A UNIVERSALITY THEOREM FOR NONNEGATIVE MATRIX FACTORIZATIONS

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Abstract. Let A be a *nonnegative matrix*, that is, a matrix with nonnegative real entries. A nonnegative factorization of size k is a representation of A as a sum of k nonnegative rank-one matrices. The space of all such factorizations is a bounded semialgebraic set, and we prove that spaces arising in this way are universal. More presicely, we show that every bounded semialgebraic set U is rationally equivalent to the set of nonnegative size-k factorizations of some matrix A up to a permutation of matrices in the factorization. Our construction is effective, and we can compute a pair (A, k) in polynomial time from a given description of U as a system of polynomial inequalities with coefficients in $\mathbb{Q}$. This result gives a complete description of the algorithmic complexity of several important problems, including the nonnegative matrix factorization, completely positive rank, nested polytope problem, and it also leads to a complete resolution of the problem of Cohen and Rothblum on nonnegative factorizations over different ordered fields.

1. Introduction

Let A be a matrix with nonnegative real entries, and let k be an integer number. The *nonnegative matrix factorization*, or NMF, is the task to approximate (or express) A with a sum of k rank-one matrices each of which has nonnegative entries. Both the general version of NMF and its exact counterpart are important tools in modern pure and applied mathematics. This problem is inherent in clustering problems and data mining [24], and its outputs can be easier to interpret than those obtained from other factorization techniques [35]. This feature leads to many real-world applications of NMF, which include image processing [53], text mining [15], music analysis [31], and audio signal processing [34]. Another notable application of NMF is a parts-based approach to representation of objects, which leads to progress in face recognition [53]. Also, NMF arises naturally in statistics and helps in studying the expectation maximization algorithms [52], describes the spaces of explanations [61], and one of its approximate versions turns out to be equivalent to a popular document clustering method known as the probabilistic latent semantic analysis [25]. A geometric approach allows one to reformulate NMF as a question on the extension complexity of a polytope and as a nested polytope problem, which leads to applications in combinatorial optimization [82], theory of computation [33], statistics [52], and quantum mechanics [17]. An interested reader is referred to a recent textbook [36] for more details on the applications of NMF.

As we see, the need of solving a particular instance of NMF can arise in a variety of applications, which motivates one to try constructing efficient algorithms for solving this problem and to explore its computational complexity.

Question 1. What is the computational complexity of NMF?

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In a relatively old paper [17], which has now become a standard reference on the topic, Cohen and Rothblum analyze the complexity of the following straightforward approach to NMF. They write an instance of NMF as a system of polynomial equations and inequalities with real unknowns and apply the quantifier elimination algorithm of Renegar [66] to this system. This approach was sharpened in a recent highly cited paper [3] and its sequel [60], which give an algorithm that solves NMF and halts in polynomial time for every fixed factorization rank k. Of course, these theoretical procedures are not applicable in practice as modern computers are unable to solve even just a 4×4 instance with a direct application of the quantifier elimination approach [79]. A landmark paper of Vavasis [81] employs the geometric perspective to prove the NP-hardness of NMF and poses a question of whether or not this problem actually belongs to NP. The question of better algorithms for NMF and several related computational tasks, including the *completely positive* matrix factorization and *nested polytope* problem, appeared in many subsequent papers and remained open. Now we need to switch to formal definitions and present our resolution of the problem before we go back to the short survey of its history in Section 3. In that section, we will also discuss other consequences of our main theorem, which apart from the above mentioned complexity results include the complete resolution of the Cohen–Rothblum problem on the nonnegative ranks over different ordered fields [17]. The remaining technical Sections 4–8 are devoted to the proofs of our results.

2. The formulation of the Universality theorem

Now we switch to a more general setting and consider an arbitrary ordered field $\mathcal{F}$. We denote by $\mathcal{F}_+$ the set of all nonnegative elements of $\mathcal{F}$, and we call an $m \times n$ matrix A over $\mathcal{F}$ *nonnegative* if every entry of it is taken from this set. A *size-k nonnegative $\mathcal{F}$-factorization* of A is a family $(A_1, \ldots, A_k)$ of rank-one matrices with entries in $\mathcal{F}_+$ such that $A_1 + \ldots + A_k = A$; any permutation of this family obviously remains a valid factorization. We do not want to think of such factorizations as different, and to speak formally, we define $\operatorname{fact}_+(A, \mathcal{F}, k)$ as the set of all tuples $(A_1, \ldots, A_k)$ satisfying $A_1 + \ldots + A_k = A$ and $A_1 \succ \ldots \succ A_k$, where $\succ$ denotes the lexicographic ordering on the space of matrices which we define as follows.

Definition 2. Assume A, B are real matrices whose rows are labeled with numbers in a finite set $I \subset \mathbb{Z}$, and the columns have labels in a finite set $J \subset \mathbb{Z}$. Also, we consider the lexicographic ordering on $I \times J$ induced by the natural ordering of $\mathbb{Z}$. We write $A \succ B$ if either $A = B$ or $A(i, j) > B(i, j)$ for the minimal $(i, j) \in I \times J$ satisfying $A(i, j) \neq B(i, j)$.

Now let $\mathcal{R}$ be a real closed extension of $\mathcal{F}$. Then $\operatorname{fact}_+(A, \mathcal{R}, k)$ is a semi-algebraic subset of $\mathcal{R}^{mnk}$ because it is naturally defined by a first-order formula involving existential quantifiers and polynomial inequalities with coefficients in $\mathcal{F}$, and by classical results of Tarski and Seidenberg [29] this formula is equivalent over $\mathcal{R}$ to some quantifier-free formula. Also, the set $\operatorname{fact}_+(A, \mathcal{R}, k)$ is bounded because the entries of the matrices in the nonnegative factorizations of A cannot exceed the maximal entry of A. We define the quantity $\operatorname{rk}_+(A, \mathcal{F})$ as the smallest k such that $\operatorname{fact}_+(A, \mathcal{F}, k)$ is non-empty and call it the *nonnegative rank* of A *with respect to* the field $\mathcal{F}$. In the case $\mathcal{F} = \mathcal{R}$, we simply call this quantity the *nonnegative rank* of A and denote it by $\operatorname{rk}_+ A$, which is possible because, according to *Tarski's transfer principle* [10, Chapter 5], its value does not depend on the choice of a real closed extension $\mathcal{R}$, see also [17].

Now we proceed with a variation of the concept of the rational equivalence [59], which we need for the formulation of the main result. Let $m \leqslant n$ be nonnegative integers, and let $\pi : \mathcal{R}^n \to \mathcal{R}^m$ be a projection mapping whose images are obtained by removing the coordinates with the indexes in a fixed subset C of cardinality $n - m$. If $U \subset \mathcal{R}^n$ is an arbitrary set, then the set

$$V = \pi(U) \subset \mathcal{R}^m$$

is an *$\mathcal{F}$-rational projection* of U if there are functions $\varphi_1, \dots, \varphi_n$ which are rational over $\mathcal{F}$ and act from $\mathcal{R}^m$ to $\mathcal{R}$ such that, for all $v = (v_1, \dots, v_m) \in V$, the preimage $\pi^{-1}(v)$ is unique and equal to

$$(\varphi_1(v), \dots, \varphi_n(v)) .$$

We say that arbitrary sets $U_1 \subset \mathcal{R}^n$ and $U_2 \subset \mathcal{R}^m$ are *strongly $\mathcal{F}$-equivalent*, and we write $U_1 \sim_{\mathcal{F}} U_2$, in the case if (U_1, U_2) belongs to the equivalence relation generated by the $\mathcal{F}$-rational projections. In order to justify our terminology, we note that any pair of strongly $\mathbb{Q}$-equivalent sets are rationally equivalent in the usual sense [59] as well, but we use this fact neither to prove the Universality theorem nor to derive any of its consequences.

In order to formulate the main result of our paper, we also need to specify the form of the input data in the computational problems that we discuss. Namely, we will assume that rational numbers are explicitly given as fractions of two integers written in the unary system. Rational matrices are supposed to be given as lists of entries expressed in the form as above, and a polynomial $f \in \mathbb{Q}[x_1, \dots, x_n]$ should be expressed as a sum $f = s_1\mu_1 + \dots + s_l\mu_l$, where s_i are rational numbers and μ_i are products of the form $\alpha_{i1} \dots \alpha_{im_i}$ with $\alpha_{ij} \in \{x_1, \dots, x_n\}$. In fact, our result can be considered for slightly more general ways to represent the input of the problem below — for instance, one can allow strict inequalities or equations and use different logical connectives between them instead of taking just the system of inequalities, and also one can use the binary or decimal systems to represent the coefficients. We decided not to spend further efforts on such generalizations and used settings suitable to get main consequences of the Universality theorem discussed in Section 3. Now we only note that our result might turn out to be tricky even to formulate if one allowed arbitrary logical connectives, as can be seen from the comparison with Section 1 of Part III in [66].

The Universality Theorem for Nonnegative Matrix Factorizations.

Fix an ordered field $\mathcal{F}$ and its real closed extension $\mathcal{R}$.

For a finite family $\Phi = \{f_1, \dots, f_t\}$ of polynomials in $\mathcal{F}[x_1, \dots, x_n]$, we define $\mathrm{Sol}(\Phi)$ as the set of all points in $\mathcal{R}^n$ satisfying the conditions $f_1 \geqslant 0, \dots, f_t \geqslant 0$.

Then there are a matrix $M(\Phi)$ over $\mathcal{F}_+$ and an integer $k(\Phi)$ such that $\mathrm{fact}_+(M, \mathcal{R}, k)$ is strongly $\mathcal{F}$-equivalent to $[0,1]^n \cap \mathrm{Sol}(\Phi)$. The construction of the pair (M, k) does not depend on the initial choice of $\mathcal{R}$ and can be performed in polynomial time if $\mathcal{F} = \mathbb{Q}$.

As explained above, the solution space of every instance of the nonnegative matrix factorization problem is a bounded semialgebraic set. The above theorem shows that every such set does in turn arise as the space of nonnegative factorizations of some matrix, which explains the word '*universality*' in its title. In other words, the space of such factorizations can be arbitrarily complicated within the collection of bounded semialgebraic sets. One can think of our result as an NMF analogue of the famous theorem of Mnëv [59] on oriented matroids and other universality results on polytopes [67], Nash equilibria [20], graph drawings [8], art galleries [1], linkages [69] and several other objects in geometry [55]. Recent linear algebraic universality results include matrix completion problems and tensor decompositions [70, 72] and positive semidefinite matrix factorizations [75].

3. Consequences of the Universality theorem

We continue the discussion of the history of the problem and describe several applications of the main result. These include (1) the complexity of NMF and (2) completely positive matrix factorization, (3) the study of nonnegative rank functions over different ordered fields and the problem of Cohen and Rothblum, and (4) the geometric question known as the *nested polytope* problem. These areas of application are considered in four different subsections below.

The computational problems considered in this paper are stated in *decision* form, that is, as yes-no questions whose answer depends on the value of the *input*. A *polynomial-time many-one reduction* (or, simply, *polynomial reduction*) from one decision problem to another is a polynomial-time computable function that converts the inputs of the first problem to the inputs of the second problem such that the answer remains the same. Two problems are *polynomial-time equivalent* if they admit polynomial reductions in both directions. As it is clear from the introduction, this paper deals with many questions involving polynomial equations and inequalities, which makes it natural to consider the following general problem.

Problem 3. The existential theory of the reals (ETR).
Input: A system S of polynomial equations and inequalities with rational coefficients.
Question: Does S have a solution in $\mathbb{R}$?

A usual definition of ETR allows one to use arbitrary logical connectives on equations and inequalities, but since the corresponding problem remains polynomial-time equivalent to Problem 3 as noted in [55], we decided to take a version that is more suitable to our further considerations. The class of problems that admit polynomial reductions to ETR is called $\exists\mathbb{R}$, and problems polynomial-time equivalent to ETR are called $\exists\mathbb{R}$-*complete* [68]. One can think of $\exists\mathbb{R}$-complete problems as a natural home for hardest questions asked in terms of systems of equations and inequalities with real unknowns, and the universality results discussed above [1, 8, 20, 55, 59, 67, 69, 70, 72, 75] give examples of such problems. It is known that

$$\mathrm{NP} \subseteq \exists\mathbb{R} \subseteq \mathrm{PSPACE},$$

where the first inclusion is standard and the second one follows from the work of Canny [13]. The presice location of $\exists\mathbb{R}$ in between NP and PSPACE is unknown, but we note that the equality $\exists\mathbb{R} = \mathrm{NP}$ is not expected by the community [69] and refer the reader to [55, 68] for detailed surveys on what is known on the complexity of ETR.

3.1. **Nonnegative matrix factorization.** The NMF problem is usually defined as follows. The input is (A, k), where A is an $m \times n$ matrix with nonnegative real entries and k is a positive integer. The output is a pair of nonnegative real matrices (B, C) of the sizes $m \times k$ and $k \times n$, respectively, such that the distance from A to BC is minimal possible [81]. Various applications may lead to the consideration of different notions of the distance, but the following problem corresponds to the equality $A = BC$, and hence it is related to NMF in any case.

Problem 4. NONNEGATIVE RANK.
Input: A nonnegative rational matrix A and a positive integer r.
Question: Is it correct that the nonnegative rank of A does not exceed r?

Remark 5. Since Problem 4 is a special case of the general NMF problem [81], any hardness result on NONNEGATIVE RANK applies to NMF as well.

The earlier progress on lower bounds on the complexity of NMF came from the paper [81], which proves the NP-hardness of NONNEGATIVE RANK. (In fact, the result of [81] is significantly stronger because it deals with NONNEGATIVE RANK restricted to matrices with $\operatorname{rank} A = r$. We also refer to [46, 74] for a short NP-harndess proof of NONNEGATIVE RANK and to [74] for additional comments.) However, known algorithms for Problem 4 were still based on quantifier elimination, which raised suspicions that it may not lie in NP.

Question 6. Is NONNEGATIVE RANK in NP?

This question was asked by Vavasis in [81] and later commented on by Gillis [37] and Chistikov–Kiefer–Marušić–Shirmohammadi–Worrell [16]. Also, several widely known references [3, 9, 18] mention the result of [81] as an *NP-completeness* proof, although, as we said, Question 6 was explicitly posed in [81] as an open problem. In case of positive answer to this question, one possible certificate for NONNEGATIVE RANK could be a factorization of A whose entries are rational numbers of small bit length; Vavasis [81] points out the relation of this approach to the problem of Cohen and Rothblum [17] discussed in the subsection 3.3 below. Our Universality theorem gives the full understanding of the computational complexity of the nonnegative rank computation. In particular, we get the answer to Question 1.

Theorem 7. NONNEGATIVE RANK *and* NMF *are* $\exists\mathbb{R}$*-complete problems.*

Proof. The naive formulations of NONNEGATIVE RANK and NMF are instances of ETR, so what we only need is to construct polynomial reductions from ETR to each of these problems. In fact, according to Remark 5, it is sufficient to focus on a reduction from ETR to NONNEGATIVE RANK.

In order to construct such a reduction, we use Theorem 3.3 in [75], which states that ETR is polynomial-time equivalent to its version restricted to a single polynomial equation $f = 0$ that is known to have no solutions outside the unit cube. We apply the Universality theorem with $\mathcal{F} = \mathcal{R} = \mathbb{R}$ and $\Phi = \{f, -f\}$, and we denote by (M, k) the resulting output pair. According to our notation, the set $\mathrm{Sol}(\Phi)$ is the zero locus of f, and the Universality theorem states that this zero locus is strongly $\mathbb{R}$-equivalent to $\mathrm{fact}_+(M, \mathbb{R}, k)$. Since our notion of strong equivalence preserves the cardinalities of the sets, we conclude that the equation $f = 0$ has a solution if and only if the set $\mathrm{fact}_+(M, \mathbb{R}, k)$ is non-empty, which means, in turn, that the nonnegative rank of M is less than or equal to k. □

As explained above, the equality $\mathrm{NP} = \exists\mathbb{R}$ is not considered likely, so Theorem 7 can be thought of as evidence that NONNEGATIVE RANK does not belong to NP, and, as such, it answers Question 6. Also, Theorem 7 gives final answers to several questions arisen in the paper [3] and its sequel [60], which explore possible transformations of the naive formulation of NONNEGATIVE RANK into a polynomial system with lesser number of variables. The authors of [3] discuss the question of how many real variables are required in such formulation, and prior to the present work we did not know that the real variables are needed at all (it has been consistent with the existing state of knowledge that NONNEGATIVE RANK admits a polynomial transformation to some combinatorial problem, say, in NP). This issue remained unresolved after the paper [60], which mentions Question 1 and discusses universality results as a means of resolving the question of algorithmic complexity. Theorem 7 implies that the naive formulation of NONNEGATIVE RANK is optimal up to polynomial reductions; this result determines the exact location of this problem in the hierarchy of complexity classes and suggests that the approach of [3, 60] is not enough to handle the general case of NONNEGATIVE RANK.

As a final remark of this subsection, we note that the situation with the complexity of NONNEGATIVE RANK is similar to the one with the so-called *positive semidefinite* matrix rank, see [30, 39] for detailed discussions. As it follows from the present paper and [75], these matrix invariants are both $\exists\mathbb{R}$-complete in general, and the papers [3, 76] show that both these quantities can be determined in polynomial time if they are bounded by a constant fixed in advance. The situation is also similar for the concept of the *cp-rank* that we introduce in the following subsection; we prove the $\exists\mathbb{R}$-completess of this invariant, and the polynomial-time algorithm for the bounded case is constructed in [28]. Finally, we recall that the $\exists\mathbb{R}$-completeness result holds for the computation of the rank of a three-way real tensor [70, 72], but still this problem is fixed parameter tractable if parametrized by the rank [27].

3.2. **Completely positive matrix factorization.** A real $n \times n$ matrix A is called *completely positive* if there exist an integer r and a real nonnegative $n \times r$ matrix B such that

$$BB^\top = A. \tag{3.1}$$

This concept gives a natural generalization of both nonnegative and positive semidefinite matrices, and the active study of it began in the 60's [21, 44, 56]. Namely, a matrix A can be completely positive only when it is *doubly nonnegative* (which means that A is both nonnegative and positive semidefinite at the same time). A standard fact known from the 60's [56] is that the cones of doubly nonnegative and completely positive $n \times n$ matrices coincide for $n \leqslant 4$ and differ for $n \geqslant 5$. Subsequent studies confirmed the relevance of the cone of completely positive matrices in mathematical programming [23], block designs in combinatorics [43], mathematical models of energy demand [40], and DNA evolution in biology [48]. An interested reader is referred to the monograph [6], which is focused on completely positive matrices and contains a more detailed account of their basic theory and applications. One quantity related to complete positivity is the *completely positive rank* of a matrix A, or just the *cp-rank* of it, which equals the smallest r realizing the decomposition (3.1).

Definition 8. The *cp-rank* of a square nonnegative real matrix A is the smallest integer r for which A is a sum of r symmetric nonnegative rank-one matrices. We define $\text{cp-rank}(A) = r$ if such an r exists, and we write $\text{cp-rank}(A) = +\infty$ otherwise.

Clearly, a matrix is completely positive if and only if it has finite cp-rank. The determination of the cp-rank is called one of the main problems in theory of completely positive matrices in the above mentioned monograph [6] and survey papers [5, 7], and algorithmic questions devoted to this problem were discussed further in [22, 28, 41, 45, 47, 62, 80]. The authors of these papers point out the importance of the problem of the cp-rank computation, and they mention that the best known algorithms for this problem still require exponential time.

Problem 9. CP-RANK.
Input: A rational square matrix S and a positive integer r.
Question: Is $\text{cp-rank}(S) \leqslant r$?

By Caratheodory's theorem, an $n \times n$ completely positive matrix has cp-rank at most $0.5n(n+1)$, and this bound is known to be asymptotically optimal [11]. Since the membership problem for the completely positive cone is NP-hard [23], the existence of a polynomial upper bound on possible cp-ranks rules out polynomial-time algorithms for their computation unless P = NP. However, this fact implies neither $\exists\mathbb{R}$-completeness nor even NP-hardness[1] of CP-RANK, which were the statements widely believed by specialists. Elbassioni and Nguyen [28] discussed the possibility that CP-RANK may not belong to NP, and Gribling, de Laat, and Laurent conjectured that CP-RANK is $\exists\mathbb{R}$-complete [41].

Theorem 10. CP-RANK *is $\exists\mathbb{R}$-complete.*

We obtain this result from our Universality theorem by reducing NONNEGATIVE RANK to CP-RANK. The reduction is not straightforward and given separately in Section 4 below.

3.3. **The problem of Cohen and Rothblum.** Let us go back to the paper [17] of Cohen and Rothblum, who gave one of the earliest detailed accounts on the topic. As far as we know, they were the first to consider the nonnegative rank functions with respect to different fields and to study their mutual behavior and computational complexity. This subsection is devoted to the following concept, which was already introduced in Section 2.

[1]For the standard definition of NP involving polynomial-time many-one reductions.

Definition 11. Let $\mathcal{F}_1 \subseteq \mathcal{F}_2$ be an extension of ordered fields, and let A be a matrix with nonnegative entries in $\mathcal{F}_1$. The *nonnegative rank* of A *with respect to* $\mathcal{F}_2$ is the smallest possible number $\mathrm{rk}_+(A, \mathcal{F}_2)$ of summands in a representation of A as a sum of rank-one matrices with nonnegative entries in $\mathcal{F}_2$.

Cohen and Rothblum ask the following question, see page 163 in [17].

Question 12. How sensitive is the nonnegative rank to the choice of the ordered field?

This question survived several solution attempts since it was posed in 1993. Different versions of it appeared in the papers of Berman–Rothblum [7], Kubjas–Robeva–Sturmfels [52], Vandaele–Gillis–Glineur–Tuyttens [79], Vavasis [81] and were discussed on several computer science conferences [4, 50], but the earliest progress showing the dependence of the underlying field came only in 2015 in a technical paper [73] of the present author. Namely, we gave an example of an ordered field extension $\mathcal{F}_1 \subset \mathcal{F}_2$ and a matrix A over $\mathcal{F}_1$ such that $\mathrm{rk}_+(A, \mathcal{F}_1) \neq \mathrm{rk}_+(A, \mathcal{F}_2)$ in the notation of Definition 11. A subsequent work [16] described a rational matrix A for which $\mathrm{rk}_+(A, \mathbb{Q}) \neq \mathrm{rk}_+(A, \mathbb{R})$, thus settling a popular special case of Question 12. Both papers [16] and [73] are based on hard calculations limiting the potential of their methods to make progress on Questions 1 and 12, but the paper [74] gave a simple proof that a rational matrix A can satisfy $\mathrm{rk}_+(A, \mathbb{Q}) \neq \mathrm{rk}_+(A, \mathbb{R})$. Our Universality theorem leads to a complete answer to Question 12.

Theorem 13. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be ordered extensions of $\mathcal{F}$. The following are equivalent:*
(1) the equality $\mathrm{rk}_+(A, \mathcal{F}_1) = \mathrm{rk}_+(A, \mathcal{F}_2)$ holds for all nonnegative matrices A over $\mathcal{F}$,
(2) any finite system $f_1 \geqslant 0, \ldots, f_k \geqslant 0$ of polynomial inequalities with

$$f_1, \ldots, f_k \in \mathcal{F}[x_1, \ldots, x_n]$$

has a solution in $(\mathcal{F}_1 \cap [0,1])^n$ if and only if it has a solution in $(\mathcal{F}_2 \cap [0,1])^n$.

Proof. The naive formulation of the property of having the nonnegative rank less than or equal to a given number consists of several polynomial inequalities whose variables are the entries of the matrices in a potential nonnegative factorization. The coefficients of the corresponding polynomials are taken from the entries of a given matrix A, and the largest entry of A can be assumed to be equal to 1 without loss of generality. This forces the variables to belong to the segment $[0,1]$, and hence the nonnegative ranks of A with respect to $\mathcal{F}_1$ and $\mathcal{F}_2$ should be equal if the statement (2) is true. In other words, this confirms the implication (2)$\to$(1).

In order to prove (1)$\to$(2), we act by contradiction and take a family $\Phi = \{f_1, \ldots, f_k\}$ violating the condition (2). Further, we apply the Universality theorem with this family Φ, and we denote by M and k the resulting matrix and factorization rank. Standard results of model theory [14, page 196, Examples] allow us to assume that the fields $\mathcal{F}$, $\mathcal{F}_1$, $\mathcal{F}_2$ lie in a common real closed extension, which we take in the role of $\mathcal{R}$ in the Universality theorem. According to the initial assumption of this paragraph, the set $\mathrm{Sol}(\Phi) \cap [0,1]^n$ either

(a) has a point with all coordinates in $\mathcal{F}_1$ but no point with all coordinates in $\mathcal{F}_2$, or

(b) has a point with all coordinates in $\mathcal{F}_2$ but no point with all coordinates in $\mathcal{F}_1$.

The Universality theorem states that $\mathrm{Sol}(\Phi) \cap [0,1]^n$ is strongly $\mathcal{F}$-equivalent to $\mathrm{fact}_+(M, \mathcal{R}, k)$, and since the $\mathcal{F}$-rational projections preserve the properties (a) and (b), we get that $\mathrm{fact}_+(M, \mathcal{R}, k)$ satisfies one of them, too. This means that the nonnegative ranks of M are different with respect to $\mathcal{F}_1$ and $\mathcal{F}_2$, which implies the desired negation of the condition (1). $\square$

The proved theorem gives an intrinsic description of the property of $(\mathcal{F}_1, \mathcal{F}_2)$ to define the same nonnegative rank function and thus a complete answer to Question 12. Detailed comments on this description do not belong to the scope of the paper, but we note that Theorem 13 gives many examples when taking an extension of a field generated by the entries of a given matrix may lead to better nonnegative factorizations. The following example demonstrates this phenomenon for subfields sandwiched between an ordered field and its real closure.

Example 14. Consider an ordered field $\mathcal{F}$ and its real closure $\mathcal{R}$. If two fields $\mathcal{F}_1 \neq \mathcal{F}_2$ satisfy $\mathcal{F} \subseteq \mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{R}$, then there is a matrix A over $\mathcal{F}_+$ such that $\operatorname{rk}_+(A, \mathcal{F}_1) \neq \operatorname{rk}_+(A, \mathcal{F}_2)$.

Proof. Let α be an element in either $\mathcal{F}_1 \setminus \mathcal{F}_2$ or $\mathcal{F}_2 \setminus \mathcal{F}_1$. Since the real closure is an algebraic extension, there exists a univariate nonzero polynomial φ with coefficients in $\mathcal{F}$ such that $\varphi(\alpha) = 0$. Since the difference of any pair of distinct roots of φ is also algebraic over $\mathcal{F}$, the absolute value of this difference cannot be less than all positive elements of $\mathcal{F}$ at the same time. Therefore, there exist a_1, a_2 in $\mathcal{F}$ such that α is a unique root of φ in the interval between a_1 and a_2. This allows an appropriate $\mathcal{F}$-linear coordinate change to transform φ into a polynomial in $\mathcal{F}[x]$ which has a root in exactly one of the sets $\mathcal{F}_1 \cap [0, 1]$ and $\mathcal{F}_2 \cap [0, 1]$. An application of Theorem 13 allows us to get a matrix A with all entries in $\mathcal{F}$ such that the nonnegative ranks of A are different with respect to $\mathcal{F}_1$ and $\mathcal{F}_2$, which completes the proof. □

We finalize this subsection with a discussion of an algorithmic question.

Question 15. Is there an algorithm computing $\operatorname{rk}_+(A, \mathbb{Q})$ for a rational input matrix A?

This question was asked in [7, 17] and remains open, but the Universality theorem shows that it is equivalent to the famous open problem on the rational solvability of Diophantine equations [49, 51, 54, 57, 64, 65]. Namely, we cannot expect a positive answer to Question 15 as the following problem is believed to be undecidable.

Problem 16. Rational Diophantine equation.
Input: A polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$.
Question: Do there exist $\xi_1, \ldots, \xi_n \in \mathbb{Q}$ such that $f(\xi_1, \ldots, \xi_n) = 0$?

Theorem 17. *Question 15 is equivalent to the decidability of Problem 16.*

Proof. The property $\operatorname{rk}_+(A, \mathbb{Q}) \leqslant k$ is naturally expressed as a system of polynomial inequalities over $\mathbb{Q}$, and the rational solvability of such systems can be reduced to the rational solvability of a single equation as follows. The inequality $x \geqslant 0$ is equivalent to

$$\exists x_1 \, \exists x_2 \, \exists x_3 \, \exists x_4 \quad x = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

because every integer is a sum of four integer squares, and a system of equations $f_1 = 0, \ldots, f_k = 0$ is equivalent to the single equation $f_1^2 + \ldots + f_k^2 = 0$. Hence, the decidability of Rational Diophantine equation implies a positive answer to Question 15.

Now we proceed with the opposite direction of the equivalence in the current theorem, which means that a positive answer to Question 15 implies the decidability of Rational Diophantine equation. In other words, we are given a polynomial $f \in \mathbb{Q}[x_1, \ldots, x_n]$ as in Problem 16, and we need to tell whether or not the equation $f = 0$ has a rational solution using the rational nonnegative rank computation as an intermediate procedure. To this end, we assume without loss of generality that the degree of f is $d > 0$ because otherwise the existence of the solutions to $f = 0$ is trivial to decide. Then we consider the *homogenization*

$$h(x_1, \ldots, x_n, y) = f\left(\frac{x_1}{y}, \ldots, \frac{x_n}{y}\right) \cdot y^d$$

and we note that

(3.2) for any $\gamma \neq 0$, one has $f(\xi_1, \ldots, \xi_n) = 0$ if and only if $h(\gamma\xi_1, \ldots, \gamma\xi_n, \gamma) = 0$.

We also define the new polynomial

$$g(u_1, \ldots, u_n, z) = h(u_1 - 1/2, \ldots, u_n - 1/2, z^2 - 1/2)$$

and, since the formula $z^2 - 1/2$ cannot vanish with $z \in \mathbb{Q}$, the rational solvability of $g = 0$ implies the rational solvability of $f = 0$ in view of the conclusion (3.2). Similarly, the rational solvability of $f = 0$ implies that the equation $g = 0$ is satisfied by some rational assignment of $(u_1, \ldots, u_n, z)$ arbitrarily close to $(1/2, \ldots, 1/2, 1/\sqrt{2})$. Therefore, the rational solvability of $f = 0$ is equivalent to the existence of a rational solution to $g = 0$ in the unit cube, which can be decided by the rational nonnegative rank computation as in the Universality theorem. □

3.4. **The *nested polytope* problem.** We switch to the geometrical point of view on nonnegative factorizations. The following question dates back to one of the earliest publications on NMF ever [78], and close relationships of these problems were explained in foundational papers of Cohen–Rothblum [17] and Yannakakis [82].

Problem 18. NESTED POLYTOPE.
Given: An integer k; the vertices of a polytope $P \subset \mathbb{R}^d$; the facet-defining inequalities of a polytope $Q \subset \mathbb{R}^d$ satisfying $P \subset Q$.
Question: Is there a polytope $N \subset \mathbb{R}^d$ with at most k vertices satisfying $P \subseteq N \subseteq Q$?

The question about algorithms for NESTED POLYTOPE is attributed to Klee in a 1989 paper [2], and further discussions of this question are contained in [19, 38, 58, 63, 81]. The NP-hardness proof for NESTED POLYTOPE appeared in [19], but the $\exists\mathbb{R}$-completeness of this problem remained an open problem [16] before being proved in a recent preprint [26]. However, polynomial reductions from[2] NONNEGATIVE RANK to NESTED POLYTOPE are quite well known to specialists, so the authors of [26] did not claim the novelty of their result and attributed it to an earlier version [71] of the present paper.

Theorem 19. NESTED POLYTOPE *is* $\exists\mathbb{R}$*-complete.*

Proof. The reduction of NONNEGATIVE RANK to NESTED POLYTOPE appears in [17], so the result follows from Theorem 7. See Appendix in [26] for another account on this reduction. □

Remark 20. I learned about the history of the nested polytope problem from [26], and I would like to thank Till Miltzow for a fruitful e-mail discussion on the topic of this subsection.

The remaining part of the paper is arranged as follows. In Section 4, we give the first illustration of our method and build a polynomial reduction from NONNEGATIVE RANK to CP-RANK, which confirms that Theorem 10 follows from the main Universailty theorem. The latter theorem is proved in Sections 5–8 with the use of a strategy to consider a more general problem in which input matrices are allowed to contain unknowns. As shown in Section 8, this generalized version of NMF can encode any system of polynomial inequalities, and Sections 5–7 give a series of *gadgets* that allow us to reduce generalized factorization problems with unknown matrices to the conventional version of NMF.

4. REDUCING NONNEGATIVE RANK TO CP-RANK

We open the technical part of the paper by showing how to deduce Theorem 10 from our Universality theorem. This section gives a relatively simple application of our technique and serves as an illustration of the proof of the main result, which is given in Sections 5–8.

From basic linear algebra, we know that the conventional rank of a matrix over a field equals the order of the largest square full rank submatrix, and thus a small perturbation of an

[2]But not the other way around, so the result of the present paper does not follow from [26].

entry outside the rows and columns corresponding to this submatrix would lead to a matrix with greater rank. As noted in [74], the second part of this statement is not true for the nonnegative rank, and it is this property that we exploit in our considerations to prove the fundamental difference between the complexities of the conventional and nonnegative rank computation.

Example 21. For arbitrary positive $\alpha \in \mathbb{R}$ and $x \in [\alpha^2, 2\alpha^2]$, we consider the matrix

$$\mathcal{C}(x,\alpha) = \begin{pmatrix} x & \alpha & \alpha & \alpha & \alpha \\ \alpha & 2 & 1 & 0 & 1 \\ \alpha & 1 & 2 & 1 & 0 \\ \alpha & 0 & 1 & 2 & 1 \\ \alpha & 1 & 0 & 1 & 2 \end{pmatrix}$$

and we denote by $\mathcal{C}_0$ its bottom-right 4×4 submatrix. Then $\text{cp-rank}(\mathcal{C}) = \text{rk}_+(\mathcal{C}_0) = 4$.

Proof. We have $\text{rk}_+(\mathcal{C}_0) = 4$ because the positive entries of $\mathcal{C}_0$ cannot be covered by three rectangles each of which has all entries positive (we refer to [32] for more information about the related quantity known as the *rectangle covering number*). The entire matrix $\mathcal{C}$ has cp-rank four because of the decomposition

$$\begin{pmatrix} a^2 & a & a & 0 & 0 \\ a & 1 & 1 & 0 & 0 \\ a & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} b^2 & 0 & b & b & 0 \\ 0 & 0 & 0 & 0 & 0 \\ b & 0 & 1 & 1 & 0 \\ b & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} a^2 & 0 & 0 & a & a \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 1 & 1 \\ a & 0 & 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} b^2 & b & 0 & 0 & b \\ b & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 1 \end{pmatrix}$$

in which $\{2a, 2b\} = \left\{\alpha - \sqrt{x-\alpha^2}, \alpha + \sqrt{x-\alpha^2}\right\}$. $\square$

A similar simple observation has lead us in [74] to a short proof of the NP-hardness result for NONNEGATIVE RANK, which has been previously given a more complicated proof by Vavasis [81]. Now we use our method to construct a polynomial reduction from NONNEGATIVE RANK to CP-RANK. For arbitrary symmetric nonnegative real matrix

$$M = \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix}$$

with a specified submatrix C and any $\alpha > 0$, we consider the matrix $\Psi(M, C, \alpha)$ defined as

$$\left(\begin{array}{ccc|ccc|cccc} & & & & & & 0 & 0 & 0 & 0 \\ & A & & & B & & \vdots & \vdots & \vdots & \vdots \\ & & & & & & 0 & 0 & 0 & 0 \\ \hline & & & & & & \alpha & \alpha & \alpha & \alpha \\ & B^\top & & & C & & \vdots & \vdots & \vdots & \vdots \\ & & & & & & \alpha & \alpha & \alpha & \alpha \\ \hline 0 & \dots & 0 & \alpha & \dots & \alpha & 2 & 1 & 0 & 1 \\ 0 & \dots & 0 & \alpha & \dots & \alpha & 1 & 2 & 1 & 0 \\ 0 & \dots & 0 & \alpha & \dots & \alpha & 0 & 1 & 2 & 1 \\ 0 & \dots & 0 & \alpha & \dots & \alpha & 1 & 0 & 1 & 2 \end{array}\right)$$

and said to appear by applying the *cp-gadget with parameter* α to the specified submatrix C of M. The matrices appearing after the subtraction of an arbitrary fixed number $\xi \in [\alpha^2, 2\alpha^2]$ from every entry in the C-block of M are said to be obtained by *utilizing* this gadget.

Lemma 22. *The cp-rank of $\Psi(M, C, \alpha)$ does not exceed four plus the smallest possible cp-rank of a matrix obtained by utilizing the gadget.*

Proof. We take a matrix H obtained by adding four zero rows and columns to an arbitrary matrix obtained by utilizing the gadget. Taking c to be the order of the C-block of M, we denote by $\mathcal{C}'$ the matrix obtained from $\mathcal{C}(x, \alpha)$ as in Example 21 by taking the c copies of the first row and first column and by adding the zero blocks to make it have the same block structure as $\Psi(M, C, \alpha)$ and H. According to the definitions, the matrix $\mathcal{C}'$ can be chosen so that $H + \mathcal{C}' = \Psi(M, C, \alpha)$, and since the cp-rank of $\mathcal{C}'$ equals four, the proof is complete. □

The following observation is standard and easy.

Observation 23. *Let M be one of the nonnegative matrices*

$$\begin{pmatrix} A & B \\ O & C \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} A & O \\ B & C \end{pmatrix}$$

over an ordered field $\mathcal{F}$. Then $\mathrm{rk}_+(M, \mathcal{F}) \geqslant \mathrm{rk}_+(A, \mathcal{F}) + \mathrm{rk}_+(C, \mathcal{F})$.

Proof. If a rank-one matrix has non-zeros in both A and C blocks, it has a non-zero at the O block, and hence it cannot be used in a nonnegative decomposition of M. □

Now we are prepared to construct a reduction from Nonnegative rank to CP-rank.

Theorem 24. *Let B be an $n \times n$ real matrix with nonnegative entries. We denote by α the smallest integer exceeding $\sqrt{ng}$, where g is the largest entry of B. We define*

$$\mathcal{B} = \begin{pmatrix} S & B \\ B^\top & S \end{pmatrix},$$

where S denotes the $n \times n$ matrix with the number $8n^2\alpha^2$ on the main diagonal and with $2\alpha^2$ everywhere else. We define $\mathcal{R}$ as the matrix obtained from $\mathcal{B}$ by repeatedly applying the cp-gadgets with parameter α to every 2×2 principal submatrix of each of the two S-blocks, and then the cp-gadgets with parameter $2n\alpha$ to every principal 1×1 submatrix of the same two blocks. Then

$$\text{cp-rank}(\mathcal{R}) = \mathrm{rk}_+(B) + 4n(n+1).$$

Proof. Step 1. We are going to show the '$\leqslant$' inequality using Lemma 22. The total number of gadgets is $n(n+1)$, so we need to show that their utilization can lead to a matrix of cp-rank at most $\mathrm{rk}_+(B)$. Working out the 2×2 gadgets, we can leave the off-diagonal entries of the S-blocks equal to arbitrary numbers in $[0, \alpha^2]$; each of the diagonal entries remains at least $8n^2\alpha^2 - 2\gamma\alpha^2 \geqslant 6n^2\alpha^2$, where $\gamma = n - 1$ is the total number of the 2×2 gadgets involving every particular diagonal entry of S. The utilization of the 1×1 gadgets allows us to subtract any number in $[4n^2\alpha^2, 8n^2\alpha^2]$ from any diagonal entry, so we can get arbitrary numbers in $[0, 2n^2\alpha^2]$ at the diagonal positions. In particular, the utilization of all the gadgets allows us to fill the S-blocks of $\mathcal{B}$ with arbitrary and possibly different numbers in $[0, ng]$; let us now see that a matrix of cp-rank $\mathrm{rk}_+(B)$ can arise in this way.

A nonnegative decomposition of B can be written as $B = u_1 v_1^\top + \ldots + u_r v_r^\top$ with nonnegative vectors $u_1, \ldots, u_r, v_1, \ldots, v_r$ of length n. Since the maximal entry of B is g, we can use scaling and assume without loss of generality that no coordinate of these vectors exceeds $\sqrt{g}$. The matrix

$$\mathcal{B}' = \begin{pmatrix} u_1 u_1^\top & u_1 v_1^\top \\ v_1 u_1^\top & v_1 v_1^\top \end{pmatrix} + \ldots + \begin{pmatrix} u_r u_r^\top & u_r v_r^\top \\ v_r u_r^\top & v_r v_r^\top \end{pmatrix}$$

has cp-rank at most r and satisfies the assumptions obtained in the first paragraph.

Step 2. We are going to proceed with the '$\geqslant$' part by demonstrating a stronger inequality $\mathrm{rk}_+(\mathcal{R}) \geqslant \mathrm{rk}_+(B) + 4n(n+1)$. We can see that the matrix $\mathcal{R}$ has the form

$$\left(\begin{array}{c|c|c|c} S & B & * & O \\ \hline B^\top & S & O & * \\ \hline * & O & D & O \\ \hline O & * & O & D \end{array}\right)$$

in which D is the block-diagonal matrix with $0.5n(n+1)$ diagonal blocks equal to the copies of the matrix $\mathcal{C}_0$ as in Example 21. The O-blocks are zero matrices of appropriate sizes, and the $*$'s stand for submatrices that do not need to be specified. The result follows from Observation 23 because the two D-blocks have nonnegative ranks $2n(n+1)$ each, so even if we remove the rows and columns corresponding to $B^\top$ from $\mathcal{R}$, we would have a lower bound of $2n(n+1) + 2n(n+1) + \mathrm{rk}_+(B)$ for the nonnegative rank of the resulting submatrix of $\mathcal{R}$. $\square$

Since every matrix can be completed to a square matrix without changing its nonnegative rank by adding several zero lines, the assumption of B being $n \times n$ does not cause any loss of generality in Theorem 24. Therefore, we have a polynomial reduction of NONNEGATIVE RANK to CP-RANK, which proves Theorem 10 modulo our main Universality theorem.

5. THE PROOF: INCOMPLETE MATRICES AND PRELIMINARY RESULTS

In this section, we generalize the concept of a *nonnegative factorization* to so-called *incomplete matrices*, which are allowed to contain unknown entries. Formally speaking, we define a *nonnegative incomplete matrix over* a field $\mathcal{F}$ as one having at every entry either an element of $\mathcal{F}_+$ or a variable ranging in a given segment $[a, b]$ with some nonnegative $a, b \in \mathcal{F}$. We assume that the same variable may occur at arbitrary number of entries of a given incomplete matrix. A completion B of such an incomplete matrix $\mathcal{B}$ is any conventional matrix that can be obtained from $\mathcal{B}$ by assigning some value to every variable within its range. We define $\mathrm{fact}_+(\mathcal{B}, \mathcal{F}, k)$ as the union of all sets $\mathrm{fact}_+(B, \mathcal{F}, k)$ over all completions B of $\mathcal{B}$. In other words, a formula $B_1 + \ldots + B_k$ belongs to $\mathrm{fact}_+(\mathcal{B}, \mathcal{F}, k)$ if the matrices $B_1, \ldots, B_k$ sum to a completion of $\mathcal{B}$, are rank-one, have entries in $\mathcal{F}_+$ and satisfy $B_1 \succ \ldots \succ B_k$ in the sense of Definition 2. If $\mathcal{R}$ is the real closure of $\mathcal{F}$, then the smallest k for which $\mathrm{fact}_+(\mathcal{B}, \mathcal{R}, k)$ is non-empty is called the *nonnegative rank* of an incomplete matrix $\mathcal{B}$.

We recall that, according to Definition 2, the rows and columns of matrices may have arbitrary labels in $\mathbb{Z}$. A matrix is said to be $I \times J$ if its rows and columns are labeled with sets $I \subset \mathbb{Z}$ and $J \subset \mathbb{Z}$, respectively. When representing matrices as tables of numbers, we may permute rows and columns, so their order may not correspond to that of I or J. In the following three examples, we assume that the ground field $\mathcal{F}$ is the real number field $\mathbb{R}$, and we assign the labels $1, 2, \ldots$ consecutively to the columns from the left to the right and to the rows from the top to the bottom.

Example 25. Consider the incomplete matrix

$$\mathcal{A} = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$$

assuming $y \in [0, 1]$. The nonnegative rank of $\mathcal{A}$ is (at least) two because the conventional rank of any completion of $\mathcal{A}$ is two. The set $\mathrm{fact}_+(\mathcal{A}, \mathbb{R}, 2)$ consists of all formal decompositions

$$\begin{pmatrix} 1 & 0 \\ u & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ v & 1 \end{pmatrix}$$

with $u, v \in \mathbb{R}$ satisfying $u \geqslant 0$, $v \geqslant 0$, $u + v \leqslant 1$.

Example 26. Another incomplete matrix

$$\mathcal{B} = \begin{pmatrix} 1 & 1 & x_1 \\ x_2 & y & 1 \\ x_3 & 2 & y \end{pmatrix}$$

with variables ranging in $[0, 2]$ has nonnegative rank one, and actually $\operatorname{fact}_+(\mathcal{B}, \mathbb{R}, 1)$ is the singleton set corresponding to $(x_1, x_2, x_3, y) = \left(\sqrt{0.5}, \sqrt{2}, 2, \sqrt{2}\right)$. In particular, a non-empty space of nonnegative factorizations of a matrix with rational known entries may consist of non-rational points only, see also [16, 74].

The following example is useful in further considerations.

Example 27. The nonnegative rank of the matrix

$$V = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

equals four, and the set $\operatorname{fact}_+(V, \mathbb{R}, 4)$ consists of the two factorizations

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

because V has no submatrix with all entries positive of size greater than 1×2 or 2×1.

In the following lemma and rest of this paper, the word 'matrix' refers to conventional matrices without unknowns unless the adjective 'incomplete' is explicitly used.

Lemma 28. *Consider a nonnegative matrix*

$$B = \left(\begin{array}{c|c} u & o \\ \hline B' & B'' \end{array} \right)$$

over an ordered field $\mathcal{F}$, where u is a non-zero row vector and o is a zero row vector. If $\operatorname{Rank}_+(B'') \geqslant r - 1$*, then in every nonnegative factorization $B = B_1 + \ldots + B_r$*
(i) for some i, the first row of B_i equals $(u|o)$,
(ii) for all $j \neq i$, a non-zero row of B_j contains at least one non-zero entry outside the u-part.

Proof. In order to express the first row of B, we should have a matrix B_i with rows collinear to a non-zero vector with all entries outside u zero. This matrix B_i does not contribute to the B'' block, so the condition $\operatorname{Rank}_+(B'') \geqslant r - 1$ forces the matrices B_j with $j \neq i$ satisfy the condition (ii). But these matrices in turn cannot contribute to the first row of B because of the zero part of it, so the condition (i) is also satisfied. □

The analogue of Lemma 28 for the columns of B can be deduced from the original version by considering the transpose of B. However, we will need an even stronger corollary of Lemma 28.

Corollary 29. *Consider a nonnegative matrix*

$$B = \left(\begin{array}{c|c|c} a & u & o \\ \hline v & C_1 & C_2 \\ \hline o & C_3 & C_4 \end{array}\right)$$

over an ordered field $\mathcal{F}$, where u, v are non-zero row and column vectors, respectively, and the o's denote the zero submatrices of appropriate sizes. If $a \neq 0$ and $\operatorname{Rank}_+(C_4) \geqslant r-1$, then in every nonnegative factorization $B = B_1 + \ldots + B_r$, one of the matrices B_t has the form

$$\left(\begin{array}{c|c|c} a & u & O \\ \hline v & W & O \\ \hline O & O & O \end{array}\right)$$

with $W = a^{-1}vu$.

Proof. According to Lemma 28, the rows of one of the matrices B_i are collinear to the first row of B, and the columns of some B_j are collinear to the first column of B. Since neither B_i nor B_j contribute to the C_4 block of B, the condition $\operatorname{Rank}_+(C_4) \geqslant r-1$ implies $i = j$. We conclude that the first row and first column of B_i should coincide with those of B, respectively, and then B_i has to have the desired form because it is rank-one. □

6. The proof: A simple gadget

The two upcoming sections give a series of *gadgets* that allow one to reduce the conventional nonnegative factorization problem to an *a priori* more general version with incomplete matrices. The following theorem is devoted to the case when a variable appears in at most one row of an incomplete matrix.

Theorem 30. *Consider a nonnegative incomplete $I \times J$ matrix*

$$\mathcal{A} = \left(\begin{array}{c|c} A & B \\ \hline c & x \ldots x \end{array}\right)$$

over a field $\mathcal{F}$ in which the A block is of size $m \times n$, the B block is $m \times t$, and c is a $1 \times n$ row vector. Assume that the variable x ranges in $[\beta - \alpha, \beta]$, where $\beta \geqslant \alpha$ are positive elements of $\mathcal{F}$, and assume that A, B, c do not involve any occurrence of x. If

$$\operatorname{rk}_+ (A \mid B) \geqslant r \quad \text{and} \quad \operatorname{rk}_+ \begin{pmatrix} A \\ c \end{pmatrix} \geqslant r, \tag{6.1}$$

then for the matrix $\mathcal{G}(\mathcal{A}, x)$ defined as

$$\mathcal{G} = \left(\begin{array}{c|c|cccc} & & 0 & 0 & 0 & 0 \\ A & B & \vdots & \vdots & \vdots & \vdots \\ & & 0 & 0 & 0 & 0 \\ \hline c & \beta \ldots \beta & \alpha & \alpha & \alpha & \alpha \\ \hline 0 \ldots 0 & \alpha \ldots \alpha & \alpha & \alpha & 0 & 0 \\ 0 \ldots 0 & 0 \ldots 0 & 0 & \alpha & \alpha & 0 \\ 0 \ldots 0 & 0 \ldots 0 & 0 & 0 & \alpha & \alpha \\ 0 \ldots 0 & 0 \ldots 0 & \alpha & 0 & 0 & \alpha \end{array}\right)$$

with four last rows having indexes $\min I - 1, \min I - 2, \min I - 3, \min I - 4$ *and four last columns having indexes* $\min J - 1, \min J - 2, \min J - 3, \min J - 4$*, the set* $\operatorname{fact}_+(\mathcal{A}, \mathcal{R}, r)$ *is an $\mathcal{F}$-rational projection of* $\operatorname{fact}_+(\mathcal{G}, \mathcal{R}, r+4)$*, where $\mathcal{R}$ is any real closed extension of $\mathcal{F}$.*

Proof. Consider a nonnegative factorization $\mathcal{G} = G_1 + \ldots + G_{r+4}$ and apply Lemma 28 consecutively to the three last rows of $\mathcal{G}$, where the use of Lemma 28 is justified by Observation 23 and the condition (6.1). This shows that the first three summands $G_1 + G_2 + G_3$ are

$$\left(\begin{array}{c|c|cccc} O & O & O & O & O & O \\ \hline O & O & \alpha_1 & 0 & 0 & \alpha_1 \\ \hline O & O & 0 & 0 & 0 & 0 \\ O & O & 0 & 0 & 0 & 0 \\ O & O & 0 & 0 & 0 & 0 \\ O & O & \alpha & 0 & 0 & \alpha \end{array}\right) + \left(\begin{array}{c|c|cccc} O & O & O & O & O & O \\ \hline O & O & 0 & 0 & \alpha_2 & \alpha_2 \\ \hline O & O & 0 & 0 & 0 & 0 \\ O & O & 0 & 0 & 0 & 0 \\ O & O & 0 & 0 & \alpha & \alpha \\ O & O & 0 & 0 & 0 & 0 \end{array}\right) + \left(\begin{array}{c|c|cccc} O & O & O & O & O & O \\ \hline O & O & 0 & \alpha_3 & \alpha_3 & 0 \\ \hline O & O & 0 & 0 & 0 & 0 \\ O & O & 0 & \alpha & \alpha & 0 \\ O & O & 0 & 0 & 0 & 0 \\ O & O & 0 & 0 & 0 & 0 \end{array}\right)$$

with some nonnegative $\alpha_1, \alpha_2, \alpha_3$, where the partition into the blocks corresponds to the partition of the matrix $\mathcal{G}$, and the O's stand for zero blocks of appropriate sizes. Now we can apply Lemma 28 to the fourth row from the bottom of $\mathcal{G} - G_1 - G_2 - G_3$ and see that

$$G_4 = \left(\begin{array}{c|c|cccc} O & O & O & O & O & O \\ \hline O & \gamma\ldots\gamma & \gamma & \gamma & 0 & 0 \\ \hline O & \alpha\ldots\alpha & \alpha & \alpha & 0 & 0 \\ O & 0\ldots0 & 0 & 0 & 0 & 0 \\ O & 0\ldots0 & 0 & 0 & 0 & 0 \\ O & 0\ldots0 & 0 & 0 & 0 & 0 \end{array}\right)$$

with some nonnegative γ. Subtracting $G_1 + G_2 + G_3 + G_4$ from $\mathcal{G}$, we get the matrix

$$\left(\begin{array}{c|ccc|cccc} & & & & 0 & 0 & 0 & 0 \\ A & & B & & \vdots & \vdots & \vdots & \vdots \\ & & & & 0 & 0 & 0 & 0 \\ \hline c & \beta-\gamma & \ldots & \beta-\gamma & * & * & * & * \\ \hline 0\ldots0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ 0\ldots0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ 0\ldots0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ 0\ldots0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

which being equal to $G_5 + \ldots + G_{r+4}$ has nonnegative rank r. But since $(A|B)$ does already have nonnegative rank at least r by the assumption (6.1), we use Observation 23 and conclude that the $*$ placeholders are all zero, which means that

$$\alpha - \alpha_1 - \gamma = 0,\ \alpha - \alpha_3 - \gamma = 0,\ \alpha - \alpha_2 - \alpha_3 = 0,\ \alpha - \alpha_1 - \alpha_2 = 0,$$

or $\alpha_2 = \gamma$, $\alpha_1 = \alpha_3 = \alpha - \gamma$. Since $\alpha_1 \geqslant 0$ and $\alpha_2 \geqslant 0$, we have $\gamma \in [0, \alpha]$, and hence $G_5 + \ldots + G_{r+4}$ is a nonnegative factorization of size r for $\mathcal{A}$ up to adding several zero rows and columns. Therefore, we have a one-to-one correspondence between the nonnegative factorizations of $\mathcal{A}$ with size r and nonnegative factorizations of $\mathcal{G}$ of size $r+4$, and the latter factorizations arise by adding four new matrices G_1, G_2, G_3, G_4 whose entries are polynomials of x with coefficients in $\mathcal{F}$. □

We say that the matrix $\mathcal{G}$ as in Theorem 30 is obtained from $\mathcal{A}$ by applying the *simple gadget* to the variable x. We finalize the section with a slightly more detailed treatment of the special case $t = 1$, which corresponds to the variable x being single. We emphasize that the following corollary does not require the assumption (6.1).

Corollary 31. *If $t = 1$ and $\mathcal{A}$, $\mathcal{G}$ are matrices as in Theorem 30, then* $\mathrm{rk}_+(\mathcal{G}) = \mathrm{rk}_+(\mathcal{A}) + 4$.

Proof. Follows from Theorem 30; we do not give a more detailed proof here because a related statement appears as Corollary 3 in [74]. □

7. The proof: A strong gadget

In this section, we give a construction allowing us to treat variables that occur in different rows and different columns of a given incomplete matrix.

Consider a nonnegative incomplete $I \times J$ matrix B over an ordered field $\mathcal{F}$ in which one of the variables is denoted by y and ranges in $[a, b]$ with $a < b$. We denote the set of entries equal to y by $\mathcal{T} = \{(i_1, j_1), \dots, (i_\tau, j_\tau)\}$, and we assume that $\mathcal{T}$ is transversal, that is, each of the sequences $i_1, \dots, i_\tau$ and $j_1, \dots, j_\tau$ has no repeating indexes. In other words, we can write

$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix},$$

where B_1 is a square $\tau \times \tau$ block with y's on the diagonal, and B_2, B_3, B_4 have no appearance of y. We consider the matrix $\Gamma(B, y)$ defined as

$$\left(\begin{array}{c|c|c} z \quad \dots \quad z & 1 \quad \dots \quad 1 & 0 \quad \dots \quad 0 \\ \hline \Lambda_\tau & I_\tau & O \\ \hline I_\tau & B_{\mathcal{T}} & B_2 \\ \hline O & B_3 & B_4 \end{array}\right) \tag{7.1}$$

in which $B_{\mathcal{T}}$ is obtained from B_1 by replacing every entry equal to y by $2b$. The blue Λ_τ block is $\tau \times \tau$ and has new pairwise different variables ranging in $[1/(2b-a), 1/b]$ on the main diagonal and zeros everywhere outside the main diagonal; the green variable z is also new, and it ranges in $[1/(2b-a), 1/b]$ as well. The cyan I_τ blocks are the $\tau \times \tau$ identity matrices. The first $\tau + 1$ rows of Γ get labels $\min I - \tau - 1, \dots, \min I - 1$, and the first τ columns are labeled $\min J - \tau, \dots, \min J - 1$ in the order they are listed in (7.1), and the remaining rows and columns preserve those labels as they had in the initial matrix B.

The following theorem gives a reduction from a relevant special case of the nonnegative rank problem on incomplete matrices to the conventional version.

Theorem 32. *Suppose $\mathcal{F}$ is an ordered field with a real closed extension $\mathcal{R}$. Let M be a nonnegative incomplete matrix over $\mathcal{F}$ partitioned into the blocks as*

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

with no variable appearances outside M_1 and such that $\operatorname{rk}_+(M_4) = \rho$. We assume that each of the variables $x_1, \dots, x_n$ appearing in M occurs at most once in every row and every column, and every such x_i ranges in $[a_i, b_i]$ with $a_i < b_i$ and occurs $\tau_i > 0$ times. We define the matrix

$$\mathcal{M} = \Gamma(\Gamma(\dots(\Gamma(M, x_1)\dots), x_{n-1}), x_n),$$

which is the outcome of the repeated application of the construction as in (7.1) intended to eliminate the variables $x_1, \dots, x_n$ in M. We define $\mathcal{M}_0$ as the matrix obtained by the repeated application the simple gadgets to all variables in $\mathcal{M}$ taken in arbitrarily fixed order.

Then $\operatorname{fact}_+(\mathcal{M}_0, \mathcal{R}, \rho + 4n + 5\tau_1 + \dots + 5\tau_n) \sim_{\mathcal{F}} \operatorname{fact}_+(M, \mathcal{R}, \rho)$.

Proof. The matrix $\mathcal{M}$ has two types of variables, green and blue. The green variables correspond to the z-variables in (7.1), and the blue variables come from the Λ_τ blocks. As we can see, one green and τ_i blue variables come after the ith iteration of Γ in the definition of $\mathcal{M}$. The $(\tau_i+1)\times\tau_i$ submatrix of $\mathcal{M}$ containing the variables appeared on the ith iteration is to be called the ith *variable block*. Our proof goes in three steps; in Step 1, we get rid of the gadgets that correspond to the blue variables and prove that

$$\operatorname{fact}_+(\mathcal{M}_0,\mathcal{R},\rho+4n+5\tau_1+\ldots+5\tau_n)\sim_{\mathcal{F}}\operatorname{fact}_+(\mathcal{M}_g,\mathcal{R},\rho+4n+\tau_1+\ldots+\tau_n), \tag{7.2}$$

where $\mathcal{M}_g$ is the matrix obtained from $\mathcal{M}$ by applying the simple gadgets to the green variables only. In Step 2, we deal with the green gadgets and prove that

$$\operatorname{fact}_+(\mathcal{M}_g,\mathcal{R},\rho+4n+\tau_1+\ldots+\tau_n)\sim_{\mathcal{F}}\operatorname{fact}_+(\mathcal{M},\mathcal{R},\rho+\tau_1+\ldots+\tau_n). \tag{7.3}$$

In Step 3, we analyze the nonnegative factorizations of $\mathcal{M}$ and confirm that

$$\operatorname{fact}_+(\mathcal{M},\mathcal{R},\rho+\tau_1+\ldots+\tau_n)\sim_{\mathcal{F}}\operatorname{fact}_+(M,\mathcal{R},\rho), \tag{7.4}$$

which would complete the proof in view of (7.2) and (7.3).

Step 1. Let $\mathcal{M}'_q$ be the matrix obtained from $\mathcal{M}$ by applying the simple gadgets to all the green variables and to a non-empty set of q blue variables. In order to check (7.2) by the induction, it is sufficient to show that, for every $q\in\{1,\ldots,\tau_1+\ldots+\tau_n\}$, we have

$$\operatorname{fact}_+(\mathcal{M}'_q,\mathcal{R},\rho+4n+\tau_1+\ldots+\tau_n+4q)\sim_{\mathcal{F}}\operatorname{fact}_+(\mathcal{M}'_{q-1},\mathcal{R},\rho+4n+\tau_1+\ldots+\tau_n+4q-4). \tag{7.5}$$

Assume that $\mathcal{M}'_{q-1}$ differs from $\mathcal{M}'_q$ by taking the simple gadget on a variable that appeared at an ith iteration of Γ; this variable is called *current* in the rest of Step 1. We define $\mathcal{M}'_{q1}$ as the matrix obtained from $\mathcal{M}'_{q-1}$ by removing the row corresponding to the current variable, and we define $\mathcal{M}'_{q2}$ in the same way but with the removal of the column corresponding to the current variable. According to Theorem 30, the condition (7.5) would follow from

$$\operatorname{rk}_+(\mathcal{M}'_{q1}),\operatorname{rk}_+(\mathcal{M}'_{q2})\geqslant\rho+4n+\tau_1+\ldots+\tau_n+4q-4. \tag{7.6}$$

To get these estimates, we apply Corollary 31 and utilize the remaining $q-1$ simple gadgets on the blue variables of $\mathcal{M}'_{q1}$ and $\mathcal{M}'_{q2}$ so the inequalities (7.6) would in turn follow from

$$\operatorname{rk}_+(\mathcal{N}_1),\operatorname{rk}_+(\mathcal{N}_2)\geqslant\rho+4n+\tau_1+\ldots+\tau_n, \tag{7.7}$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are the matrices obtained from $\mathcal{M}_g$ by removing the row containing the current variable and the column containing it, respectively. In order to prove (7.7), we take the submatrices of $\mathcal{N}_1$ and $\mathcal{N}_2$ having, as diagonal blocks, the M_4 block coming from the initial matrix M, the n V-blocks[3] from the green gadgets, all the blue Λ-blocks except the ith one, and the ith cyan I-block (the bottom-left one for $\mathcal{N}_1$ and the upper-right one for $\mathcal{N}_2$). Using Observation 23, we get a lower bound of $\rho+4n+\tau_1+\ldots+\tau_n$ for the nonnegative ranks of these submatrices, which leads to the desired bound (7.7).

Step 2. The green gadgets are treated in a way similar to Step 1. We take $\mathcal{M}''_q$ as one of the matrices obtained from $\mathcal{M}$ by applying the simple gadgets to a set of q green variables, and we arrange it that $\mathcal{M}''_{q-1}$ differs from $\mathcal{M}''_q$ by taking the simple gadget on a variable that appeared at an ith iteration of Γ. In order to prove (7.4) by the induction, we need to show

$$\operatorname{fact}_+(\mathcal{M}''_q,\mathcal{R},\rho+\tau_1+\ldots+\tau_n+4q)\sim_{\mathcal{F}}\operatorname{fact}_+(\mathcal{M}''_{q-1},\mathcal{R},\rho+\tau_1+\ldots+\tau_n+4q-4). \tag{7.8}$$

Similarly to Step 1, we define $\mathcal{M}''_{q1}$ as the matrix obtained from $\mathcal{M}''_{q-1}$ by removing the row corresponding to the ith green variable, and we define $\mathcal{M}''_{q2}$ in the same way but with the removal of the columns corresponding to the ith green variable. To complete the proof of (7.8) by a reference to Theorem 30, we need the inequalities

$$\operatorname{rk}_+(\mathcal{M}''_{q1}),\operatorname{rk}_+(\mathcal{M}''_{q2})\geqslant\rho+\tau_1+\ldots+\tau_n+4q-4,$$

[3]By a V-block we mean a positive multiple of the matrix from Example 27.

which follow from Observation 23 because we can consider the submatrices of $\mathcal{M}''_{q1}$ and $\mathcal{M}''_{q2}$ having, as diagonal blocks, the M_4 block coming from the initial matrix M, the $(q-1)$ occurrences of the V-blocks from the green gadgets, all the blue Λ-blocks except the ith one, and the ith upper-right cyan I-block.

Step 3. For any blue variable of $\mathcal{M}$, we apply Corollary 29 to the row and column containing this variable. Therefore, the first $\tau_1+\ldots+\tau_n$ matrices of any factorization in $\operatorname{fact}_+(\mathcal{M},\mathcal{R},\rho+\tau_1+\ldots+\tau_n)$ should have non-zero entries in the 3×2 submatrices of the form

$$\begin{pmatrix} z_i & 1 \\ z_i & 1 \\ 1 & 1/z_i \end{pmatrix} \tag{7.9}$$

and zeros everywhere else, which means in particular that the ith green and blue variables take the same value $z_i \in [1/(2b_i-a_i), 1/b_i]$. Subtracting all such matrices from $\mathcal{M}$, we get a matrix that differs from the initial matrix M only by (i) the addition of several zero rows and columns and (ii) substitutions $x_i \to 2b_i - 1/z_i$. This gives a one-to-one correspondence between the nonnegative factorizations of M with size ρ and the nonnegative factorizations of $\mathcal{M}_0$ with size $\rho+\tau_1+\ldots+\tau_n$, and the latter factorizations arise by adding $\tau_1+\ldots+\tau_n$ new matrices whose non-zero entries are as in (7.9), so in particular they are $\mathcal{F}$-rational functions of the variables involved in M. This implies the condition (7.4) and completes the proof. $\square$

8. The proof: Encoding a polynomial equation

Now we are going to complete the proof of the Universality theorem using Theorem 32. The following lemmas explain how to express the polynomial inequality $f \geqslant 0$ in terms of the factorization spaces of incomplete matrices. Namely, the first of these lemmas encodes linear combinations, the second one allows us to take the product of two variables, and the third one resolves inequalities. We assume that the rows and columns of the matrices S, P, Q in these lemmas are labeled by consecutive integers from the top to the bottom and from the left to the right, respectively. In the rest of our paper, we consider an arbitrary ordered field $\mathcal{F}$, and we work with nonnegative factorizations over an arbitrary real closed extension $\mathcal{R}$.

Lemma 33. *Assume $y_1 = 1$ and $b, s_1, \ldots, s_l, N$ are positive elements of $\mathcal{F}$ satisfying $b \leqslant s_1$ and $N \geqslant s_1+\ldots+s_l$. Assume $y_2,\ldots,y_l, z_1,\ldots,z_l$ are variables ranging in $[0,1]$, and let L be a variable ranging in $[b,N]$. Let $\mathcal{S} = \mathcal{S}(s_1+s_2y_2+\ldots+s_ly_l = L)$ be the matrix obtained from*

$$S = \left(\begin{array}{c|ccc} y_1 & & & \\ \vdots & & I_l & \\ y_l & & & \\ \hline L & s_1 & \ldots & s_l \\ \hline z_1 & & & \\ \vdots & & I_l & \\ z_l & & & \end{array}\right)$$

by repeatedly applying the simple gadgets to every blue variable. Then
(1) if we remove all the rows and columns of $\mathcal{S}$ which contain variables (that is, which contain $y_2,\ldots,y_l$ and L), we get a matrix with nonnegative rank $5l$;
(2) for any assignment $\gamma_2,\ldots,\gamma_l \in [0,1]$ of the variables $y_2,\ldots,y_l$, the matrix obtained from $\mathcal{S}$ by realizing this assignment admits a unique nonnegative factorization $\Phi(\gamma_2,\ldots,\gamma_l)$ of size $5l$. This factorization requires $L = s_1+s_2\gamma_2+\ldots+s_l\gamma_l$, and the rule $\Phi(\gamma_2,\ldots,\gamma_l)\to(\gamma_2,\ldots,\gamma_l)$ defines an $\mathcal{F}$-rational projection of the set $\operatorname{fact}_+(\mathcal{S},\mathcal{R},5l)$.

Proof. Removing from $\mathcal{S}$ the rows and columns containing $y_1, \ldots, y_l$ and L, we get a block-triangular matrix that has, as diagonal blocks, the l positive scalar multiples of the matrix V as in Example 27 and one additional $l \times l$ unity block (the V-blocks come because of the gadgets, and the unit block corresponds to the bottom-right block of S). Therefore, this matrix has nonnegative rank $4l + l = 5l$, which implies the statement (1).

Concerning the condition (2), our first step is a proof that

$$\text{fact}_+(S, \mathcal{R}, l) \text{ is an } \mathcal{F}\text{-rational projection of } \text{fact}_+(\mathcal{S}, \mathcal{R}, 5l). \tag{8.1}$$

To this end, we consider the matrix $\mathcal{S}_q$ obtained from S by applying the simple gadgets to the variables $z_1, \ldots, z_q$ with any $q \in \{1, \ldots, l\}$. This gives $\mathcal{S} = \mathcal{S}_l$, and we also assume $\mathcal{S}_0 = S$. In order to confirm the condition (8.1) by the induction, we need to check that

$$\text{fact}_+(\mathcal{S}_q, \mathcal{R}, l + 4q) \text{ is an } \mathcal{F}\text{-rational projection of } \text{fact}_+(\mathcal{S}_{q-1}, \mathcal{R}, l + 4q - 4) \tag{8.2}$$

holds for all q. The condition (8.2) can be checked with the application of Theorem 30 justified because the matrix obtained from $\mathcal{S}_{q-1}$ by removing the row and column containing z_q has the nonnegative rank equal to $l + 4q - 4$, which is the case by the argument similar to the consideration of the condition (1). Therefore, the condition (8.1) is confirmed, and we can focus on the set $\text{fact}_+(S, \mathcal{R}, l)$. Indeed, we can have $\operatorname{rank} S \leqslant l$ only if

$$z_1 = y_1, \ \ldots, \ z_l = y_l \text{ and } L = s_1 y_1 + \ldots + s_l y_l,$$

and, in this case, every corresponding assignment $\gamma_2, \ldots, \gamma_l \in [0, 1]$ of the variables $y_2, \ldots, y_l$ gives a unique nonnegative $\mathcal{F}$-factorization of size l represented as

$$\left(\begin{array}{c|c} \gamma_1 \mathbf{e}_1 & \mathbf{E}_{11} \\ \hline \gamma_1 s_1 & s_1 \mathbf{g}_1 \\ \hline \gamma_1 \mathbf{e}_1 & \mathbf{E}_{11} \end{array}\right) + \ldots + \left(\begin{array}{c|c} \gamma_l \mathbf{e}_l & \mathbf{E}_{ll} \\ \hline \gamma_l s_l & s_l \mathbf{g}_l \\ \hline \gamma_l \mathbf{e}_l & \mathbf{E}_{ll} \end{array}\right) \tag{8.3}$$

with $\gamma_1 = 1$. Here, the notations $\mathbf{e}_i$, $\mathbf{g}_i$, $\mathbf{E}_{ii}$ stand for the ith unit column vector, the ith unit row vector, and the (i, i) matrix unit, respectively. The factorization (8.3) has entries equal to every of the numbers $\gamma_2, \ldots, \gamma_l$, and all the other entries can be represented as $\mathcal{F}$-rational functions of $(\gamma_2, \ldots, \gamma_l)$ not depending on a particular choice of $(\gamma_2, \ldots, \gamma_l)$. Therefore, the mapping sending the factorization (8.3) into the tuple $(\gamma_2, \ldots, \gamma_l)$ is an $\mathcal{F}$-rational projection of $\text{fact}_+(S, \mathcal{R}, l)$, and hence a comparison with the condition (8.1) completes the proof. □

Lemma 34. *Let u_1, u_2, u_3, z_1, z_2 be variables ranging in $[0, 1]$. Let $\mathcal{P} = \mathcal{P}(u_1 u_2 = u_3)$ be the matrix obtained from*

$$P = \begin{pmatrix} u_3 & u_1 & z_1 \\ u_2 & 1 & 1 \\ z_2 & 1 & 1 \end{pmatrix}$$

by applying the simple gadgets to each of the blue variables. Then
(1) if we remove all the rows and columns of $\mathcal{P}$ which contain variables (that is, which contain u_1, u_2, u_3), we get a matrix with nonnegative rank 9;
(2) for any assignment $\nu_1, \nu_2 \in [0, 1]$ of the variables u_1, u_2, the matrix obtained from $\mathcal{P}$ by realizing this assignment admits a unique nonnegative factorization $\Psi(\nu_1, \nu_2)$ of size 9. This factorization requires $\nu_1 \nu_2 = u_3$, and the rule $\Psi(\nu_1, \nu_2) \to (\nu_1, \nu_2)$ defines an $\mathcal{F}$-rational projection of the set $\text{fact}_+(\mathcal{P}, \mathcal{R}, 9)$.

Proof. The statement (1) and the fact that $\text{fact}_+(P, \mathcal{R}, 1)$ is an $\mathcal{F}$-rational projection of $\text{fact}_+(\mathcal{P}, \mathcal{R}, 9)$ are justified similarly to the previous theorem. In order to complete the proof, we check that P has rank one if and only if $z_1 = u_1$, $z_2 = u_2$, and $u_1 u_2 = u_3$. □

Lemma 35. *Assume $0 < b < N$ are elements of $\mathcal{F}$, and L, R, λ are variables ranging in $[b, N]$. Let $\mathcal{Q} = \mathcal{Q}(L \geqslant R)$ be the matrix obtained from*

$$Q = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & \lambda & N \\ 1 & 1 & 1 & 1 & L & R \end{pmatrix}$$

by applying the simple gadget to the blue variable. Then
(1) if we remove all the rows and columns of $\mathcal{Q}$ which contain variables (that is, that contain L and R), we get a matrix with nonnegative rank 8;
(2) for any assignment $L', R' \in [b, N]$ of the variables L, R, the matrix obtained from $\mathcal{Q}$ by realizing this assignment admits a nonnegative factorization of size 8 if and only if $L' \leqslant R'$. In this case, this factorization $\Upsilon(L', R')$ is unique, and the rule $\Upsilon(L', R') \to (L', R')$ defines an $\mathcal{F}$-rational projection of the set $\operatorname{fact}_+(\mathcal{Q}, \mathcal{R}, 8)$.

Proof. Similarly to the previous two lemmas, the situation reduces to the consideration of the set $\operatorname{fact}_+(Q, \mathcal{R}, 4)$. Using Lemma 28, we conclude that the first three matrices in every element of $\operatorname{fact}_+(Q, \mathcal{R}, 4)$ have the form

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1-x & 1-x & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & x & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-x & 1-x & 0 & 0 \end{pmatrix}$$

for some $x \in [0, 1]$, and hence the remaining fourth matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & L'/x & R'/x \\ x & 0 & 0 & x & L' & R' \end{pmatrix}$$

shows that $R'/x = N$ and $L'/x = \lambda$, or

$$x = R'/N \quad \text{and} \quad \frac{L'}{R'} = \frac{\lambda}{N} \in \left[\frac{b}{N}, 1\right].$$

Then we have $x \in [0, 1]$ and $L' \leqslant R'$ as expected, and, for every appropriate (L', R') the corresponding factorization is unique, and it satisfies the desired property. □

We are almost ready to complete the proof of the main result, which we do by applying Theorem 32 to the direct sum of several appropriate matrices as in the three lemmas above.

Remark 36. The *direct sum* D of (possibly incomplete) matrices $A_1, \ldots, A_s$ is the block-diagonal matrix having these matrices as diagonal blocks. We assume that each row corresponding to a matrix A_i has smaller index than any row corresponding to a matrix A_j provided that $i < j$. If $\operatorname{rk}_+(D) = r$ and $\operatorname{rk}_+(A_i) = r_i$ for all i, then

$$r \geqslant r_1 + \ldots + r_s, \tag{8.4}$$

and if a nonnegative factorization F of some completion of D has the size $r_1 + \ldots + r_s$, then F is the sum of nonnegative factorizations of the corresponding completions of the $A_1, \ldots, A_s$ blocks of the sizes $r_1, \ldots, r_s$, respectively.

Remark 37. The inequality (8.4) holds with the equality if D is a matrix without variables. Otherwise, this inequality may be strict in some cases. For instance, we can take a variable $x \in [1, 2]$, and then the nonnegative ranks of the incomplete matrices

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & x \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & 2 \\ 1 & x \end{pmatrix}$$

are both equal to one, but since there does not exist any x for which A_1 and A_2 are of rank one simultaneously, the direct sum of A_1 and A_2 has nonnegative rank three.

Now we can complete the proof of the main result.

The proof of the Universality theorem. For any polynomial $f \in \mathcal{F}[x_1, \ldots, x_n]$, we can write the inequality $f \geqslant 0$ in an equivalent form

$$s_1 + s_2\mu_2 + \ldots + s_l\mu_l \geqslant s_{l+1} + s_{l+2}\mu_{l+2} + \ldots + s_r\mu_r,$$

where the s_i's are positive elements in $\mathcal{F}$, and the μ_i's are non-empty products of the form

$$\alpha_{i1} \ldots \alpha_{im_i}$$

with $\alpha_{ij} \in \{x_1, \ldots, x_n\}$. We introduce a constant $N = s_1 + \ldots + s_r$ and new variables L, R and v_{ij}, where $i \in \{2, 3, \ldots, r\} \setminus \{l + 1\}$ and $j \in \{2, 3, \ldots, m_i\}$; we also set $v_{i1} = \alpha_{i1}$.

Now we are going to construct an incomplete matrix $\mathcal{H}(f)$ depending on the variables

$$(v_{ij}),\ x_1, \ldots, x_n,\ L,\ R.$$

We assume that the v_{ij}'s and x_k's range in $[0, 1]$, and that L, R range in $[\min\{s_1, s_{l+1}\}, N]$. We define $\mathcal{H}(f)$ as the direct sum of the following matrices:

(i) the matrix $\mathcal{Q}(L \geqslant R)$,
(ii) the matrix $\mathcal{S}(L = s_1 + s_2v_{2m_2} + \ldots + s_lv_{lm_l})$,
(iii) the matrix $\mathcal{S}(R = s_{l+1} + s_{l+2}v_{l+2,m_{l+2}} + \ldots + s_rv_{rm_r})$, and
(iv) the matrices $\mathcal{P}(v_{ij} = v_{i,j-1}\alpha_{ij})$ with $i \in \{2, 3, \ldots, r\} \setminus \{l + 1\}$ and $j \in \{2, \ldots, m_i\}$.

According to Lemmas 33, 34 and 35, the matrix obtained from $\mathcal{H}(f)$ by removing all the rows and columns in which at least one variable occurs has nonnegative rank

$$\rho(f) = 8 + 5r + 9(m_2 + m_3 + \ldots + m_l) + 9(m_{l+2} + m_{l+3} + \ldots + m_r) - 9(r - 2).$$

Now we construct $\mathcal{H} = \mathcal{H}(\Phi)$ as the direct sum of $\mathcal{H}(f_1), \ldots, \mathcal{H}(f_t)$ over all polynomials $f_1, \ldots, f_t \in \Phi$ as in the formulation of the Universality theorem. The matrix obtained from $\mathcal{H}$ by removing all rows and columns with variables has rank $\rho_0 = \rho(f_1) + \ldots + \rho(f_t)$, and Lemmas 33, 34 and 35 show that the completions $\hat{H}$ of $\mathcal{H}$ having the nonnegative rank equal to ρ_0 are uniquely determined by the assignment $(\xi_1, \ldots, \xi_n)$ of values to the variables $(x_1, \ldots, x_n)$. Moreover, the completion corresponding to such an assignment has a nonnegative factorization $\Xi(\xi_1, \ldots, \xi_n)$ of the desired size ρ_0 if and only if we have $(\xi_1, \ldots, \xi_n) \in [0, 1]^n$ and $(\xi_1, \ldots, \xi_n)$ is a simultaneous solution to the inequalities $f_1 \geqslant 0, \ldots, f_t \geqslant 0$, and whenever it exists the factorization $\Xi(\xi_1, \ldots, \xi_n)$ is unique, and the mapping

$$\Xi(\xi_1, \ldots, \xi_n) \to (\xi_1, \ldots, \xi_n)$$

is an $\mathcal{F}$-rational projection of $\operatorname{fact}_+(\mathcal{H}, \mathcal{R}, \rho_0)$. In particular, the set $\operatorname{fact}_+(\mathcal{H}, \mathcal{R}, \rho_0)$ is strongly $\mathcal{F}$-equivalent to the set of all those simultaneous solutions of the inequalities $f_1 \geqslant 0, \ldots, f_t \geqslant 0$ which have all variables in $[0, 1]$. It remains to apply Theorem 32 to transform $\mathcal{H}(\Phi)$ into a matrix without unknowns and complete the proof of the Universality theorem. $\square$

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