with strong and higher-order nonlinearity exhibiting exact and explicit general periodic solutions in terms of sinusoidal function. When the function $\vartheta(x) = 0$, the equation (1) turns into

$$\ddot{x} + u(x)\dot{x}^2 + h(x) = 0 \tag{4}$$

which is known as the quadratic Lienard type equation [3,15, 16-19]. The equation (4) has been the subject of a vast literature [3, 15-19]. Many interesting results have been obtained from the rich and various study for the quadratically dissipative Lienard type equation (4). In [16] the harmonic periodic solution but with amplitude-dependent frequency has been for the first time, obtained for the equation (4) under appropriate choice of functions $u(x)$ and $h(x)$. The equation (4) is investigated in [17] under the framework of the generalized Sundman transformation formalism. Thus, the authors in [17] have successfully shown the existence of a class of equations of type (4) containing differential equations which can exhibit harmonic periodic solution but with amplitude-dependent frequency. It is observed recently in [20] that this class of quadratic Lienard type equations highlighted by Akande et al. [17] can admit the Jacobi elliptic functions as general periodic solution. Despite these results, one can notice that there is a very limited number of equations of type (2), (3) and (4) that have exact and explicit general periodic solutions. In this context the problem of finding exact and explicit general periodic solutions for a mixed Lienard equation of type (1) becomes very more difficult to solve as it includes simultaneously different kinds of nonlinearity. Although the equations of type (1) have been intensively investigated in the literature [1-3], they have no known exact and general periodic solutions. It is then very interesting to study the existence of periodic solution of equations of type (1) but also to calculate explicitly this solution. Thus, one can ask whether there are functions $u(x)$, $\vartheta(x)$ and $h(x)$ that ensure exact and explicit general periodic solutions for equations of type (1). The objective in this paper is to show the existence of such functions ensuring the calculation of periodic solutions. In this perspective, we review briefly the theory of nonlinear differential equations introduced recently in [9,20,21] by Monsia and his group (section 2) and calculate explicitly the sinusoidal periodic solution of the equation of type (1) of interest (section 3). A conclusion is drawn for the work finally.

2. Review of the theory

According to [9, 20, 21] the mixed Lienard type differential equation corresponding to the first integral

$$b = g(x)\dot{x} + a\, f(x)x^{\ell} \tag{5}$$

can be read

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 + a\,\ell x^{\ell-1}\frac{f(x)}{g(x)}\,\dot{x} + ab\,x^{\ell}\,\frac{f'(x)}{g^2(x)} - a^2\,x^{2\ell}\,\frac{f'(x)f(x)}{g^2(x)} = 0 \tag{6}$$

where a, b and ℓ are arbitrary parameters and $g(x) \neq 0$ and $f(x)$ are arbitrary functions of x. The prime stands for derivative with respect to the argument. When $f(x) = x^{-\ell}$, the equation (6) reduces to

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 + \frac{a\ell}{xg(x)}\,\dot{x} + (a-b)\frac{a\,\ell}{xg^2(x)} \tag{7}$$

Under $b = 0$, the equation (7) becomes

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 + \frac{a\ell}{xg(x)}\,\dot{x} + \frac{a^2\,\ell}{xg^2(x)} = 0 \tag{8}$$

The equation (8) is of the type (1) when $u(x) = \frac{g'(x)}{g(x)}, \vartheta(x) = \frac{a\ell}{xg(x)},$ and $h(x) = \frac{a^2\ell}{xg^2(x)}$. Now the problem to solve is to find the appropriate function $g(x)$ which ensures the sinusoidal solution for the equation (8).

3. Equation of interest and its solution

Let us consider $g(x) = (\mu^2 - x^2)^{-\frac{1}{2}}$, where $\mu \succ 0$. Then the equation (8) reduces to the desired mixed Lienard type equation

$$\ddot{x} + \frac{x}{\mu^2 - x^2}\dot{x}^2 + a\ell\frac{\sqrt{\mu^2 - x^2}}{x}\,\dot{x} + \frac{a^2\ell\mu^2}{x} - a^2\ell x = 0 \tag{9}$$

where $\ell \neq 0,$ and the corresponding first integral (5) takes the form

$$g(x)\dot{x} + a = 0 \tag{10}$$

such that

$$\int \frac{dx}{\sqrt{\mu^2 - x^2}} = -a(t + K) \tag{11}$$

where K is a constant of integration. The evaluation of the integral in (11) is immediate to give

$$\sin^{-1}\left(\frac{x}{\mu}\right) = -a(t + K) \tag{12}$$

from which one can secure the desired sinusoidal periodic solution of the equation (9) in the definitive form

$$x(t) = \mu \sin[-a(t + K)] \tag{13}$$

The formula (13) is harmonic and isochronous periodic solution when $a \prec 0.$ In this case the equation (9) and the linear harmonic oscillator equation

$$\ddot{x} + a^2 x = 0 \tag{14}$$

have identical solutions where the amplitude of the oscillations of the equation (14) is choosen to be $\mu.$ The application of $\ell = 0,$ transforms the equation (9) into

$$\ddot{x} + \frac{x}{\mu^2 - x^2}\dot{x}^2 = 0 \tag{15}$$

which has been proved in [19] to have sinusoidal and isochronous periodic solution as the linear harmonic oscillator equation.

Conclusion

A mixed Lienard type equation is studied in this paper. We show for the first time the existence of sinusoidal periodic solution for this equation and calculate it explicitly. The obtained solution can be used to describe harmonic and isochronous oscillations of nonlinear dynamical systems.

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